Supplemental Material for "Quantum geometry induced microwave enhancement of flat-band superconductivity"

Adiabatic Elimination of Remote Bands

Here we provide the alternative, and equivalent, procedure for obtaining the effective velocity operator $\check{\Gamma}$. We closely follow Refs. 35 and 36. We first consider the normal state with $\Delta = 0$ though we retain the Nambu space, so that we have full second-quantized BdG Hamiltonian of

$$H = \sum_{\mathbf{k},\mathbf{k}'} \Psi^\dagger_{\mathbf{k}'} \left[\check{\tau}_3 \hat{\mathcal{H}}_{\mathbf{k}-\mathbf{A}(t)\check{\tau}_3} \delta_{\mathbf{k}',\mathbf{k}} + \check{\tau}_3 \hat{V}^{\text{imp}}_{\mathbf{k}',\mathbf{k}} \right] \Psi_{\mathbf{k}}. \tag{S1}$$

Here $\Psi_{\mathbf{k}}$ is the Nambu annihilation operator for electrons in the multiband basis. Here we have used the Peierl's substitution on the normal-state Bloch Hamiltonian to obtain the effect of the gauge potential, and $\hat{V}^{\text{imp}}_{\mathbf{k}',\mathbf{k}}$ is the impurity scattering potential and is assumed to respect time-reversal symmetry. Even if this is a trivial identity matrix in the orbital basis, such that $\hat{V}^{\text{imp}}_{\mathbf{k}',\mathbf{k}} = V^{\text{imp}}_{\mathbf{k}',\mathbf{k}} \hat{1}$, this will still obtain nontrivial overlaps between Bloch bands of the form

$$U^{\beta\alpha}_{\mathbf{k}'\mathbf{k}} \equiv \langle \beta \mathbf{k}' | \alpha \mathbf{k} \rangle V^{\text{imp}}_{\mathbf{k}',\mathbf{k}}. \tag{S2}$$

The Bloch bands are denoted by $|\alpha\mathbf{k}\rangle$ with Bloch band eigenvalue $\epsilon_{\alpha\mathbf{k}}$.

It is useful to partition these bands in to two sets; the "flat bands" with energies $|\epsilon_{\alpha\mathbf{k}}| < \epsilon_c$ within a cutoff window of the Fermi level, and "remote bands with energies $|\epsilon_{\alpha\mathbf{k}}| > \epsilon_c$ outside this cutoff window. Presumably ϵ_c is large as compared to the energy scales of the flat-band band-width or superconductivity.

To this end, we introduce the projection operators, defined for every $\mathbf{k}$ point separately, as

$$\hat{\mathcal{P}}_{\mathbf{k}} = \sum_{\alpha} |\alpha\mathbf{k}\rangle\langle\alpha\mathbf{k}| \theta(|\epsilon_{\alpha\mathbf{k}}| < \epsilon_c), \tag{S3}$$

and its complement $\hat{\mathcal{Q}}_{\mathbf{k}} = \hat{1} - \hat{\mathcal{P}}_{\mathbf{k}}$. This involves the logical indicator function $\theta(x)$ which is one if x is true, and 0 if x is false.

Now, we perform Schrieffer-Wolff elimination of the electrons in these remote bands. For the linear case at hand, this is easiest done by taking the Heisenberg equations of motion. The equations of motion for the Nambu electrons is obtained as

$$\left[i\partial_t - \hat{\mathcal{H}}_{\mathbf{k}} \check{\tau}_3 \right] \psi_{\mathbf{k}} = -\mathbf{A}(t) \cdot \check{\mathbf{v}}_{\mathbf{k}} \psi_{\mathbf{k}} + \hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'} \check{\tau}_3 \psi_{\mathbf{k}'}. \tag{S4}$$

Here we have gone to the Bloch band basis, with $\psi_{\mathbf{k}}$ the Nambu-Bloch spinor for electrons. We have also expanded up to linear order in the perturbing electromagnetic potential, and defined the velocity operator

$$\check{\mathbf{v}}_{\mathbf{k}} = \check{\tau}_0 \partial_{\mathbf{k}} \hat{\mathcal{H}}_{\mathbf{k}}. \tag{S5}$$

This has matrix elements

$$\langle \alpha\mathbf{k} | \check{\mathbf{v}}_{\mathbf{k}} | \beta\mathbf{k} \rangle = \mathbf{v}^{\alpha\beta}_{\mathbf{k}}. \tag{S6}$$

We can now solve this equation by solving first for the remote bands; if we are only interested in low-energy properties, we can neglect $i\partial_t \ll \epsilon_c$, so that

$$-\hat{\mathcal{Q}}_{\mathbf{k}} \hat{\mathcal{H}}_{\mathbf{k}} \check{\tau}_3 \psi_{\mathbf{k}} = -\hat{\mathcal{Q}}_{\mathbf{k}} \mathbf{A}(t) \cdot \check{\mathbf{v}}_{\mathbf{k}} \hat{\mathcal{P}}_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\mathbf{k}'} \hat{\mathcal{Q}}_{\mathbf{k}} \hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'} \check{\tau}_3 \hat{\mathcal{P}}_{\mathbf{k}'} \psi_{\mathbf{k}'}. \tag{S7}$$

We have neglected the terms diagonal in $\hat{\mathcal{Q}} \cdot \hat{\mathcal{Q}}$ since we assume the bands are intrinsically unoccupied as $\epsilon_c \gg T$, so that $\hat{\mathcal{Q}}\psi$ can be regarded as near zero, and all occupation of the remote bands comes from the lower bands virtually scattering in to and out of them. We can then solve this and insert it in to equations of motion for the flat bands. Note $[\hat{\mathcal{Q}}_{\mathbf{k}}, \hat{\mathcal{H}}_{\mathbf{k}}] = 0$.

This is split between intra-flat band terms and terms which arise from perturbative dressing by the remote bands at second order, with

$$\begin{aligned}\left[i\partial_t - \hat{\mathcal{H}}_{\mathbf{k}}\check{\tau}_3\right]\hat{\mathcal{P}}_{\mathbf{k}}\psi_{\mathbf{k}} &= -\hat{\mathcal{P}}_{\mathbf{k}}\mathbf{A}(t)\cdot\check{\mathbf{v}}_{\mathbf{k}}\hat{\mathcal{P}}_{\mathbf{k}}\psi_{\mathbf{k}} + \sum_{\mathbf{k}'}\hat{\mathcal{P}}_{\mathbf{k}}\hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'}\check{\tau}_3\hat{\mathcal{P}}_{\mathbf{k}'}\psi_{\mathbf{k}'} \\ &- \mathbf{A}(t)\cdot\hat{\mathcal{P}}_{\mathbf{k}}\check{\mathbf{v}}_{\mathbf{k}}\hat{\mathcal{Q}}_{\mathbf{k}}\left(-\check{\tau}_3\hat{\mathcal{Q}}_{\mathbf{k}}\hat{\mathcal{H}}_{\mathbf{k}}\hat{\mathcal{Q}}_{\mathbf{k}}\right)^{-1}\left[-\mathbf{A}(t)\cdot\hat{\mathcal{Q}}_{\mathbf{k}}\check{\mathbf{v}}_{\mathbf{k}}\hat{\mathcal{P}}_{\mathbf{k}}\psi_{\mathbf{k}} + \hat{\mathcal{Q}}_{\mathbf{k}}\sum_{\mathbf{k}'}\hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'}\check{\tau}_3\hat{\mathcal{P}}_{\mathbf{k}'}\psi_{\mathbf{k}'}\right] \\ &+ \sum_{\mathbf{k}'}\hat{\mathcal{P}}_{\mathbf{k}}\hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'}\check{\tau}_3\hat{\mathcal{Q}}_{\mathbf{k}'}\left(-\check{\tau}_3\hat{\mathcal{Q}}_{\mathbf{k}'}\hat{\mathcal{H}}_{\mathbf{k}'}\hat{\mathcal{Q}}_{\mathbf{k}'}\right)^{-1}\left[-\mathbf{A}(t)\cdot\hat{\mathcal{Q}}_{\mathbf{k}'}\check{\mathbf{v}}_{\mathbf{k}'}\hat{\mathcal{P}}_{\mathbf{k}'}\psi_{\mathbf{k}'} + \hat{\mathcal{Q}}_{\mathbf{k}'}\sum_{\mathbf{k}''}\hat{V}^{\text{imp}}_{\mathbf{k}',\mathbf{k}''}\check{\tau}_3\hat{\mathcal{P}}_{\mathbf{k}''}\psi_{\mathbf{k}''}\right]. \quad \text{(S8)}\end{aligned}$$

Of the four terms generated by perturbation, we see the last one is second order in impurity scattering and therefore will only renormalize the already-present impurity scattering. We therefore ignore this term as it is not qualitatively important. Likewise, we see that there is a term generated at second order in the velocity operator. While this is in principle an interesting and important term to retain for studying the nonlinear response, or computing superfluid density for instance, we see that this will not generate the relevant inelastic scattering processes at second order since it does not scatter the momentum.

We therefore retain only the two cross terms, which we can use to define an effective velocity operator with matrix elements

$$\check{\mathbf{\Gamma}}_{\mathbf{k}\mathbf{k}'} = \hat{\mathcal{P}}_{\mathbf{k}}\check{\mathbf{v}}_{\mathbf{k}}\hat{\mathcal{Q}}_{\mathbf{k}}\left(-\check{\tau}_3\hat{\mathcal{Q}}_{\mathbf{k}}\hat{\mathcal{H}}_{\mathbf{k}}\hat{\mathcal{Q}}_{\mathbf{k}}\right)^{-1}\hat{\mathcal{Q}}_{\mathbf{k}}\hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'}\check{\tau}_3\hat{\mathcal{P}}_{\mathbf{k}'} + \hat{\mathcal{P}}_{\mathbf{k}}\hat{V}^{\text{imp}}_{\mathbf{k},\mathbf{k}'}\check{\tau}_3\hat{\mathcal{Q}}_{\mathbf{k}'}\left(-\check{\tau}_3\hat{\mathcal{Q}}_{\mathbf{k}'}\hat{\mathcal{H}}_{\mathbf{k}'}\hat{\mathcal{Q}}_{\mathbf{k}'}\right)^{-1}\hat{\mathcal{Q}}_{\mathbf{k}'}\check{\mathbf{v}}_{\mathbf{k}'}\hat{\mathcal{P}}_{\mathbf{k}'}. \quad \text{(S9)}$$

The explicit expression in terms of matrix elements is

$$\mathbf{\Gamma}^{\alpha\beta}_{\mathbf{k},\mathbf{k}'} \equiv \langle\alpha\mathbf{k}|\check{\mathbf{\Gamma}}_{\mathbf{k}\mathbf{k}'}|\beta\mathbf{k}'\rangle = -\sum_{\gamma:\ |\epsilon_{\gamma\mathbf{k}}|>\epsilon_c}\mathbf{v}^{\alpha\gamma}_{\mathbf{k}}\frac{1}{\epsilon_{\gamma\mathbf{k}}}U^{\gamma\beta}_{\mathbf{k},\mathbf{k}'} - \sum_{\gamma:\ |\epsilon_{\gamma\mathbf{k}'}|>\epsilon_c}U^{\alpha\gamma}_{\mathbf{k},\mathbf{k}'}\frac{1}{\epsilon_{\gamma\mathbf{k}'}}\mathbf{v}^{\gamma\beta}_{\mathbf{k}'}. \quad \text{(S10)}$$

This corresponds to a coherent sum over the possible scattering pathways which couple to a remote band at second order and scatter back off of the external vector potential. This then renormalizes the effective Hamiltonian for the flat-bands by introducing additional couplings to the vector potential of the form

$$\left[i\partial_t - \hat{\mathcal{H}}_{\mathbf{k}}\check{\tau}_3\right]\hat{\mathcal{P}}_{\mathbf{k}}\psi_{\alpha\mathbf{k}} = -\mathbf{A}(t)\cdot\left[\hat{\mathcal{P}}_{\mathbf{k}}\check{\mathbf{v}}_{\mathbf{k}}\hat{\mathcal{P}}_{\mathbf{k}}\psi_{\mathbf{k}} + \check{\mathbf{\Gamma}}_{\mathbf{k},\mathbf{k}'}\check{\mathcal{P}}_{\mathbf{k}'}\psi_{\mathbf{k}'}\right]. \quad \text{(S11)}$$

This has the benefit of only involving the low-energy degrees of freedom in the flat-band, valid up to first order in the external vector potential. Crucially, the second term leads to intra-flat band scattering. The matrix element for scattering is then obtained in terms of the impurity potential as

$$|\mathbf{\Gamma}^{\alpha\beta}_{\mathbf{k}\mathbf{k}'}|^2 = |V^{\text{imp}}_{\mathbf{k}',\mathbf{k}}|^2\left|\sum_{\gamma:\ |\epsilon_{\gamma\mathbf{k}}|>\epsilon_c}\mathbf{v}^{\alpha\gamma}_{\mathbf{k}}\frac{1}{\epsilon_{\gamma\mathbf{k}}}\langle\gamma\mathbf{k}|\beta\mathbf{k}'\rangle + \sum_{\gamma:\ |\epsilon_{\gamma\mathbf{k}'}|>\epsilon_c}\langle\alpha\mathbf{k}|\gamma\mathbf{k}'\rangle\frac{1}{\epsilon_{\gamma\mathbf{k}'}}\mathbf{v}^{\gamma\beta}_{\mathbf{k}'}\right|^2 \equiv N_{\text{imp}}V_0^2\Phi^{\alpha\beta}_{\mathbf{k},\mathbf{k}'}. \quad \text{(S12)}$$

For short-range impurity scattering we have the impurity averaged result of $\overline{|V^{\text{imp}}_{\mathbf{k}',\mathbf{k}}|^2} = N_{\text{imp}}V_0^2$ for the momentum scattering matrix elements. In the main text we will mostly be concerned with the case where we have only one flat band of interest at the Fermi level, in which case we simply drop the matrix elements on Φ. Futhermore, if one derives this using standard perturbation theory, the modification is only in $-1/\epsilon_{\gamma\mathbf{p}} \to \frac{1}{\epsilon_{\mu\mathbf{p}'}-\epsilon_{\gamma\mathbf{p}}}$ where $\mu, \mathbf{p}, \mathbf{p}'$ are as appropriate for that term in the summation, where $\mu\mathbf{p}'$ will run over the currently considered flat band. The corrections due to this are small in $\epsilon_{\mu\mathbf{p}}/\epsilon_c$ and therefore are controlled.

Superconducting gap enhancement in the Landau-Ginzburg regime

In this section, we briefly describe the enhancement of superconducting order in the Landau-Ginzburg regime. We start with the self-consistent gap equation, see Eq. (2) in the main text, and write the superconducting gap under an external drive as $\Delta \approx \Delta_0 + \delta\Delta$. Next, we collect the terms which capture the effect of microwave radiation such that

$$\frac{\delta\Delta}{\Delta_0}\left(\frac{1}{g} - \sum_{\mathbf{k}}\frac{1-2f^0_{\mathbf{k}}}{2E_{\mathbf{k}}}\right) = -\sum_{\mathbf{k}}\frac{\delta f_{\mathbf{k}}}{2E_{\mathbf{k}}} \quad \text{(S13)}$$

where as described in the main text $f^0_{\mathbf{k}}$ is the equilibrium distribution and $\delta f_{\mathbf{k}}$ is the correction under microwave drive.

To proceed, we show that for $T \lesssim T_c$, the bracket in Eq. (S13) is related to coeffcients a_0 and b_0 of the Ginzburg Landau free energy: $\mathcal{F}[\Delta_0] = a_0\Delta_0^2 + b_0\Delta_0^4/2$. This can be done in 2-steps. First, we write the quasiparticle energy denominator on the left hand side of Eq. (S13) as $E_{\mathbf{k}}^{-1} \approx |\epsilon_{\mathbf{k}} - \mu|(1 - \Delta^2/2|\epsilon_{\mathbf{k}} - \mu|^2)$. Second, we compare the respective terms with a_0 and b_0 which can be obtained following the standard thermodynamics of superconductors for $T \lesssim T_c$ which was recently formulated for flat band superconductors [13]. We assume momentum independent and real order parameter. The order parameter is self consistently obatined by solving $\partial\mathcal{F}[\Delta_0]/\partial\Delta_0 = 0$ which gives

$$a_0 + b_0\Delta_0^2 = 0. \tag{S14}$$

Here, $a_0 = 1/g - \beta^{-1}\sum_{\mathbf{k},n} G_e(\mathbf{k},\omega_n)G_h(\mathbf{k},\omega_n)$ and $b_0 = (\beta^{-1}/2)\sum_{\mathbf{k},n} G_e^2(\mathbf{k},\omega_n)G_h^2(\mathbf{k},\omega_n)$ with $G_{e/h}(\mathbf{k},\omega_n) = (\pm i\omega_n - \epsilon_{\pm\mathbf{k}} + \mu)^{-1}$ being the single particle/hole Mastsubara Green's function; $\sum_n$ denoted sum over Matsubara frequency. Performing the Matsubara sums, we can re-write Eq. (S13) as

$$\frac{\delta\Delta}{\Delta_0}\left(a_0 + 2b_0\Delta_0^2\right) = -\sum_{\mathbf{k}} \frac{\delta f_{\mathbf{k}}}{E_{\mathbf{k}}} \tag{S15}$$

and utilizing Eq. (S14) we obtain

$$\frac{\delta\Delta}{\Delta_0} = a_0^{-1}\frac{\delta f_{\mathbf{k}}}{E_{\mathbf{k}}} \tag{S16}$$

which is Eq. (9) of the main text. Using $1/g = \tanh[\beta_c(\epsilon_{\mathbf{k}} - \mu)/2]/2(\epsilon_{\mathbf{k}} - \mu)$, for $T \lesssim T_c$ we obtain

$$a_0 = \nu_F\left(\frac{T - T_c}{T_c}\right)\tanh\left(\frac{\beta_c W}{4}\right) \tag{S17}$$

for an isotropic flat band with $\epsilon_{\mathbf{k}} \in [-W/2, W/2]$ satisfying $\beta_c W \lesssim 1$.

Continuum model for strained TBG-hBN heterostructure

In this section, we detail how we simulated the electronic structure of TBG using the continuum model. For TBG, we define the lattice structure as in Ref. [37]. In each graphene layer the primitive (original) lattice vectors are $\mathbf{a}_1 = a_{\mathrm{G}}(1, 0)$ and $\mathbf{a}_2 = a_{\mathrm{G}}(1/2, \sqrt{3}/2)$ with $a_{\mathrm{G}} = 0.246$ nm being the lattice constant. The corresponding reciprocal space lattice vectors are $\mathbf{b}_1 = (2\pi/a_{\mathrm{G}})(1, -1/\sqrt{3})$ and $\mathbf{b}_2 = (2\pi/a_{\mathrm{G}})(0, 2/\sqrt{3})$, and Dirac points are located at $K_\zeta = -\zeta(2\mathbf{b}_1 + \mathbf{b}_2)/3$. For a twist angle θ (accounting for the rotation of layers), the lattice vectors of layer l are given by $\mathbf{a}_{l,i} = R(\mp\theta/2)\mathbf{a}_i$, $\mp$ for $l = 1, 2$ respectively, and $R(\theta)$ represents rotation by an angle θ about the normal. Also, from $\mathbf{a}_{l,i}.\mathbf{b}_{l',j} = 2\pi\delta_{ij}\delta_{ll'}$ we can check that the reciprocal lattice vectors become $\mathbf{b}_{l,i} = R(\mp\theta/2)\mathbf{b}_i$ with corresponding Dirac points now located at $\mathbf{K}_{l,\zeta} = -\zeta(2\mathbf{b}_{l,1} + \mathbf{b}_{l,2})/3$.

At small angles, the slight mismatch of the lattice period between two layers gives rise to long range moiré superlattices. The reciprocal lattice vectors for these moiré superlattices are given as $\mathbf{g}_i = \mathbf{b}_{1,i} - \mathbf{b}_{2,i}$. The superlattice vectors $\mathbf{L}$, can then be found using $\mathbf{g}_i.\mathbf{L}_j = 2\pi\delta_{ij}$, where $\mathbf{L}_1$ and $\mathbf{L}_2$ span the moiré unit cell with lattice constant $L = \mathbf{L}_1 = \mathbf{L}_2 = a_{\mathrm{G}}/[2\sin\theta/2]$.

Next, when the moiré superlattice constant is much longer than the atomic scale, the electronic structure can be described using an effective continuum model for each valley $\zeta = \pm$. The total Hamiltonian is block diagonal in the valley index, and for each valley, the effective Hamiltonian in the continuum model is written in terms of the sublattice and layer basis (A_1, B_1, A_2, B_2)

$$H_\zeta = \begin{pmatrix} H_{1,\zeta}(\mathbf{p}) & T_\zeta^\dagger \\ T_\zeta & H_{2,\zeta}(\mathbf{p}) \end{pmatrix} \tag{S18}$$

where $H_{l,\zeta} = -\hbar v_F R(\pm\theta/2)\mathbf{p}.(\zeta\sigma_x, \sigma_y)$ is the Hamiltonian for each layer with $\hbar v_F/a_{\mathrm{G}} = 2135.4$ meV, and

$$T_\zeta = \begin{pmatrix} t_{\mathrm{AA}} & t'_{\mathrm{AB}} \\ t'_{\mathrm{AB}} & t_{\mathrm{AA}} \end{pmatrix} + \begin{pmatrix} t_{\mathrm{AA}} & t'_{\mathrm{AB}}e^{-i\zeta\frac{2\pi}{3}} \\ t'_{\mathrm{AB}}e^{i\zeta\frac{2\pi}{3}} & t_{\mathrm{AA}} \end{pmatrix} e^{i\zeta\mathbf{g}_1.\mathbf{r}} + \begin{pmatrix} t_{\mathrm{AA}} & t'_{\mathrm{AB}}e^{i\zeta\frac{2\pi}{3}} \\ t'_{\mathrm{AB}}e^{-i\zeta\frac{2\pi}{3}} & t_{\mathrm{AA}} \end{pmatrix} e^{i\zeta(\mathbf{g}_1+\mathbf{g}_2).\mathbf{r}} \tag{S19}$$

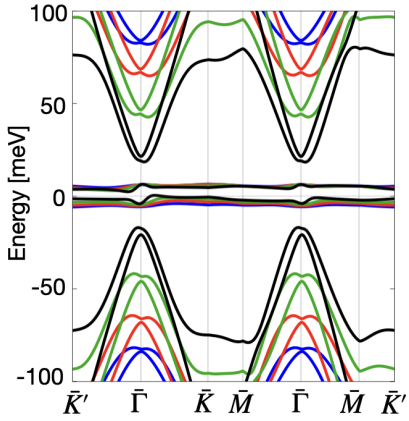

FIG. S1. TBG bandstructure for $t_{AA} = 20$ meV (blue), $t_{AA} = 40$ meV (red), $t_{AA} = 60$ meV (green) and $t_{AA} = 80$ meV (black). Additionally, we use $\delta_1 = \delta_2 = 5$ meV and strain of 0.1%.

where $\mathbf{g}_i$ is the reciprocal lattice vector of mBZ. We use the tunnelling parameter $t'_{\rm AB} = 97.5$ meV, and use different values for t_{AA}. When hBN is aligned with the graphene layers, C_2 symmetry is broken modifying the layer Hamiltonians $H_{l,\zeta}$. This can be described by introducing a sublattice staggered potential Δ_l so that the Hamiltonian for each layer $H_{l,\zeta}(\mathbf{p}) \to H_{l,\zeta}(\mathbf{p}) + \Delta_l \sigma_z$.

Finally, the presence of a uniaxial heterostrain in TBG of magnitude χ can be described by the linear strain tensor

$$\mathcal{E}_l = \mathcal{F}(l)\chi \begin{pmatrix} -\cos^2\varphi + \nu\sin^2\varphi & (1+\nu)\cos\varphi\sin\varphi \\ (1+\nu)\cos\varphi\sin\varphi & \nu\cos^2\varphi - \sin^2\varphi \end{pmatrix} \tag{S20}$$

where $\mathcal{F}(l = 1, 2) = \mp 1/2$, $\nu = 0.165$ is the Poisson ratio of graphene and φ gives direction of the applied strain. The strain tensor satisfies general transformations in each layer, $\mathbf{a}_l \to \mathbf{a}'_l = [1 + \mathcal{E}_l]\mathbf{a}_l$ and $\mathbf{b}_l \to \mathbf{b}'_l \approx [1 - \mathcal{E}_l^T]\mathbf{b}_l$ for real and reciprocal lattice vectors respectively. The strain induced geometric deformations affect the interlayer coupling and further changes the electron motion via gauge field $\mathbf{A}_l = \sqrt{3}\beta/2a_{\rm G}(\mathcal{E}_l^{xx} + \mathcal{E}_l^{yy}, -2\mathcal{E}_l^{xy})$, where $\beta = 3.14$. As a result, we have $\mathbf{p} \to \mathbf{p}_{l,\zeta} = [1 + \mathcal{E}_l^T][\mathbf{k} - \mathcal{K}_{l,\zeta}]$ with $\mathcal{K}_{l,\zeta} = [1 - \mathcal{E}_l^T]\mathbf{K}_{l,\zeta} - \zeta\mathbf{A}_l$.

The effective TBG Hamiltonian modified by the effects of strain and hBN alignment with graphene layers via sublattice staggered potential, can be re-written as

$$\mathcal{H}_\zeta = \begin{pmatrix} H_{1,\zeta}(\mathbf{p}_{1,\zeta}) + \Delta_1\sigma_z & T_\zeta^\dagger \\ T_\zeta & H_{2,\zeta}(\mathbf{p}_{2,\zeta}) + \Delta_2\sigma_z \end{pmatrix} \tag{S21}$$

Note that for a given $\mathbf{q}$ in the mBZ, the 4×4 Hamiltonian in Eq. (S21) is cast into a multiband eigensystem problem as the interlayer coupling leads to hybridisation of the eigenstates at Bloch vectors $\mathbf{q}$ and $\mathbf{p}' = \mathbf{p} + \mathbf{g}$, where $\mathbf{g} = m_1\mathbf{g}_1 + m_2\mathbf{g}_2$ and $m_{1,2} \in \mathbb{Z}$. We truncate the size of the matrix by defining a circular cut-off $|\mathbf{p} - \mathbf{p}'| < 4|\mathbf{g}_1|$. For a given Bloch vector $\mathbf{p}$, this gives us 61 sites in reciprocal space, and a corresponding matrix of size 244×244 which is then diagonalized to obtain eigenvalues and eigenvectors. TBG bandstructure with strain and hBN encapsulation is shown in Fig. at different values of t_{AA}.

Numerical evaluation of $\delta\Delta/\Delta_0$ in TBG-heterostructure

The change in superconducting order $\delta\Delta/\Delta_0$ is calculated using Eq. (9) where the change in distribution function under drive is calculated using collision integral in Eq. (3). Two Riemann sums, over $\mathbf{k}'$ and $\mathbf{k}$ in Eq. (3) and Eq. (9), respectively are calculated over discrete grid in (k_x, k_y) plane of mBZ. Two sums make the problem computationally heavy for which we choose a coarse grid of 50×50 points, and include only two flat bands with the nearest remote bands in our calculation. Further, we choose $\Delta_0 = 0.1$ meV and approximate δ-function as a Lorentzian with phenomenological energy broadening of 0.1 meV.