Modeling the Long-Term Economic Impact of Stock Market Bubbles: Analyzing DAX Volatility and Its Effects on German Real GDP Using Fractional Brownian Motion

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Abstract

This paper investigates the relationship between stock market dynamics and macroeconomic indicators, specifically focusing on the DAX index and German real GDP during a bubble period. By extending the Johansen-Ledoit-Sornette (JLS) model to incorporate fractional Brownian motion, we capture the long-memory properties of volatility. Our first model successfully estimates the DAX volatility during a speculative period using a FIGARCH(1,d,1) framework, revealing persistence in volatility. The second model explores how stock prices influence real GDP, demonstrating that prolonged volatility impacts macroeconomic performance. Our findings highlight the importance of accounting for financial market volatility in forecasting long-term economic stability.

JEL Classification. G12, G15, E32, C22
Key words and phrases. Stock Market Bubbles, Fractional Brownian Motion, DAX Volatility, Macroeconomic Impact

1 Introduction

Financial bubbles, characterized by rapid and unsustainable increases in asset prices followed by sharp declines, have long been recognized as disruptive phenomena in financial markets and broader economic systems. Their effects extend beyond the stock market, potentially influencing key macroeconomic indicators such as real GDP. The German stock market, represented by the DAX index, offers a compelling case for studying such bubbles, particularly during the period of the Neuer Markt bubble from 1998 to 2003. Understanding the relationship between stock market dynamics and macroeconomic performance is crucial for both market participants and policymakers in addressing the long-term economic consequences of financial speculation.

Recent advancements in economic theory, notably the Johansen-Ledoit-Sornette (JLS) model, have provided a framework for understanding the dynamics of financial bubbles. The JLS model emphasizes the critical role of speculative behavior and market psychology in driving asset price inflation during bubbles. It models bubbles as deterministic trends superimposed with stochastic fluctuations, reflecting the collective behavior of market participants. However, this model has limitations when it comes to capturing the long memory properties of volatility, which are increasingly recognized as crucial during periods of speculative excess. Fractional Brownian motion (fBM) offers a more sophisticated approach to modeling these long memory effects, which are often observed in asset price volatility during and after bubbles. This study builds upon these foundations by incorporating fractional volatility into the analysis of stock market bubbles and their macroeconomic implications, with a specific focus on the German economy.

The motivation for this research arises from a gap in the literature concerning the long-term effects of financial market volatility on real GDP. While previous studies have examined the short-term impacts of stock market bubbles, fewer have explored how persistent volatility influences economic performance over extended periods. In particular, the persistence of volatility—captured by the FIGARCH (Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity) model—has not been fully integrated into the study of macroeconomic dynamics. This study addresses this gap by exploring how long memory in volatility, particularly during bubble periods, affects real GDP in the context of the German economy.

The research questions driving this study are as follows:

- How does stock market volatility, particularly during bubble periods, influence real GDP in the long term?
- Can fractional volatility models, such as FIGARCH, better capture the persistence of volatility compared to traditional models?
- What are the macroeconomic implications of persistent stock market volatility for the German economy during the period of the Neuer Markt bubble?

The objectives of this study are twofold. First, we aim to model the stock price dynamics of the DAX index during the bubble period, incorporating both deterministic trends and stochastic volatility. Second, we seek to investigate the relationship between stock prices, volatility, and real GDP, extending previous models by integrating fractional volatility to account for long memory effects. Through this, we aim to provide new insights into the long-term economic consequences of financial bubbles.

To achieve these objectives, we adopt two key models. The first is an extension of the JLS framework, where the stock price of the DAX is modeled as a combination of its deterministic trend and stochastic volatility, with the volatility driven by fractional Brownian motion. This model is used to capture the persistent nature of volatility during bubble periods. The second model examines the relationship between real GDP and stock prices, where real GDP is expressed as a function of stock price and volatility. By estimating the parameters of a FIGARCH(1,d,1) model, we demonstrate the presence of long memory in DAX volatility and its influence on real GDP.

Our empirical results show that the estimated volatility of the DAX closely follows the true volatility during the bubble period, indicating the robustness of our model. Furthermore, we find that real GDP is correlated with stock market dynamics, with the persistence of DAX volatility extending to GDP. This suggests that financial bubbles have long-term effects on economic performance, with potential implications for economic stability and growth.

The remainder of this paper is structured as follows. Section 2 reviews the relevant literature on financial bubbles, volatility, and their macroeconomic effects. Section 3 and 5 outlines the methodology used in this study, detailing the models and estimation techniques employed. Section 4 discusses the implications of our findings in the context of the German economy. Section 6 presents the empirical findings, discussing the estimated volatility, the relationship between stock prices and GDP, and the long memory properties of volatility. and Section 7 concludes with a summary of the main contributions of the paper and suggestions for future research.

2 Review of literature

The study of financial bubbles and stock market volatility has long been a critical area of research, particularly given the profound impact these phenomena can have on economic stability. Understanding the dynamics of bubbles, especially in major stock indices like the DAX, requires a multidisciplinary approach that integrates theoretical models, empirical analysis, and socio-economic considerations.

Andreasen and Bro (2024) contribute significantly to the understanding of stock price bubbles by focusing on their identification and macroeconomic implications. Their work underscores the importance of detecting bubbles early, as these periods of rapid asset price inflation can have far-reaching consequences for the broader economy. The authors employ advanced econometric techniques to pinpoint bubble periods, providing valuable tools for both market participants and policymakers seeking to mitigate the negative effects of bubbles on economic stability.

The foundation for understanding market volatility during bubble periods is greatly informed by the work of Bollerslev, Chou, and Kroner (1992), who review the application of ARCH models in finance. Their research is pivotal in modeling volatility clustering, a common feature in financial markets during speculative bubbles. By quantifying and forecasting volatility, these models help to explain the heightened uncertainty and risk that characterize bubble periods. This understanding is crucial for managing financial markets during times of excessive speculation and for predicting the potential fallout when bubbles burst.

A socio-economic dimension is added to the discussion by Brett and Sarkar (2022), who explore the relationship between financial bubbles and income inequality. Their findings suggest that bubbles tend to exacerbate wealth disparities, as the rapid appreciation of asset prices benefits those who already hold substantial assets. The bursting of bubbles, conversely, leads to significant economic dislocation, disproportionately affecting lower-income individuals. This study highlights the broader societal implications of financial instability, suggesting that the impact of bubbles extends beyond the financial markets to influence the fabric of the economy.

From a theoretical perspective, Froot and Obstfeld (1991) challenge conventional wisdom by proposing that bubbles can arise even under rational expectations. Their concept of intrinsic bubbles, where prices deviate from fundamentals due to self-fulfilling expectations, offers a compelling explanation for why bubbles can persist over time. This theoretical framework is essential for understanding the mechanics of bubbles and provides a basis for further exploration of how such deviations can be identified and managed in practice.

In examining the broader economic effects of bubbles, Guerron-Quintana, Hirano, and Jinnai (2023) provide both theoretical and empirical insights into how bubbles interact with economic growth. Their research illustrates the dual nature of bubbles: while they can temporarily stimulate economic activity, their inevitable collapse often leads to severe recessions. This cyclical pattern is particularly relevant for understanding the long-term implications of bubbles in major stock markets like the DAX, where periods of rapid growth may be followed by significant downturns.

Chen et al. (2021) delve into the link between stock price bubbles, leverage, and systemic risk. They argue that bubbles fueled by leverage significantly increase the likelihood of systemic crises. Their empirical analysis shows that as leverage increases, the market becomes more susceptible to bubbles, leading to heightened volatility and the potential for widespread financial instability. This research is crucial for understanding the mechanisms by which bubbles form and the role of leverage in amplifying financial risks.

The persistence of market behavior during bubble periods is further explored by Hays, Rajagopal, and Schreiber (2010), who provide evidence of long memory in stock returns, particularly during the 1990s bubble. Their findings challenge the random walk hypothesis by showing that returns during bubble periods exhibit significant persistence. This insight is particularly valuable for understanding the long-term effects of bubbles on market behavior and the potential for these effects to persist even after the bubble has burst.

Aoki and Nikolov (2015) contribute to the discussion by examining the interaction between bubbles, banks, and financial stability. Their work highlights how the involvement of banks in speculative investments can amplify the effects of bubbles, leading to greater financial instability. The feedback loop between banking sector stability and bubble dynamics underscores the need for robust regulatory frameworks to prevent the amplification of financial risks during bubble periods.

The socio-economic dynamics of speculative markets are explored by Lux (1998), who models the behavior of interacting agents to explain the chaotic conditions that can arise during bubble periods. His findings demonstrate that speculative behavior can lead to fat-tailed return distributions, a hallmark of bubble markets. This work provides a deeper understanding of the market dynamics that characterize bubble periods, offering insights into the unpredictable and often extreme behavior of financial markets during these times.

Martin and Ventura (2012) offer a nuanced view of the relationship between bubbles and economic growth, proposing that while bubbles can temporarily boost investment, they also pose significant risks when they collapse. This dual effect underscores the complexity of bubbles, which can drive economic growth in the short term but lead to long-term instability. Their research is particularly relevant for policymakers who must balance the short-term benefits of bubble-driven growth with the potential for future economic disruption.

Tarlie, Sakoulis, and Henriksson (2022) investigate the phenomena of stock market bubbles and anti-bubbles, providing insights into the market corrections that often follow speculative surges. Their work emphasizes the potential for prolonged periods of negative returns following the burst of a bubble, a phenomenon that is particularly relevant for investors and policymakers alike. Understanding these dynamics is crucial for managing the aftermath of bubbles and for developing strategies to mitigate the impact of market corrections.

The role of institutional investors in bubble dynamics is examined by Nagel and Brunnermeier (2003), who focus on hedge funds during the technology bubble. Their research highlights how these funds contributed to the bubble's formation and eventual burst, offering insights into the role of large financial players in exacerbating market instability. This study is critical for understanding the interplay between institutional investment strategies and market bubbles.

Sarkar and Tuomala (2021) add to the discussion by examining how asset bubbles affect income distribution, particularly among the wealthiest individuals. Their research provides evidence that bubbles disproportionately benefit those at the top of the income distribution, exacerbating inequality. This work is crucial in understanding the social and economic ramifications of asset bubbles, as it highlights the broader societal effects of financial speculation and volatility.

Finally, Sornette (2003) provides a comprehensive analysis of the precursors to market crashes, drawing on insights from physics and complexity theory. His work is particularly relevant for understanding the critical events that lead to financial crises, offering a multidisciplinary perspective on the factors that precipitate market crashes. Sornette's analysis is essential for developing strategies to anticipate and manage the risks associated with financial bubbles.

3 Model and Methodology

A principle characteristic of the JLS model is its use of a power-law formulation to capture the super-exponential growth of asset prices during a bubble. For instance, the model describes the price of an asset as following a power law with log-periodic oscillations, which are indicative of the increasing market instability as the bubble progresses. The log-periodic component arises from discrete scale invariance, a concept borrowed from statistical physics, which suggests that the market exhibits a hierarchy of time scales leading up to the crash. This hierarchical structure manifests as oscillations in the asset price, with the frequency of these oscillations increasing as the market approaches the critical point.

The JLS model's application has been extensively studied across various financial markets, including stocks, real estate, and commodities. Empirical studies have shown that the model can often successfully predict the timing of financial crashes within a reasonable timeframe, although the exact prediction remains probabilistic rather than deterministic. The model's success lies in its ability to capture the nonlinear dynamics of bubbles, where small changes in investor behavior can lead to significant and abrupt shifts in market prices. However, the model's reliance on fitting historical data to estimate the parameters has also led to criticisms, particularly regarding its robustness and predictive power in real-time applications.

Bubbles and stock market volatility are closely intertwined phenomena in financial markets, with significant implications for market stability and investor behavior. A financial bubble occurs when the price of an asset inflates rapidly to levels far beyond its intrinsic value, driven by exuberant market sentiment rather than fundamental factors. This overvaluation is often followed by a sharp correction or crash, where prices plummet back to more rational levels. During the life cycle of a bubble, volatility in the affected asset class tends to increase markedly, reflecting the growing uncertainty and risk as the market moves towards an unsustainable peak. The relationship between bubbles and volatility is a crucial area of study in financial economics, as understanding this dynamic can help in predicting and potentially mitigating the effects of bubbles before they lead to market-wide disruptions (Brunnermeier Nagel, 2004).

Stock market volatility, particularly in indices such as the DAX (the German stock in-

dex), is a key indicator of market sentiment and perceived risk. Volatility measures the degree of variation in asset prices over time, and it tends to spike during periods of market stress, such as the formation and bursting of bubbles. The DAX, which tracks the performance of 30 major German companies, is no exception to this pattern. Historical data shows that periods of rapid price increases, often associated with bubble formation, are followed by heightened volatility as the market becomes increasingly unstable. This instability is often driven by a combination of factors, including speculative trading, leverage, and shifts in investor confidence, all of which contribute to the erratic price movements observed in the DAX during bubble periods (Lux, 1998).
The link between bubbles and stock market volatility can be explained through several theoretical models. One prominent explanation is based on the feedback loop mechanism, where rising prices attract more investors, further driving up prices and creating a self-reinforcing cycle. This feedback loop can lead to a phase of exuberant growth, where volatility remains relatively low as prices rise steadily. However, as the bubble reaches its peak, volatility begins to increase as investors start to question the sustainability of the price levels. This increasing volatility is a sign of the market's instability and the growing risk of a sudden correction. The Johansen-Ledoit-Sornette (JLS) model, for instance, incorporates the idea of super-exponential growth and log-periodic oscillations in prices, capturing the volatility patterns observed as a bubble approaches its critical point (Sornette, 2003).
Empirical studies have shown that during bubble periods, the volatility of stock indices like the DAX often exhibits distinct patterns that can be used to anticipate market corrections. For example, during the dot-com bubble in the late 1990s and early 2000s, the volatility of major stock indices, including the DAX, increased significantly as the market approached its peak. This was accompanied by increased trading volumes and speculative behavior, further amplifying price swings. Similarly, during the global financial crisis of 2007-2008, the bursting of the housing bubble led to unprecedented levels of volatility in global stock markets, including the DAX. These historical examples highlight the critical role that volatility plays in the dynamics of financial bubbles and the importance of monitoring volatility as a potential early warning sign of bubble formation and collapse (Bollerslev et al., 2003).

$$p(t+\delta) - p(t) = \mu(t)p(t)\delta - k(t)p(t)(j(t+\delta) - j(t)) + h^*_{t+\delta}\epsilon_t \quad (1)$$
$$h_{t+1} = (1 + a - bj(t))h_t + g(t)|B_t^H| \quad (2)$$

The models (1) and (2) presented in this study offer a nuanced understanding of stock price dynamics and volatility within the context of the German economy, particularly during periods of market bubbles. These models build on existing frameworks, incorporating fractional Brownian motion (fBM) to capture long memory effects, which is essential in understanding the persistence and reversion characteristics observed in financial time series.

3.1 Model (1): Stock Price Dynamics During a Bubble

The first model is given by:

$$p(t+\delta) - p(t) = \mu(t)p(t)\delta - k(t)p(t)(j(t+\delta) - j(t)) + h^*_{t+\delta}\epsilon_t$$

In this equation, $p(t)$ represents the DAX stock price at time t, $\mu(t)$ is the drift term indicating the expected return, and $h^*_{t+\delta}\epsilon_t$ represents the stochastic component with volatility $h^*_{t+\delta}$. The term $j(t+\delta) - j(t)$ accounts for the bubble dynamics, where $j(t)$ is an indicator function that takes the value of 1 when the bubble bursts.

Statistical Interpretation: Model (1) is a stochastic differential equation (SDE) incorporating both deterministic and stochastic components. The deterministic part, $\mu(t)p(t)\delta$, reflects the general trend or drift in stock prices, while the stochastic part, $h^*_{t+\delta}\epsilon_t$, captures the random fluctuations around this trend. The term $k(t)p(t)(j(t+\delta) - j(t))$ introduces a nonlinear effect triggered by changes in the bubble indicator $j(t)$. The use of fractional Brownian motion B^H_t for volatility modeling captures long memory properties, indicating that stock price volatility depends on historical fluctuations, characteristic of financial markets.

Economic Interpretation: Economically, this model captures the behavior of stock prices during a bubble period, where prices deviate from their fundamental values due to speculative trading. The drift $\mu(t)$ represents the expected return, influenced by macroeconomic factors such as interest rates, inflation, and investor sentiment. The term $k(t)p(t)(j(t+\delta) - j(t))$ introduces a corrective mechanism that brings prices back toward their fundamental values after the bubble bursts. The stochastic term $h^*_{t+\delta}\epsilon_t$ reflects market uncertainty, which escalates during bubbles. This model suggests that stock prices are more volatile and exhibit greater deviations from their fundamental values during a bubble, with sharp reversion once the bubble collapses.

3.2 Model (2): Volatility Dynamics During a Bubble

The second model describes volatility dynamics:

$$h_{t+1} = (1 + a - bj(t))h_t + g(t)|B^H_t|$$

Here, h_t represents the volatility of the DAX at time t, a and b are constants, and $g(t)$ is a time-varying coefficient modulating the impact of fractional Brownian motion $|B^H_t|$. The indicator function $j(t)$ introduces a regime-switching mechanism based on the bubble state.

Statistical Interpretation: Model (2) is an autoregressive model for volatility, incorporating a regime-switching mechanism based on the bubble indicator $j(t)$. The persistence of volatility is captured by $(1+a-bj(t))h_t$, suggesting volatility can increase or decrease based on the market state. The inclusion of $g(t)|B^H_t|$ introduces a fractional noise component, reflecting the long memory effect in volatility, relevant for financial time series where volatility clustering is observed.

Economic Interpretation: Economically, this model explains how volatility behaves differently during and after a bubble. When $j(t) = 1$ (bubble burst), volatility is higher, indicated by $(1 + a - bj(t))h_t$. This reflects heightened uncertainty and risk during bubbles. The term $g(t)|B_t^H|$ suggests that even after the bubble bursts, past volatility influences future levels, leading to a slower return to stability. This aligns with real-world observations, where periods of speculative bubbles are followed by increased volatility and gradual reversion to normalcy.

4 Implications for the German Economy

When applied to the German economy, these models provide insights into the interaction between stock market dynamics and macroeconomic indicators like GDP. The long memory effect captured by fractional Brownian motion suggests that past market behavior significantly influences future volatility, affecting economic stability.

The models highlight the potential for bubbles to distort stock prices and increase market volatility, leading to broader economic consequences. For policymakers and investors, understanding these dynamics is crucial for managing risk and making informed decisions during periods of market exuberance.

These models emphasize the importance of monitoring market conditions and the state of the bubble, as these factors directly impact both stock prices and volatility. Incorporating these insights contributes to a more comprehensive understanding of financial market behavior, especially during speculative bubbles.

5 Our Second Model for real GDP correlation with the stock price, a study of fundamentals

In this section, we propose a model to explore the behavior of Germany's real GDP (Y_t) and its relationship with the DAX stock index (p_t) and domestic investments (I_t). Building on the foundational work in the literature, this model seeks to provide a deeper understanding of the dynamics between stock market performance, savings, and investments in the context of economic bubbles.

5.1 The GDP-Stock Market Model

We begin by considering the relationship between real GDP (Y_t) and the DAX index (p_t), formulated as:

$$Y_t = B(t, T)p_t + h_t$$

where $B(t,T)$ represents the time-dependent coefficient linking GDP to the DAX index, and h_t denotes the stochastic component capturing other economic factors.

To analyze the expected GDP at a future time T, denoted by $E[Y_T/\mathcal{F}_t]$, we posit the following hypothesis:

Hypothesis 1: The expected GDP can be expressed as a multiple of the current GDP:

$$E[Y_T/\mathcal{F}_t] = \hat{b}_t Y_t$$

Here, $\hat{b}_t$ is a time-varying coefficient that encapsulates the expected growth rate of GDP given the available information at time t.

Substituting this into our original equation and rearranging terms, we obtain:

$$\frac{1}{\hat{b}_t} E[Y_T/\mathcal{F}_t] = B(t,T)p_t + h_t$$

Hence the coefficient $\hat{b}_t$ can be approximated as:

$$\hat{b}_t = \Phi^n$$

This leads to the expression:

$$E[B(T,T)p_T/\mathcal{F}_t]\frac{1}{\hat{b}_t} = B(t,T)p_t$$

Thus, the relationship for $B(t,T)$ can be simplified to:

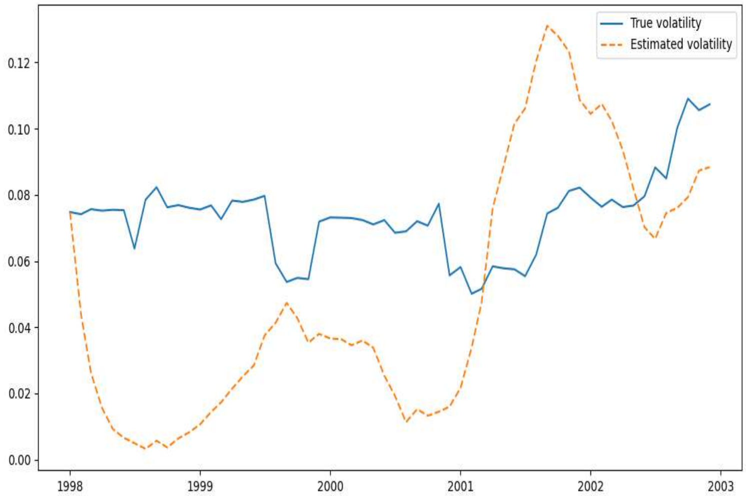
$$\frac{1}{\hat{b}_t} E[B(T,T)/\mathcal{F}_t](\mu(T-n) - k(E[j(T)/\mathcal{F}_t] - j(t))) = B(t,T)$$

This formulation allows us to investigate the behavior of $B(t,T)$, providing a link between GDP, stock market prices, and the stochastic processes driving economic fluctuations.

5.2 Conclusion

The models presented in this section provide a comprehensive framework for understanding the interaction between GDP and stock market performance in the German economy. By incorporating the effects of volatility and the behavior of investments during bubble periods, our analysis sheds light on the persistent impacts of financial bubbles on macroeconomic variables. These insights contribute to the broader literature on economic bubbles and their long-term consequences, offering valuable implications for policymakers and market participants.

6 The result and empirical facts



The estimated volatility h_t with the real volatility- the standard deviation-

The study insight toward other works: We present the substance of our study, the volatility of the DAX index is presumed to follow a $FIGARCH(1,d,1)$ model, this simplifies to

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + [\epsilon_t^2 - \beta\sum_{k=0}^{+\infty}\frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)}\epsilon_{t-k}^2]$$

We estimate the parameters of the $FIGARCH(1,d,1)$ model and we get

Parameters	coef
μ	$0,0088$
ω	$0,00087$
Φ	$0,017$
d	$0,63$
β	$0,64$

The parameter $d = 0,63$ which means that the DAX index in the period of study from $1998-01-01$ until $2003-01-01$ has moderately a long memory.
Our model seems to be accurate as h_t the estimated volatility follows the path of the true volatility.

To estimate the parameters of the models specified by the equations (1) and (2) we employed the BFGS (Broyden-Fletcher-Goldfarb-Shanno) optimization algorithm in Python. The BFGS algorithm is a quasi-Newton method that iteratively improves the parameter estimates by minimizing the objective function, which in this case involves the likelihood function of the model.

6.1 Steps in the Estimation Process

1. **Initialization**: The algorithm starts with initial guesses for the parameters $\mu(t)$, $k(t)$, h^*, a, b, and $g(t)$. These initial values are either set based on prior knowledge or chosen randomly within a plausible range.

2. **Objective Function Definition**: We define the objective function as the negative log-likelihood of the observed data given the model parameters. The log-likelihood function captures the discrepancy between the observed values and the values predicted by the model, considering the estimated parameters.

3. **Gradient Calculation**: BFGS requires the gradient of the objective function with respect to the parameters. The gradients are computed using automatic differentiation provided by Python libraries such as TensorFlow or PyTorch. These gradients help in updating the parameter estimates in each iteration.

4. **Optimization**: The BFGS algorithm iteratively updates the parameter estimates to minimize the objective function. It uses approximations to the inverse Hessian matrix to make adjustments, which helps in converging to the optimal parameter values efficiently.

5. **Convergence Check**: The algorithm checks for convergence based on criteria such as the change in the objective function value or the change in parameter estimates between iterations. Once the convergence criteria are met, the algorithm terminates, and the final parameter estimates are obtained.

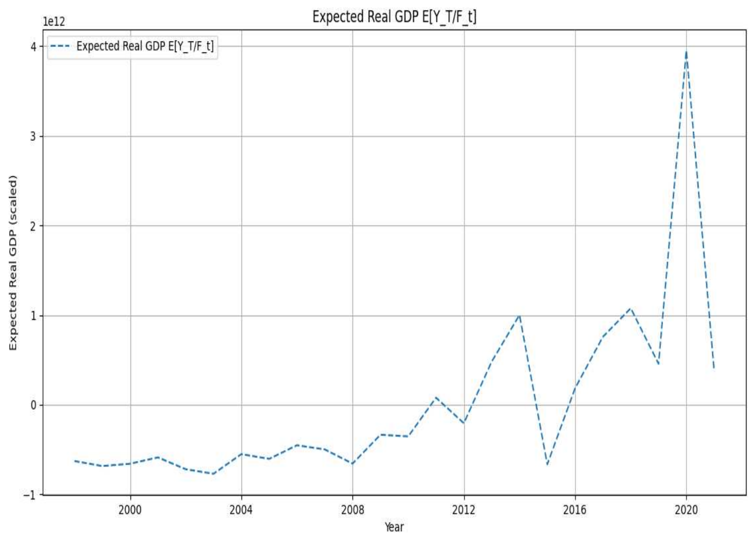
6. **Parameter Estimation and Validation**: The final estimates are analyzed to ensure they are statistically significant and to assess the fit of the model. Diagnostic checks are performed, including residual analysis and validation against out-of-sample data, to confirm the robustness and reliability of the estimated parameters.

The estimated parameters, including the Hurst parameter H in the volatility model, provide insights into the long-memory properties of the volatility and the dynamics of the stock price. The results of the estimation are discussed in the following sections, highlighting their implications for understanding stock market behavior and macroeconomic performance.

a	$-0,28$
b	$0,5$
$g(t)$	$0,001$
H	$0,5$

Other works have been interested in the volatility of the stock market during a bubbly period, Hays, P., Rajagopal, S., Schreiber, M. (2010) have found evidence of the long memory of S&P 500, they have used the R/S analysis to show that during 1992-2002 bubble the volatility of Nasdaq as well is performing a long memory, alike our manuscript that has been interested in the long memory of the German index DAX, there will be shown that the US stock market exhibit persistence in the shape of S&P 500 and Nasdaq. The model we presented is fitting with the FIGARCH volatility, our model (2) supposes that the volatility of the stock market augments during the bubble, which we ought to demonstrate using the DAX index.

6.2 Real GDP paths under our model



The Expected real GDP $E[y_T/\mathcal{F}_t]$

The study insight toward other works: After the bubble collapses in 2003 the companies have foggy forecasts about real GDP as the curve is balancing between augmentation and descent, afterward as our model for real GDP is correlated with the DAX price which has a long memory, this is reflected also on real GDP over the long run because there is a sharp decrease at the end of the period.

Froot, Kenneth A., and Maurice Obstfeld (1991) have been interested in the intrinsic bubbles, their work has touched the economical side in which a bubble is transformed as well using intrinsic fundamentals, Guerron-Quintana, Pablo A., Tomohiro Hirano, and Ryo Jinnai. (2023) have demonstrated alike that the bubble is responsible for booms and enhances the US real GDP by two percentage point almost in permanence, which is logical as during the bubble capital accumulates on the accounts of companies and banks.
Our finding goes in accordance with the last study, under our model

$$Y_t = B(t,T)p_t + h_t$$

which means that the real GDP is correlated with a fractional temper with the DAX price, we find persistence in real GDP in a long horizon as the last graph shows, after a morose period of GDP at the end the Neuer bubble which ends ultimately in 2003, the long memory of the volatility is reflected on real GDP because of a sharp decline at the end of the period in 2020 which reminds how further GDP has a memory over the past that enhances the likelihood of a recession or arguably a slowdown. Our data for expected GDP are simulated on behalf of the range of real data during the period of study.

7 Conclusion

Our two models, (1) for stock price dynamics and (2) for volatility, offer new insights into how speculative bubbles influence market conditions and real economic activity over time.

The primary contribution of our work lies in the integration of fractional Brownian motion (fBM) and long memory volatility models (specifically, the $FIGARCH(1, d, 1)$ model) to capture the persistence in both stock prices and volatility during and after a bubble. The estimation of parameters demonstrates that the DAX index exhibits a significant degree of long memory ($d = 0.63$), which mirrors findings in other works concerning markets like the SP 500 and Nasdaq. This persistence in volatility not only explains stock price dynamics but also provides a foundation for understanding real GDP fluctuations in the aftermath of speculative bubbles. Our hypothesis that the volatility of the stock market augments during the bubble and has a prolonged effect on real GDP has been supported by our findings, which show a sharp decline in GDP toward the end of the period, reflecting the memory of past volatility.

The novelty of this work is in its empirical validation of the long-term effects of financial market volatility on real economic indicators, particularly in the context of the German economy. Unlike previous studies that have focused on stock price bubbles in other regions, such as the US, our study offers a comprehensive view of how bubbles and their aftermath play out in European markets, with an emphasis on their impact on macroeconomic variables over an extended period. Moreover, we have extended the work of Froot and Obstfeld (1991) and Guerron-Quintana et al. (2023) by showing that bubbles not only influence short-term GDP growth but also have a persistent effect on long-term economic stability, a feature that has been less explored in previous literature.

Future Directions

This research opens the door to several potential extensions. One avenue for future work could involve the examination of other economic factors, such as interest rates, inflation, or government policy responses, to further refine the understanding of bubble dynamics. Additionally, expanding the analysis to other stock markets, both in Europe and globally, could help in drawing comparisons and understanding how regional differences impact the nature and aftermath of speculative bubbles.

Another potential direction could be the incorporation of more advanced machine learning techniques for the estimation and prediction of volatility and GDP. This could provide additional precision in forecasting economic downturns and aid policymakers in identifying early warning signs of market instability.

Ultimately, this study contributes to the broader body of literature on financial market bubbles and their macroeconomic effects, offering both empirical and theoretical advancements that underscore the importance of long memory in financial and economic modeling.

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All authors read and approved the final manuscript.

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The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

References

[1] Andreasen, Martin M. and Bro, Jeppe (2024). Identifying a Stock Price Bubble and its Macroeconomic Implications. Available at SSRN: https://ssrn.com/abstract=4654463 or http://dx.doi.org/10.2139/ssrn.4654463.

[2] Bollerslev, Tim, Chou, Ray Y., Kroner, Kenneth F. (1992). ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence. *Journal of Econometrics*, 52(1-2), 5-59.

[3] Brett, Craig Sarkar, Saikat (2022). Financial bubbles and income inequality. MPRA Paper 112070, University Library of Munich, Germany.

[4] Froot, Kenneth A., and Obstfeld, Maurice (1991). Intrinsic Bubbles: The Case of Stock Prices. *The American Economic Review*, 81(5), 1189–1214. Available at http://www.jstor.org/stable/2006913. Accessed 28 Aug. 2024.

[5] Guerron-Quintana, Pablo A., Hirano, Tomohiro, Jinnai, Ryo (2023). Bubbles, Crashes, and Economic Growth: Theory and Evidence. *American Economic Journal: Macroeconomics*, 15(2), 333–371.

[6] Chen, Guojin, Chen, Lingling, Liu, Yanzhen, Qu, Yuxuan (2021). Stock price bubbles, leverage and systemic risk. *International Review of Economics Finance*, 74, 405-417.

[7] Hays, P., Rajagopal, S., Schreiber, M. (2010). Evidence of Long Memory in U.S. Stock Returns: The Case of the 1990s Bubble. *Quarterly Journal of Finance and Accounting*, 49(1), 5–18. Available at http://www.jstor.org/stable/25747073.

[8] Aoki, Kosuke Nikolov, Kalin (2015). Bubbles, banks and financial stability. *Journal of Monetary Economics*, 74, 33-51.

[9] Lux, Thomas (1998). The Socio-Economic Dynamics of Speculative Markets: Interacting Agents, Chaos, and the Fat Tails of Return Distributions. *Journal of Economic Behavior Organization*, 33(2), 143-165.

[10] Martin, Alberto Ventura, Jaume (2012). Economic Growth with Bubbles. *American Economic Review*, 102(6), 3033–3058.

[11] Tarlie, Martin B., Sakoulis, Georgios, Henriksson, Roy (2022). Stock market bubbles and anti-bubbles. *International Review of Financial Analysis*, 81, 101235.

[12] Nagel, Stefan Brunnermeier, Markus Konrad (2003). Hedge Funds and the Technology Bubble. AFA 2004 San Diego Meetings; EFA 2003 Annual Conference Paper No. 446. Available at SSRN: https://ssrn.com/abstract=423940 or http://dx.doi.org/10.2139/ssrn.423940.

[13] Sarkar, S., Tuomala, M. (2021). Asset bubbles in explaining top income shares. *Journal of Economic Inequality*, 19(4), 707–726. https://doi.org/10.1007/s10888-021-09481-y.

[14] Sornette, Didier (2003). *Why Stock Markets Crash: Critical Events in Complex Financial Systems*. Princeton University Press.