

Supplementary materials for: “Quantum Simulation of Oscillatory Unruh Effect with Superposed Trajectories”

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EXPERIMENT SETUP

We trap a $^{40}\text{Ca}^+$ ion in a cryogenic linear Paul trap operated at a temperature of approximately 10 K, with an ambient magnetic field of 0.69 mT. The ion trap is illustrated in Fig. S1 (a). Electric quadrupole transitions between $|S_{1/2}\rangle$ and $|D_{5/2}\rangle$ levels of the ion are driven with 729 nm laser pulses. Resonant carrier and sideband transitions [1] are utilized during the experiment. The radial motional mode frequency is $\omega_r = 2\pi \times 1.12$ MHz ($2\pi \times 0.88$ MHz), for the implementation of single(superposed) trajectory. The difference in motional mode frequency is due to a change in the experimental setup. However, the motional frequency should not affect the implementation of the desired effective Hamiltonian, since the 729 nm laser sideband interactions are calibrated near resonance of the transitions. The frequency and phase of the 729 nm laser are modulated with acoustic optic modulators (AOMs). The radio frequency signals driving AOMs are generated by combining a homemade direct digital synthesizer (DDS) and arbitrary wave generator (AWG) with a power splitter, then power amplified and sent to AOMs. Thus, we can apply either simple rectangular pulses of single-frequency laser by DDS, or a time-dependent modulation by AWG for simulating the modulation of coupling strengths due to relativistic trajectories.

Before the simulation process, a cooling sequence including Doppler cooling, electromagnetically-induced-transparency (EIT) cooling, and sideband cooling is applied to reach the ground motional state for the radial mode of average phonon number $\bar{n} < 0.1$. Optical pumping with σ_+ polarized 397 nm laser initializes the ion to the $|S_{1/2}, +1/2\rangle$ state. After the simulation process, measurement of fluorescence by Photomultiplier Tube (PMT) is utilized to deduce the total population on both the $|S_{1/2}, +1/2\rangle$ and $|S_{1/2}, -1/2\rangle$ states. Each detection process lasts for 500 μs . Typical average counts for dark and bright states are 1 and 40, respectively. A threshold of 20 counts is sufficient to distinguish the dark $|D_{5/2}\rangle$ and bright $|S_{1/2}\rangle$ states. For simulation with single trajectory, we map $|g\rangle$ and $|e\rangle$ states to the fine structure sub-levels $|S_{1/2}, +1/2\rangle$ and $|D_{5/2}, +1/2\rangle$, respectively (see Fig. S1 (b)). After the cooling process, the ion state is initialized to

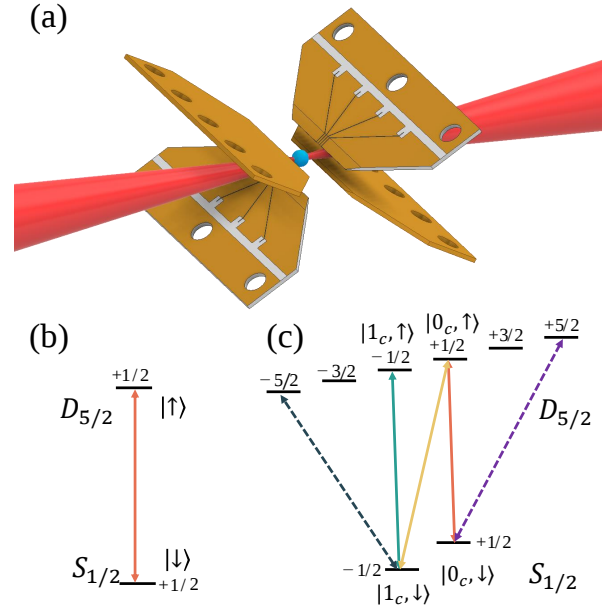


FIG. S1. (a) Illustration of the ion trap. Here trapped ion (blue ball), blade electrodes, and radial 729 nm laser beam (red beam) are shown. (b,c) Level diagram with relevant laser transitions, representing the implementation of single trajectory (b) and superposed trajectories (c). Arrows with different colors represent independent transitions used during the experiment, as depicted in solid arrows. Red and blue sidebands for the transitions of $|S_{1/2}, -1/2\rangle \rightarrow |D_{5/2}, -1/2\rangle$ and $|S_{1/2}, +1/2\rangle \rightarrow |D_{5/2}, +1/2\rangle$ are driven to implement the model. Carrier transitions are driven to implement preparation of the superposed control qubit and interference of trajectories before detection. Dashed arrows show pulses that remove unwanted populations to facilitate readout.

$|S_{1/2}, +1/2\rangle |n=0\rangle$, corresponding to the $|g, 0\rangle$ state in the model. For the implementation of superposed trajectories in the simulation, we map $|0_c, g\rangle$, $|1_c, g\rangle$, $|0_c, e\rangle$, $|1_c, e\rangle$ states to the fine structure sub-levels $|S_{1/2}, +1/2\rangle$, $|S_{1/2}, -1/2\rangle$, $|D_{5/2}, +1/2\rangle$, $|D_{5/2}, -1/2\rangle$, respectively (see Fig. S1 (c)).

THEORETICAL ANALYSIS

The analytical results for the trajectories superposition model are provided in the main text, following the previous works [3, 4]. Here we give detailed derivations below as a theoretical analysis in the ideal case, without considering decoherence terms. First, we begin with a simpler model considering a single trajectory. In the interaction picture with respect to $H_0 = \hbar\omega_p a^\dagger a + \frac{\hbar\omega_q}{2}\sigma_z$, the evolution of Eq. (4) in the main text is determined by the interaction part:

$$\begin{aligned} \frac{V_I}{\hbar g_0} = & \sin(kx(t))[(\sigma_+ a e^{i(\omega_q - \omega_p)t} + \sigma_- a^\dagger e^{-i(\omega_q - \omega_p)t}) \\ & + (\sigma_+ a^\dagger e^{i(\omega_q + \omega_p)t} + \sigma_- a e^{-i(\omega_q + \omega_p)t})]. \end{aligned} \quad (S1)$$

As defined in the main text, the detector oscillates with the trajectory $x(t) = \bar{x} + A \sin(\omega t)$. Jacobi-Anger expansion is applied to expand the modulation of coupling strength $\sin(kx(t))$ as

$$\begin{aligned} \sin(k\bar{x} + kA \sin(\omega t)) &:= \sin(\bar{u} + u \sin(\omega t)) \\ &= \sin \bar{u} [J_0(u) + 2 \sum_{n=1}^{\infty} J_{2n}(u) \cos(2n\omega t)] \\ &\quad + 2 \cos \bar{u} \sum_{n=1}^{\infty} J_{2n-1}(u) \sin((2n-1)\omega t), \end{aligned} \quad (S2)$$

where J_n is the Bessel function of the first kind. We have defined $\bar{u} = k\bar{x}$ and $u = kA$.

Substituting Eq. (S2) into Eq. (S1), the detector's oscillatory motion induces various frequency components for different terms. Specifically, resonances occur when oscillation frequency ω meets $\pm(\omega_q \pm \omega_p) + n\omega = 0$ with $n \in \mathbb{Z}$.

For the resonant case, $\omega = \omega_p + \omega_q$ and $\bar{u} = 2\pi$ are taken, which cancels $\sin \bar{u}$ terms in Eq. (S2). With the rotating-wave approximation, Eq. (S1) in this resonant case can be effectively reduced to

$$\begin{aligned} \frac{V_I}{\hbar g_0} = & [(\sigma_+ a + \sigma_- a^\dagger) + (\sigma_+ a^\dagger e^{i(\omega_q + \omega_p)t} + \sigma_- a e^{-i(\omega_q + \omega_p)t})] \\ & \times 2 \left[\sum_{n=1}^{\infty} J_{2n-1}(u) \sin((2n-1)(\omega_q + \omega_p)t) \right] \\ & \approx 2J_1(u) \sin((\omega_q + \omega_p)t) \\ & \times (\sigma_+ a^\dagger e^{i(\omega_q + \omega_p)t} + \sigma_- a e^{-i(\omega_q + \omega_p)t}) \\ & \approx J_1(u) (i\sigma_+ a^\dagger - i\sigma_- a). \end{aligned} \quad (S3)$$

This effective Hamiltonian could be modulated with u , which is related to the oscillation amplitude. So this model leads to non-zero transition element $|g, n\rangle \langle e, n+1|$ and $|e, n+1\rangle \langle g, n|$. When the initial state is the ground state $|g, 0\rangle$, the transition only occurs between $|g, 0\rangle$ and $|e, 1\rangle$. So

the system evolution could be derived as

$$\begin{aligned} |\psi(t)\rangle &= \cos(\Omega_r t) |g, 0\rangle - i \sin(\Omega_r t) e^{i\phi} |e, 1\rangle, \\ \langle \sigma_z \rangle(t) &= -\cos(2\Omega_r t), \quad \langle N \rangle(t) = \frac{1 - \cos(2\Omega_r t)}{2}, \end{aligned} \quad (S4)$$

where Rabi frequency Ω_r is $g_0 J_1(u)$. The relative phase ϕ is $\pi/2$. In this case, both expectation values would behave like the Rabi flopping form.

For the simulation of superposed trajectories, two trajectories are set as the resonant condition. So analysis above is applicable to each trajectory in the trajectories superposition model. For subspace of $|0_c\rangle\langle 0_c|$ and $|1_c\rangle\langle 1_c|$, the evolution gives $|\psi_0(t)\rangle = \cos(\Omega_0 t) |g, 0\rangle + \sin(\Omega_0 t) |e, 1\rangle$ and $|\psi_1(t)\rangle = \cos(\Omega_1 t) |g, 0\rangle + \sin(\Omega_1 t) |e, 1\rangle$. Combining these results, we could obtain the evolution of the state as

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}(|0_c\rangle |\psi_0(t)\rangle + |1_c\rangle |\psi_1(t)\rangle) \\ &= \cos(\Omega_- t) |+_c\rangle |\psi_+(t)\rangle - \sin(\Omega_- t) |-_c\rangle |\psi_-(t)\rangle, \end{aligned} \quad (S5)$$

where $\Omega_{0/1} = g_0 J_1(u_{0/1})$, $\Omega_{\pm} = (\Omega_0 \pm \Omega_1)/2$. Superposed states are $|\psi_+(t)\rangle = \cos(\Omega_+ t) |g, 0\rangle + \sin(\Omega_+ t) |e, 1\rangle$ and $|\psi_-(t)\rangle = \sin(\Omega_+ t) |g, 0\rangle - \cos(\Omega_+ t) |e, 1\rangle$. Define projectors $\hat{P}_i = |i_c\rangle\langle i_c|$ for convenience. Then the expectation of considered observable could be derived. Derived analytical results of the expectation values are

$$\begin{aligned} \langle \hat{P}_{0/1} \otimes \sigma_z \rangle(t) &= -\cos(2\Omega_{0/1} t)/2, \\ \langle \hat{P}_{0/1} \otimes N \rangle(t) &= [1 - \cos(2\Omega_{0/1} t)]/4, \\ \langle \hat{P}_+ \otimes \sigma_z \rangle(t) &= -\cos^2(\Omega_- t) \cos(2\Omega_+ t), \\ \langle \hat{P}_+ \otimes N \rangle(t) &= \cos^2(\Omega_- t) [1 - \cos(2\Omega_+ t)]/2, \\ \langle \hat{P}_- \otimes \sigma_z \rangle(t) &= \sin^2(\Omega_- t) \cos(2\Omega_+ t), \\ \langle \hat{P}_- \otimes N \rangle(t) &= \sin^2(\Omega_- t) [1 + \cos(2\Omega_+ t)]/2. \end{aligned} \quad (S6)$$

ADDITIONAL ANALYSIS FOR SINGLE TRAJECTORY SIMULATION

Our simulation scheme for the single trajectory model could be extended to a wide range of trajectories. Here, two specific trajectories in [3, 4] are implemented. These results further verify our model and theory analysis.

Different motion amplitudes are implemented for the resonant condition ($\omega = \omega_p + \omega_q$) discussed in the main text. Here, we set different oscillation amplitudes A to obtain $u = \{0, 0.3, 0.6, 1.5\}$. The oscillation frequency $\omega = 2\pi \times 300$ kHz and strength $g_0 = 2\pi \times 1.86$ kHz are the same as in the main text. Experiment and numerical simulation results are shown in Fig. S2 (a,c). When the detector remains stationary in the middle of the cavity ($u = 0$), there is no acceleration ($a = 0$) so the system stays at its ground state. For the other three $u > 0$ cases, the detector excitation $\langle \sigma_z \rangle$ and photon number $\langle N \rangle$ take a Rabi-flopping pattern. The ratio

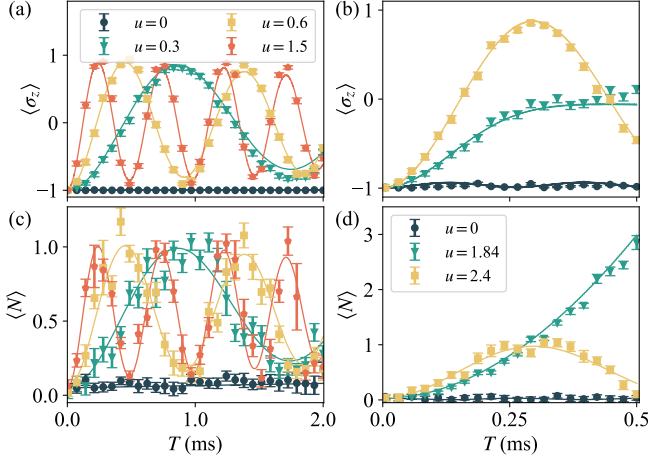


FIG. S2. Theoretical and experimental result for addition single trajectory model. (a, c) shows the evolution of the detector excitation and photon number when $\omega = \omega_p + \omega_q$. (b, d) shows the evolution of the detector excitation and photon number when $\omega = \omega_p = \omega_q$. Solid lines represent numerical simulation with decoherence. Data points with error bars represent the experimental results. As discussed in the text, different evolution forms are induced by different trajectory forms.

of flopping frequencies for different u follows our prediction ($J_1(u) = \{0.148, 0.287, 0.558\}$). We measure the expectation of σ_z and N during the evolution of 2 ms. Data points for σ_z are measured with 500 repetitions. Blue sideband and red sideband scans of 250 μ s are measured with 300 repetitions to obtain the average phonon number. The cutoff of the maximum phonon number is set as $n_{\text{cut}} = 2$.

Another interesting condition is when $\omega = \omega_p = \omega_q$. We set the initial position of the detector as $L/8$ away from the middle of the cavity so that $\bar{u} = k\bar{x} = 2\pi + \pi/2$. In this case, the effective Hamiltonian could be derived similarly to the resonant condition case:

$$\begin{aligned} \frac{V_I}{\hbar g_0} &= [(\sigma_+ a + \sigma_- a^\dagger) + (\sigma_+ a^\dagger e^{2i\omega_0 t} + \sigma_- a e^{-2i\omega_0 t})] \\ &\times [J_0(u) + 2 \sum_{n=1}^{\infty} J_{2n}(u) \cos(2n\omega_0 t)] \\ &\approx J_0(u)(\sigma_+ a + \sigma_- a^\dagger) + J_2(u)(\sigma_+ a^\dagger + \sigma_- a). \end{aligned} \quad (\text{S7})$$

This effective Hamiltonian contains both Jaynes-Cummings (JC) and anti-Jaynes-Cummings (anti-JC) terms that could be modulated with u . Here, we set specific oscillation amplitudes around $A = \{0, 0.146, 0.191\} \times L$ for the trajectory to obtain $u = \{0, 1.84, 2.40\}$.

For $u = 0$, we have $J_0(u) = 1$ and $J_2(u) = 0$, corresponding to JC model only. This model does not excite the system from ground state $|g, 0\rangle$.

For $u = 1.84$, the effective Hamiltonian has both the JC term and the anti-JC terms with equal coupling strength, with

$J_0(u) = J_2(u) = 0.403$. The effective Hamiltonian Eq. (S7) could be written as

$$\begin{aligned} V_I/\hbar &= g_0[J_0(u)(\sigma_+ a + \sigma_- a^\dagger) + J_2(u)(\sigma_+ a^\dagger + \sigma_- a)] \\ &= \Omega_c(|+_q\rangle\langle+_q| - |-_q\rangle\langle-_q|)(a + a^\dagger), \end{aligned} \quad (\text{S8})$$

where the Rabi frequency $\Omega_c = g_0 J_0(u) = g_0 J_2(u)$ and $|\pm_q\rangle = (|g\rangle \pm |e\rangle)/\sqrt{2}$. This Hamiltonian could be considered as a state-dependent displacement along the opposite direction determined by the detector state $|+_q\rangle$ and $|-_q\rangle$. If the initial state $|g, 0\rangle = (|+_q\rangle + |-_q\rangle)|0\rangle/\sqrt{2}$ is prepared, then the state evolution subjected to the above effective Hamiltonian could be easily derived to obtain $|\psi(t)\rangle = (|+_q\rangle|\alpha\rangle + |-_q\rangle|-\alpha\rangle)/\sqrt{2}$. Here $|\alpha\rangle = \mathcal{D}(\alpha)|0\rangle$ is the coherent state, with the displacement operator $\mathcal{D}(\alpha) = \exp\{\alpha a^\dagger - \alpha^* a\}$ and the displacement $\alpha(t) = i\Omega_c t$. In this case, the system evolution could be written as

$$\begin{aligned} |\psi(t)\rangle &= (|+_q\rangle|\alpha(t)\rangle + |-_q\rangle|-\alpha(t)\rangle)/\sqrt{2}, \\ \langle\sigma_z\rangle(t) &= -e^{-2|\alpha(t)|^2} = -e^{-2\Omega_c^2 t^2}, \\ \langle N\rangle(t) &= |\alpha(t)|^2 = \Omega_c^2 t^2. \end{aligned} \quad (\text{S9})$$

For $u = 2.40$, the JC term decreases to zero ($J_0(u) = 0$) while the anti-JC term is non-zero ($J_2(u) = 0.432$). So the system evolution is the same as (S4) with Rabi frequency $\Omega_r = g_0 J_2(u)$ and $\phi = 0$, leading to a Rabi-flopping pattern.

Experiment and numerical simulation results are shown in Fig. S2 (b,d). In the experiment, we set $\omega = \omega_p = \omega_q = 2\pi \times 150$ kHz and $g_0 = 2\pi \times 1.86$ kHz. For $u = 0$, no excitation occurs. For $u = 1.84$, photon displacement occurs with Gaussian decay on the detector. For $u = 2.4$, both detector and photon take Rabi-flopping firm. Due to the fast excitation of phonon in entangled displacement situation ($u = 1.84$), the cutoff of the maximum phonon number is set as $n_{\text{cut}} = 14$. The maximum value of duration T is restricted to 0.5 ms as well.

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- [1] D. J. Wineland, C. Monroe, W. M. Itano, D. Leibfried, B. E. King, and D. M. Meekhof, Experimental issues in coherent quantum-state manipulation of trapped atomic ions, *J. Res. Natl. Inst. Stand. Technol.* **103**, 259 (1998).
- [2] Y. Pan, J. Zhang, E. Cohen, C.-w. Wu, P.-X. Chen, and N. Davidson, Weak-to-strong transition of quantum measurement in a trapped-ion system, *Nat. Phys.* **16**, 1206 (2020).
- [3] S. Felicetti, C. Sabín, I. Fuentes, L. Lamata, G. Romero, and E. Solano, Relativistic motion with superconducting qubits, *Phys. Rev. B* **92**, 064501 (2015).
- [4] C. Sabín, B. Peropadre, L. Lamata, and E. Solano, Simulating superluminal physics with superconducting circuit technology, *Phys. Rev. A* **96**, 032121 (2017).