Farey tree locking of terahertz semiconductor laser frequency combs

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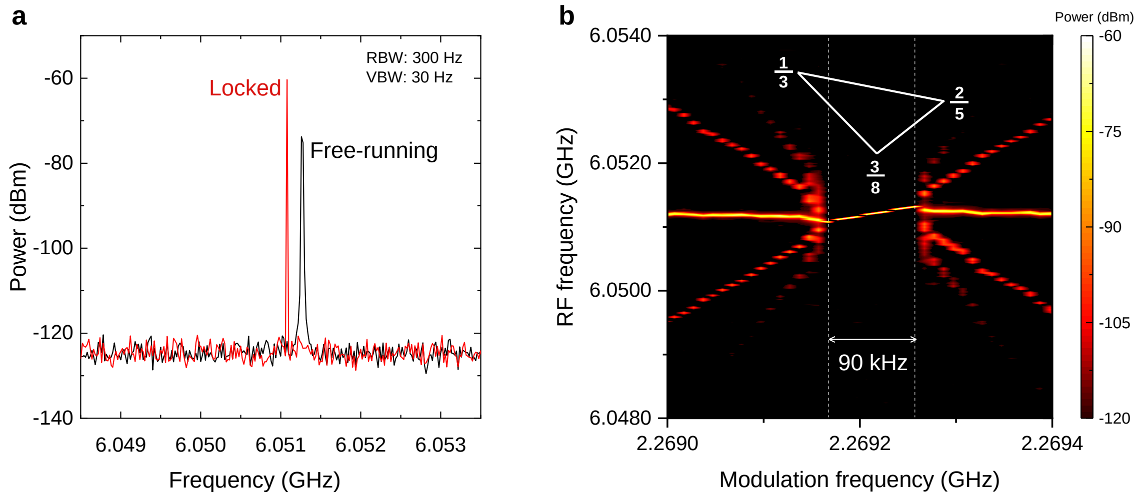

Fig. S1. Predicted Farey tree locking state with a Farey fraction 3/8. (**a**) Intermode beatnote measured in free-running (black) and Farey tree locking (red) state. Under the Farey tree locking condition, the intermode beatnote signal is significantly narrowed and the signal-to-noise ratio is improved by 12 dB compared with that measured in free-running mode. (**b**) Intermode beatnote map of the THz QCL comb as a function of modulation frequency (f_{mod}) measured near the winding number of 3/8. The step size of f_{mod} is 10 kHz. It shows a clear transition from an unstable region to the Farey tree locking region. The locking bandwidth is 90 kHz.

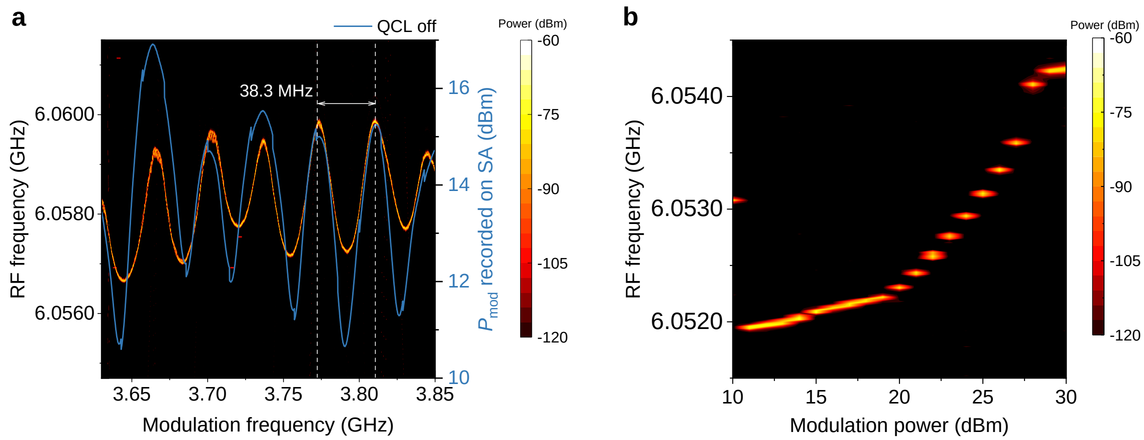

Fig. S2. (a) Periodic oscillation of intermode beatnote frequency and injected RF power with the change of modulation frequency. The orange curve shows the measured intermode beatnote frequency as a function of modulation frequency. The blue line depicts the modulation power (P_{mod}) recorded on the spectrum analyzer (SA) as a function of modulation frequency. The measurement of the blue line is based on the experimental setup in Fig. 1a of the main paper when the QCL is switched off. It can be seen that the two curves show similar oscillating behaviors with the modulation frequency and the period is 38.3 MHz. (**b**) Intermode beatnote map as the modulation power is varied. In this measurement, the modulation frequency is set to 3.7825 GHz and its power is greater than 10 dBm. We can see that f_{rep} monotonically increases with an increase in modulation power (step size of 1 dBm). An increase of 1.93 MHz in repetition frequency is observed as the modulation power increases from 20 to 30 dBm.

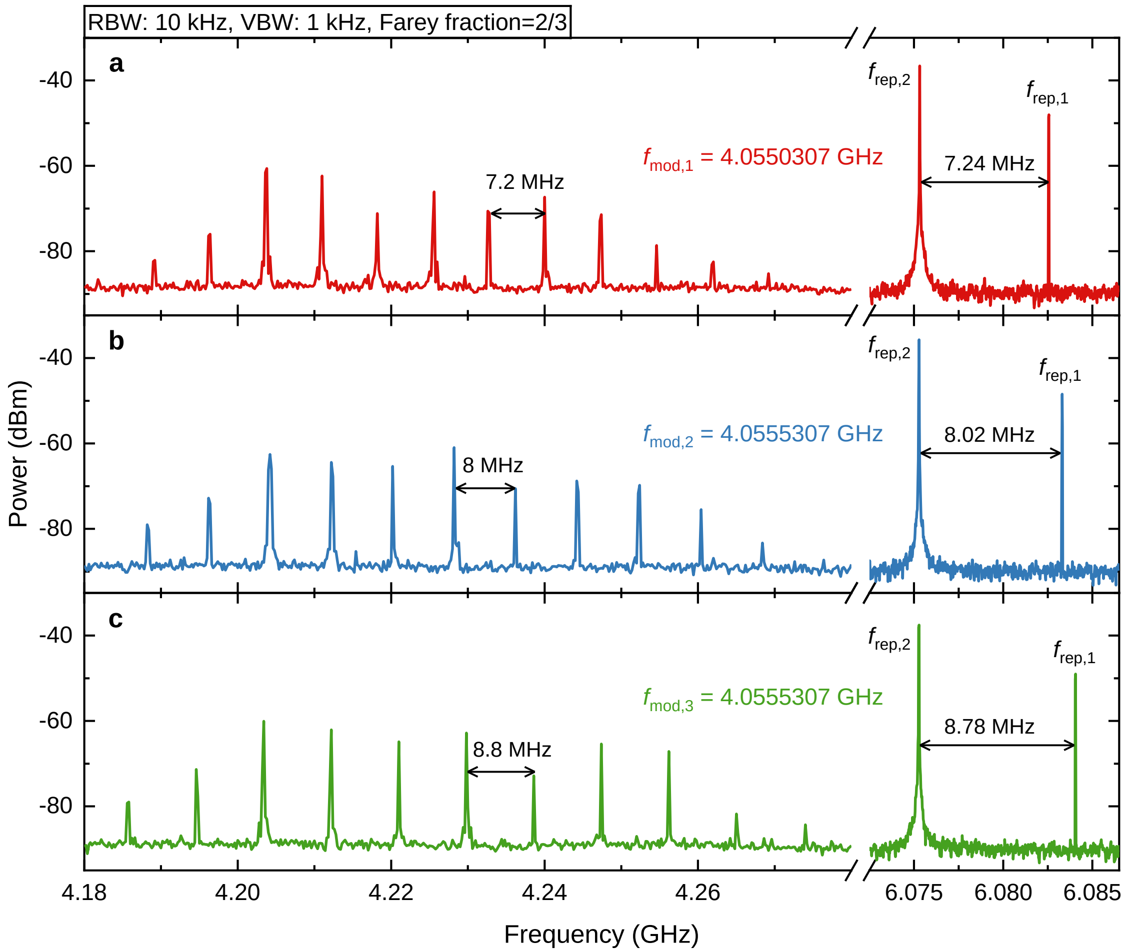

Fig. S3. Dual-comb spectra with tunable repetition frequencies. In the experiment, a Farey tree locking with a Farey fraction of 2/3 is implemented on Comb 1 to lock its repetition frequency ($f_{rep,1}$); while, Comb 2 is operated in free-running mode. (**a**) Dual-comb spectrum measured with $f_{mod,1}$=4.0550307 GHz, resulting in a repetition frequency difference (Δf_{rep}) of 7.24 MHz which is equal to the dual-comb line spacing. (**b**) Dual-comb spectrum measured with $f_{mod,2}$=4.0555307 GHz ($f_{mod,2}$=$f_{mod,1}$+0.5 MHz). The frequency tuning in modulation frequency results in an increase of dual-comb line spacing compared to the results shown in (a). (**c**) Dual-comb spectrum measured with $f_{mod,3}$=4.0560307 GHz ($f_{mod,3}$=$f_{mod,2}$+0.5 MHz). It is expected that the dual-comb line spacing is accordingly tuned as the modulation frequency is changed.

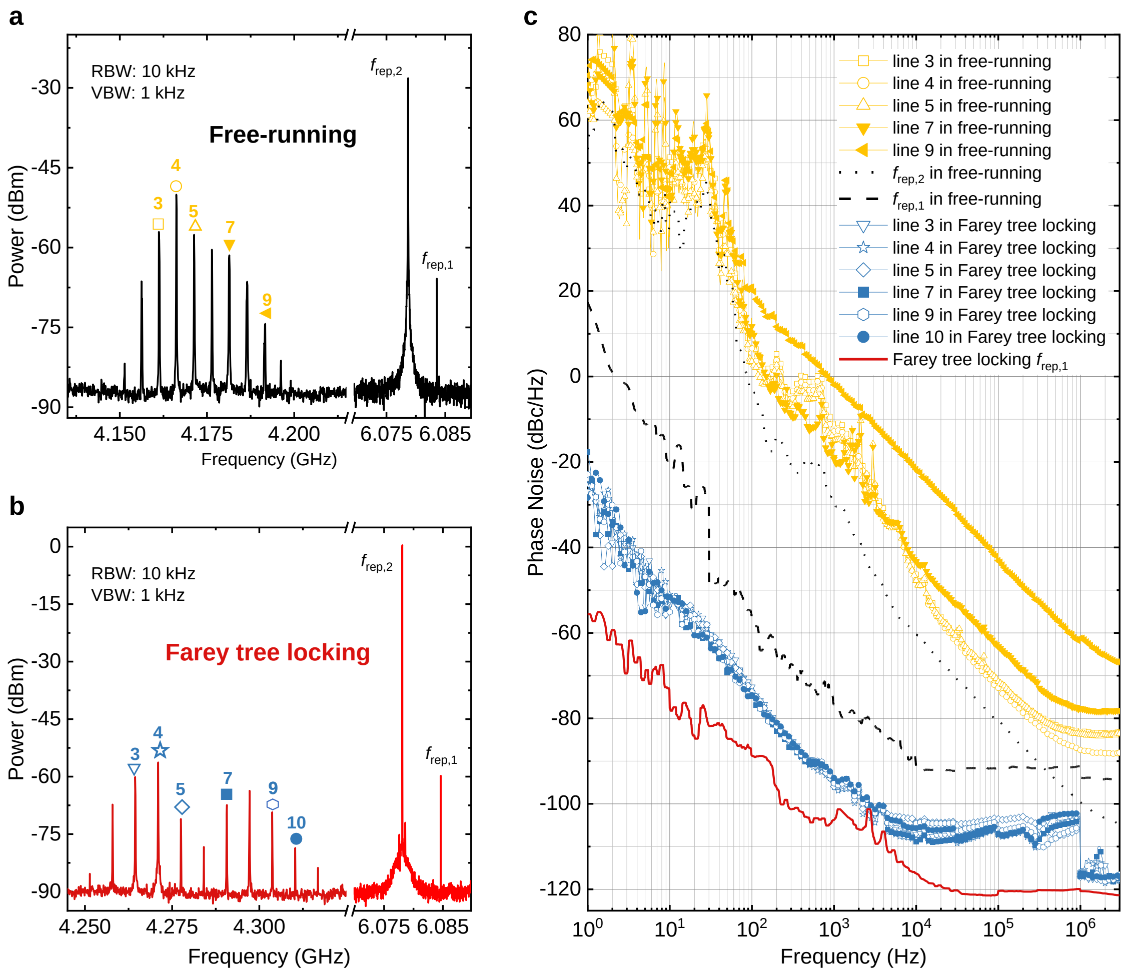

Fig. S4. (**a**) Dual-comb and intermode beatnote spectra recorded in free-running mode, which is the same as Fig. 7b of the main paper. The dual-comb lines are labeled from left to right as the index from 1 to 10, with lines 3, 4, 5, 7, and 9 specifically marked by different scatters. (**b**) Dual-comb and intermode beatnote spectra recorded under the Farey tree locking condition which is the same as Fig. 7c of the main paper. The dual-comb lines 3, 4, 5, 7, 9, and 10 are specifically marked using different scatters. (**c**) Phase noise spectra of the free-running dual-comb lines 3, 4, 5, 7, and 9 marked in (a) and the locked dual-comb lines 3, 4, 5, 7, 9, and 10 marked in (b). Phase noise spectra of dual-comb lines with the Farey tree locking exhibit consistent low phase noise performance. The offset frequency range is set from 1 Hz to 3 MHz (with the function of "low-pass filter" switched on and a cutoff frequency of 10 MHz). As a reference, the phase noise spectra of the free-running $f_{rep,1}$ (dotted line), free-running $f_{rep,2}$ (dashed line), and locked $f_{rep,1}$ (solid line) are also plotted. Note that we display every tenth point when plotting this figure to enhance the clarity of the date.

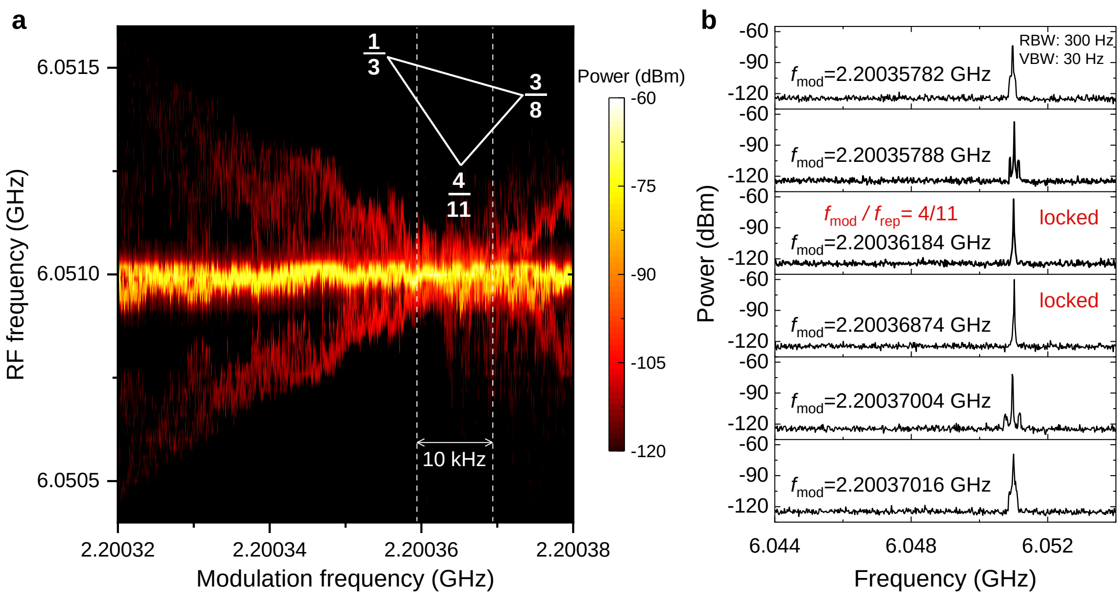

Fig. S5. Devil's staircase 4/11 fractal structure revealed by improving the frequency tuning accuracy. (**a**) Intermode beatnote map around winding number of 4/11. The modulation frequency is scanned step by step with a 20 Hz step size between the Farey fractions 1/3 and 3/8. The two vertical white dashed lines indicate the Farey tree locking bandwidth of 10 kHz around the winding number of 4/11. (**b**) Typical intermode beatnote spectra showing the evolution from free-running to Farey tree locking and finally back to free-running states at the Farey fraction 4/11.

Table S1: Farey tree locking plateaus extracted from Fig. 6a of the main paper.

Farey fraction	$f_{mod,left}$ (GHz)	$f_{mod,right}$ (GHz)	Plateau width (MHz)
1/3	2.016880000	2.016980000	0.1
3/8	2.269167250	2.269257250	0.09
2/5	2.419920000	2.421120000	1.2
1/2	3.025976773	3.034093656	8.1
4/7	3.457748600	3.457816500	0.0679
3/5	3.631600000	3.631800000	0.2
2/3	4.033100000	4.038600000	5.5
3/4	4.540400000	4.541900000	1.5
4/5	4.844400000	4.844900000	0. 5
5/6	5.046774000	5.047004000	0. 23
1	6.012372627	6.117987013	105.6