Methods

Setup. The experimental setup consists of nesting a high index doped silica[1–3], integrated ring-resonator, with a free-spectral range of ~48.9 GHz, a 1.3 million Q-factor, and a positive (focusing) Kerr nonlinearity of about 200 times that of silica, in an add-drop configuration into an amplifying, polarisation-maintaining, fibre cavity. Our samples use a glued fibre array directly on the chip (Fig. 1b), making the setup practical. Each coupling port has 1.5 dB losses, hence the a total input-output coupling losses are 3dB. Because the generated solitons have a very high conversion efficiency, the chip's input power was generally below 100 mW (see Supplementary Section S2); hence we operate well below the damage threshold of standard optical glue.

For Figs. 1-3 we used a fibre cavity with a free-spectral range of ~95 MHz and a microcavity sample with linewidth <120 MHz. The results of Fig. 4 are for a longer cavity with free-spectral range ~77 MHz and a microcavity sample with linewidth<150 MHz. We used two different microresonators with similar properties. In both cases, the fibre cavity includes a ~1-meter polarisation maintaining optical amplifier and a free-space section containing a motorised delay line, polarisation control optics to govern the cavity losses, and a 12 nm wide bandpass filter.

The optical amplifier's gain medium is a highly doped Erbium fibre (Amonics LTD). The system presently reacts from the "cold" state in a few seconds, compatible with the bulk nature of our commercial amplifier. The pump power[4–6] changes its nonlinear response and this process has been used in other works to control self-organisation in multimode fibres[7,8]. However, it typically only provides a small variation in the total system focusing nonlinearity, primarily dominated by a large focusing thermal effect[4–6]. As we discuss further in the Supplementary Materials, we can exploit this small change because, in our double recirculating cavity design, nonlinearities of the same type (focusing in our case, since the thermal nonlinearity dominates in both the microcavity and the laser loop) effectively cancel each other, resulting in a relatively small net variation in the nonlinearity. In addition, the loss becomes a key control parameter because it changes the balance of the optical energy between the microresonator and the laser cavity, allowing the system to operate under a different effective nonlinearity.

Dependence of the nonlinear gain refractive index on pump power. Erbium amplifiers have a resonance around 1538 nm and display a strong, step like nonlinear dispersion which changes sign around the resonance and increase in magnitude with pump power until saturation. Specifically, the well-known spectral response of refractive index and gain of Erbium shows a resonant behaviour around 1538 nm, with the classical, step-like response of the refractive index ruled by Kramers and Kronig relations. In particular, the jump in the refractive index response increases with the magnitude of the gain, resulting in a decrease of the refractive index for wavelengths longer than the resonance and an increase for shorter wavelengths.

Notably, because the gain saturates with the circulating laser power within the fibre cavity, this relationship means that the refractive index dependence with circulating laser power is defocusing for wavelengths shorter than 1538 nm and focusing for longer. In our experiment, using the intracavity 12 nm filter, we select this portion of the spectral gain. Because the displacement of the refractive index directly depends on the gain, and hence on the pump power, the nonlinearity provided by the gain can be controlled with the 980 nm pump. Remarkably, the gain material (and in general *any* gain material) provides then a practical degree of freedom to directly modify the slow nonlinear response of the system.

The modelling reported in Supplementary section S1 and S3 exactly describes this behaviour and is obtained from the very general Maxwell-Bloch relationships which, practically, are the simplest approximation of any gain material. Hence any gain material can be, in principle, adapted for this purpose.

Data acquisition. We simultaneously characterised the operating state of our microcomb laser with several instruments, including an optical spectrum analyser (Anritsu), a fast oscilloscope (Lecroy) to retrieve the radio-frequency spectrum, as well as by recording the intracavity power at several locations within the cavity. An autocorrelator (Femtochrome), is used to record the temporal traces and to discriminate single soliton states (with a single pulse within a period of 20 ps) from multiple soliton states. Typical two-soliton traces are reported in Fig. 1j,l and 4a. Here the autocorrelation shows the typical signature (three peaks) of two identical but not equidistant pulses, as discussed in Ref[21]. Further, we measured the absolute frequency of the oscillating microcomb laser lines using laser scanning spectroscopy, in the same configuration as in Ref[1,3] using a metrological optical frequency comb (Menlo Systems) with the addition of a gas-cell for referencing the absolute frequency axis.

The full dataset used to construct the map shown in Fig. 4, with its three repetitions reported in the Supplementary, is retrieved by an automated procedure over approximately 10 hours per map, during which all measurements are acquired for ~3500 individual settings within the defined parameter ranges (EDFA pump power and fibre-cavity delay length). We first set the amplifier pump power to zero, then fixed the cavity length, and eventually ramped the amplifier pump power up to the first value. Next, we waited for a few seconds for the system to reach the stationary state before obtaining measurements from the instruments previously listed. Next, we increased the pump power in steps of ~1.3 mW, and we repeated the process until reaching the maximum pump power in the range. Upon completing the set, we turned off the amplifier and repeated the procedure for the next delay stage setting until we probed every point of the working regime.

For the measurements presented in Fig. 4, we maintained the environment's temperature in the surroundings of the microresonator photonic chip with a PID controlled Peltier heater to within ±1°C throughout the experiment. We repeated the data taking four times with the same range of parameters, with the repetitions reported in Supplementary Section S2. It is clear from these repetitions that the soliton regime appears consistently within the same region in the parameter space, yielding the same number of solitons. Across the observed soliton range, the usable output power varies up to 10 mW, with single and two soliton states continuously present throughout. For different set of losses, we obtain two and three solitons states with energies reaching up to 30 mW. This range, especially considering the overall optical power, is quite exceptional and promising for further applications of this laser. It already meets the power requirements of many metrological and telecommunications applications without the need for amplification, which would not be amenable to sustain broadband pulses.

Finally, we notice that some single soliton states coexist with a few blue detuned modes near 1535 nm. The laser scanning spectroscopy measurements reported in Supplementary Section S2 reveal that these resonances contain two oscillating lines - one red-detuned (belonging to the soliton) and one blue-detuned (belonging to a superimposed state) for a couple of comb modes in this region. These states appear superimposed over the single comb state and represent the most visible variation in the comb spectra, otherwise unchanged. Hence, these states represent an independent perturbation that, remarkably, does not affect the quality of the soliton state. The spectra and autocorrelation of the two soliton states indicate that the spacing of these pulses within the microcavity are not generally equidistant. Amongst our extensive set of experimental data, we have observed a random distribution of the distances of the two soliton states, often evolving in time. This confirms the localised behaviour of these pulses.

Characterisation of the soliton spectra and numerical fitting. The general properties of a single soliton state in our system are summarised in the Extended Fig. E1, which shows an comprehensive characterisation of the different output ports of the microcavity, along with numerical fitting and radio frequency noise at the given repetition rate.

The experimental data are numerically fit with the mean-field model used in Refs[3,9,10], which consists of a coupled system of dissipative nonlinear equations[11–15]. In the Supplementary, this model is expanded to add the description of the slow, energy-dependent nonlinearities which explain the findings stemming from the experiments reported in the paper.

A lossy nonlinear Schrödinger equation models the evolution of the variable a for the microcavity field in the time and space coordinates t and x, normalised respectively to the fibre cavity roundtrip and microcavity length. The field in the main amplifying loop is b_0 and represents the leading supermode (i.e. the set of modes filtered by the microcavity). A generic supermode is represented by the field b_q

$$\partial_t a = \frac{i\zeta_a}{2}\partial_{xx} a + i\,|a|^2 a \; - \kappa a + \sqrt{\kappa}\sum_{q=-N}^{N} b_q, \tag{1}$$

$$\partial_t b_q = \frac{i\zeta_b}{2}\partial_{xx} b_q + \sigma_6 \partial_{6x} b_q + 2\pi\, i\,(\Delta - q) b_q + g\, b_q - \sum_{p=-N}^{N} b_p + \sqrt{\kappa} a. \qquad (2)$$

Here ∂_{xx} and ∂_{6x} are a second and sixth order derivatives. The parameter Δ represents the normalised frequency detuning between the two cavities, g is the normalised gain while the group-velocity-dispersion coefficients are $\zeta_{a,b}$, with values of $\zeta_a = 1.25 \times 10^{-4}$ and $\zeta_b = 2.5 \times 10^{-4}$. As the gain is taylored with a 12 nm flat-top filter, we use a sixth order derivative to reproduce the gain dispersion, with $\sigma_6 = (1.5 \times 10^{-4})^3$. The coupling coefficient is $\kappa = 1.5\pi$. Further details are reported in Supplementary section S2. These parameters are used to fit the experimental spectra in the Extended Fig. E1 with a numerical mode-solver that provides the nonlinear eigenmodes of the system, including the soliton functions, as in Refs[3,9,10]. In particular, the field measured at the 'drop' port directly reports the microcavity internal field a. The field at the output port, which is the 'through' port of the microcavity, can be theoretically evaluated with $c(t) \approx b_0 - \sqrt{\kappa} a$.To fit the experimental data, we included an additional component $\alpha\, b_0$ to this value, where α is a coefficient that accounts for the non-ideal transmission of the microcavity and the polarisation interference at resonance. The numerical fit of a typical experimental spectrum of this output is reported in Fig. E1 a,b and d,e for the soliton spectra in Fig.1 c and Fig.2 f. Here, we also present the experimental measurement of the gain+12 nm filter bandwidth, showing how the soliton spectrum well exceeds the amplification spectrum.

For the states in Fig. E1 a,b and d,e, the input powers to the microcavity were 44 and 63 mW, respectively. The 'through' output powers were instead 4 mW and 5 mW. The second output ('drop port') was reconnected to the amplifier leading to off-chip emitted powers of 6.5 mW and 8.3 mW, respectively. Part of the light was extracted for characterisation with a beam splitter. The total cavity operated with ~10 dB gain. When accounting for the on-chip losses of 3 dB, we estimated that the microresonator operated with 31 mW and 44 mW on-chip input powers. The on-chip 'through' output powers were 5.7 mW and 7.1 mW, while the on-chip 'drop' port powers were 9.3 and 11.8 mW. This results in an on-chip nonlinear conversion efficiency of about 20 and 30 % at the 'through' and 'drop' ports, respectively.

Finally, Fig. E1 g,h reports the RF characterisation around the repetition rate frequency. A small portion of the output 'through' port signal was processed with an electro-optic modulator, leading to additional sidebands around each of the original comb lines. Since the electro-optic modulator was driven in saturation with a GPS-referenced microwave oscillator, several harmonic sidebands were generated, whose frequency distance from the comb lines was a multiple of the modulating signal frequency[16]. We considered the third harmonic sidebands, and we set the modulation frequency such that the interaction between adjacent comb-lines produced a f_0=500 MHz beat-note. We revealed it with an amplified photo-detector and analysed it with an Electrical Spectrum Analyser (ESA). Quite evidently, in all the ESA traces, the repetition rate beat-note SNR is more than 40 dB, being here limited mainly by the ESA noise floor.

References

1. Bao, H. *et al.* Laser cavity-soliton microcombs. *Nat. Photonics* **13**, 384–389 (2019).
2. Peccianti, M. *et al.* Demonstration of a stable ultrafast laser based on a nonlinear microcavity. *Nat. Commun.* **3**, (2012).
3. Bao, H. *et al.* Turing patterns in a fiber laser with a nested microresonator: Robust and controllable microcomb generation. *Phys. Rev. Res.* **2**, 023395 (2020).
4. Barmenkov, Yu. O., Kir'yanov, A. V. & Andrés, M. V. Resonant and thermal changes of refractive index in a heavily doped erbium fiber pumped at wavelength 980nm. *Appl. Phys. Lett.* **85**, 2466–2468 (2004).

5. Janos, M. & Guy, S. C. Signal-induced refractive index changes in erbium-doped fiber amplifiers. *J. Light. Technol.* **16**, 542–548 (1998).

6. Thirstrup, C., Shi, Y. & Palsdottir, B. Pump-Induced Refractive Index and Dispersions in Er3+ Doped fibers. *J. Light. Technol.* **14**, 732–738 (1996).

7. Bochove, E. J., Cheo, P. K. & King, G. G. Self-organization in a multicore fiber laser array. *Opt. Lett.* **28**, 1200–1202 (2003).

8. Sperber, T., Billault, V., Dussardier, B., Gigan, S. & Sebbah, P. Gain As Configurable Disorder: Adaptive Pumping for Control of Multimode Fiber Amplifiers and Lasers. *arXiv:2008.04085* (2020).

9. Bao, H. *et al.* Laser cavity-soliton microcombs. *Nat. Photonics* **13**, 384–389 (2019).

10. Cutrona, A. *et al.* Temporal cavity solitons in a laser-based microcomb: a path to a self-starting pulsed laser without saturable absorption. *Opt. Express* **29**, 6629–6646 (2021).

11. Scroggie, A. J., Firth, W. J. & Oppo, G.-L. Cavity-soliton laser with frequency-selective feedback. *Phys. Rev. A* **80**, 013829 (2009).

12. Paulau, P. V., Gomila, D., Colet, P., Malomed, B. A. & Firth, W. J. From one- to two-dimensional solitons in the Ginzburg-Landau model of lasers with frequency-selective feedback. *Phys. Rev. E* **84**, 036213 (2011).

13. Atai, J. & Malomed, B. A. Stability and interactions of solitons in two-component active systems. *Phys. Rev. E* **54**, 4371–4374 (1996).

14. Malomed, B. A. Solitary pulses in linearly coupled Ginzburg-Landau equations. *Chaos Interdiscip. J. Nonlinear Sci.* **17**, 037117 (2007).

15. Atai, J. & Malomed, B. A. Bound states of solitary pulses in linearly coupled Ginzburg-Landau equations. *Phys. Lett. A* **244**, 551–556 (1998).

16. Rolland, A. *et al.* Non-linear optoelectronic phase-locked loop for stabilization of opto-millimeter waves: towards a narrow linewidth tunable THz source. *Opt. Express* **19**, 17944 (2011).