Supplementary Material for "Quantum Phases of Time Order in Many-Body Ground States"

Tie-Cheng Guo[1] and Li You[1,2]

[1] *State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China*
[2] *Frontier Science Center for Quantum Information, Beijing, China*

This supplementary provides supporting material and related details for the presentation of the main text. It is organized as follows: in Sec. I we extend the discussion of time order to finite temperature; in Sec. II, we present calculation details related to the spin-1 atomic Bose-Einstein condensate (BEC) example considered; as a more straightforward approach to understand numerical results, we present a variational approach for treating the polar ground state of a spin-1 BEC in Sec. III. Finally, we give the details about ground state and eigen-energy calculation in the non-Hermitian quantum many-body model with multi-body interaction in Sec. IV.

I. TIME ORDER AT FINITE TEMPERATURE

At finite temperature T, excited states will be populated, which can be taken into account with the Gibbs ensemble $\hat{\rho} \equiv e^{-\beta \hat{H}}/Z$, where $Z \equiv \mathrm{Tr}\, e^{-\beta \hat{H}}$ denotes the partition function and $\beta \equiv 1/T$ the inverse temperature. We then find

$$\begin{aligned} f(t) &\to \lim_{V\to\infty} \mathrm{Tr}\left(e^{i\hat{H}t}\hat{\phi}(0)e^{-i\hat{H}t}\hat{\phi}(0)\hat{\rho}\right) \\ &= \lim_{V\to\infty} \sum_{k=0}^{\infty} \langle\psi_k| e^{i\hat{H}t}\hat{\phi}(0)e^{-i\hat{H}t}\hat{\phi}(0)\frac{e^{-\beta\hat{H}}}{Z}|\psi_k\rangle \\ &= \lim_{V\to\infty} \sum_{k=0}^{\infty} \frac{1}{Z} e^{i\epsilon_k t - \beta\epsilon_k} \langle v_k| e^{-i\hat{H}t}|v_k\rangle \\ &= \lim_{V\to\infty} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{Z} c_{jk} e^{-\beta\epsilon_k} e^{-i(\epsilon_j-\epsilon_k)t}, \end{aligned} \tag{1}$$

where $|v_k\rangle$ is the eigen-state twisted vector for $|\psi_k\rangle$, and c_{jk} its associated weight. Analogously, for the non-Hermitian case, we find

$$\begin{aligned} f(t) &= \lim_{V\to\infty} \sum_{k=0}^{\infty} \frac{1}{Z} e^{i\epsilon_k t - \beta\epsilon_k} \langle v_k^{(l)}| e^{-i\hat{H}t}|v_k^{(r)}\rangle \\ &= \lim_{V\to\infty} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{Z} c_{jk} e^{-\beta\epsilon_k} e^{-i(\epsilon_j-\epsilon_k)t}, \end{aligned} \tag{2}$$

where $|v_k^{(l)}\rangle$ and $|v_k^{(r)}\rangle$ are the left and right twisted vectors for eigen-state $|\psi_k\rangle$, c_{jk} is the corresponding weight.

It is easily noted that $f(t)$ at finite temperature contains contributions from all eigen-states of the quantum many-body system $\hat{H}$, with a temperature dependent weight factor for different energy level, but $f(t)$ remains to include contributions from different periodic functions. Hence the quantum phase classification task essentially remains the same (including its possible reference to $F(t)$) as is shown in the Letter for the ground state. At finite temperature, due to thermal excitations to ground state, the temporal behavior will be more complex thus opening up for more interesting possibilities, e.g., to control *time order* phases and to study crossover or driven phase transitions between different time order phases.

II. TIME ORDER IN A SPIN-1 ATOMIC BEC

For typical interaction parameters of a spin-1 BEC (e.g., of ground state ^{87}Rb or ^{23}Na atoms) in a tight trap, spin domain formation is energetically suppressed when the atom number is not too large as spin-dependent interaction strength is much weaker than spin-independent interaction [1–4]. This facilitates a single-spatial-mode approximation

(SMA) by assuming all spin states share the same spatial wave function $\phi(\boldsymbol{r})$, which effectively decouples the spatial degrees of freedom from the spin and results in the following Hamiltonian [2, 5]

$$\hat{H} = \frac{c_2}{2N}\left[\left(2\hat{N}_0 - 1\right)\left(\hat{N}_1 + \hat{N}_{-1}\right) + 2\left(\hat{a}_1^\dagger \hat{a}_{-1}^\dagger \hat{a}_0 \hat{a}_0 + \text{h.c.}\right)\right] - p\left(\hat{N}_1 - \hat{N}_{-1}\right) + q\left(\hat{N}_1 + \hat{N}_{-1}\right), \tag{3}$$

for the model many body system, where $\hat{a}_{m_F}(m_F = 0, \pm 1)$ is the annihilation operator of the ground manifold state $|F = 1, m_F\rangle$ with corresponding number operator $\hat{N}_{m_F} = \hat{a}_{m_F}^\dagger \hat{a}_{m_F}$. p and q are linear and quadratic Zeeman shifts which could be tuned independently in experiments [6], while c_2 denotes spin exchange interaction strength. The total particle number operator $\hat{N} = \hat{N}_1 + \hat{N}_0 + \hat{N}_{-1}$ as well as the longitudinal magnetization operator $\hat{F}_z = \hat{N}_1 - \hat{N}_{-1}$ are both conserved. Thus, linear Zeeman shift can be set to $p = 0$ effectively.

As discussed in the main text, a suitable order parameter for this model system is $\hat{n}_{\text{sum}} \equiv \hat{N}_{\text{sum}}/N$ ($\hat{N}_{\text{sum}} = \hat{N}_1 + \hat{N}_{-1} = N - \hat{N}_0$), which measures the fractional atomic population in the states $|1, 1\rangle$ and $|1, -1\rangle$ and N assumes the role of system size. Following our formulation and denoting the system energy eigen-state by $|\psi_i\rangle$ ($i = 0, 1, 2, \cdots$) with increasing eigen-energy ϵ_i, the ground state twisted vector becomes $|v\rangle \equiv \hat{n}_{\text{sum}}|\psi_0\rangle = \sum_{i=0}^{\infty} a_i|\psi_i\rangle$, with $a_i = \langle\psi_i|v\rangle$ its expansion coefficient on the eigen-state $|\psi_i\rangle$. We find

$$f(t) = \lim_{N\to\infty} \langle \hat{n}_{\text{sum}}(t)\hat{n}_{\text{sum}}(0)\rangle = \lim_{N\to\infty} \sum_{j=0}^{\infty} b_j e^{-i(\epsilon_j - \epsilon_0)t}, \tag{4}$$

where $b_j \equiv |a_j|^2$ is the weight of the ground state twisted vector, $b \equiv \sum_{j=0}^{\infty} b_j$ the total weight, and

$$F(t) = \lim_{N\to\infty} \left\langle \hat{N}_{\text{sum}}(t)\hat{N}_{\text{sum}}(0)\right\rangle = \lim_{N\to\infty} \sum_{j=0}^{\infty} B_j e^{-i(\epsilon_j - \epsilon_0)t}, \tag{5}$$

where $A_i = N\langle\psi_i|v\rangle$, $B_j \equiv |A_j|^2$ is the weight of the enlarged ground state twisted vector, and $B \equiv \sum_{j=0}^{\infty} B_j$ the total weight.

Our study below is for the zero magnetization $F_z = 0$ subspace and employs exact diagonalization (ED) to calculate eigen-states as well as eigen-energies. The overall time order phase diagram for spin-1 BEC is shown in the main Letter. For ferromagnetic interaction $c_2 < 0$, the critical quadratic Zeeman energy $q/|c_2| = 2$ splits the whole region into time trivial order (TT) phase for smaller q that observes TTS, and the generalized time crystalline (gTC) order phase for $q/|c_2| > 2$ where TTS is spontaneously broken. The latter (gTC phase) is found to coincide with the ground state polar phase. The available computation resource limits the calculation to a finite system size, which prevents us from mapping out the exact details in the immediate neighborhood of $q = 2$, where further elaboration is need for its time order properties. On the other hand, for antiferromagnetic interactions, we find $q = 0$ separates TT phase from and gTC order.

In Figure 1, the weights for the ground state as well as for the low-lying excited states are shown as functions of q for a typical system size of $N = 10000$. Only the ground state weight b_0 is nonvanishing in the $q < 2$ ($q < 0$) region for ferromagnetic (antiferromagnetic) interactions, but the total weight b is zero in the $q > 2$ ($q > 0$) region for ferromagnetic (antiferromagnetic) interaction, which prompts us to examine further the enlarged weights B_i corresponding to the bulk order parameter. For ground and the first excited states, the volume enlarged weights $B_{0,1}$ are found to be nonvanishing, although both decrease as q increases and grow with N as q approaches $q = 2$ ($q = 0$) for ferromagnetic (antiferromagnetic) interaction. However, as mentioned above, limited to a system size of $N = 10000$ by computation resource in the ED calculation, we cannot exactly map out the behavior near $q = 2$ ($q = 0$) for ferromagnetic (antiferromagnetic) interaction. This consequently leaves empty for q in region $[2.0, 2.02]$ ($[0, 0.01]$) for ferromagnetic (antiferromagnetic) interaction.

The dependence on system size N is clearly revealed by Fig. 2, with the enlarged weights in the gTC regime attain fixed values as system approaches thermodynamic limit ($N \to \infty$). In regions away from $q = 2$ ($q = 0$) for ferromagnetic (antiferromagnetic) interaction, ED numerics can always approach thermodynamic limit except for the immediate neighbourhood near $q = 2$ ($q = 0$), where we infer with confidence the tendencies to divergence of the weights $B_{0,1}$ as q approaches $q = 2$ ($q = 0$).

The time evolution of two-time auto-correlation function $F(t)$ is plotted in Fig. 3 (a) for ferromagnetic and (c) for antiferromagnetic interactions, while Figs. 3 (b) and (d) display energy gaps between ground and the first excited states as a function of q for ferromagetic and antiferromagnetic interactions respectively at a system size of $N = 5000$. The behavior of $F(t)$ is quantitatively consistent with that of the weights $B_i(q)$ ($i = 0, 1$) shown in Fig. 1 and the energy gap $\epsilon_1 - \epsilon_0$ shown in Figs. 3 (b) and (d).

At finite temperature, excited states come into play by also contributing to the correlation function. We find the gTC order hosted in the polar phase persists for both ferromagnetic and antiferromagnetic interactions. The corresponding

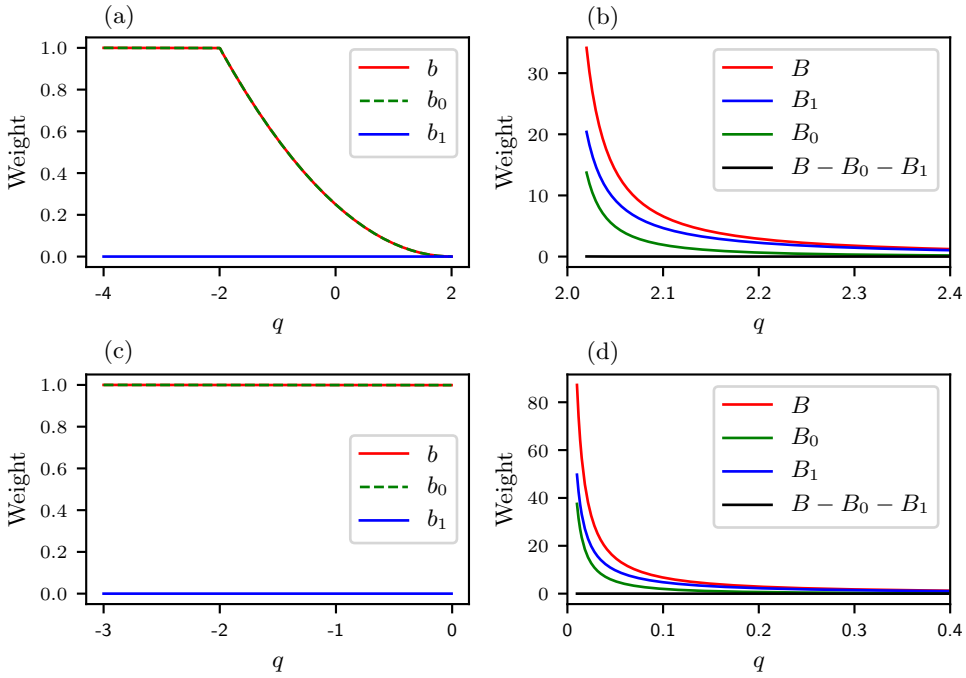

FIG. 1. Weights of ground state twisted vector in the ground and low-lying excited states as functions of q at system size $N = 10000$. The upper panel is for ferromagnetic interaction, where weights b_i for $q < 2$ are shown in (a) while weights B_i for $q > 2$ are shown in (b). The lower panel is for antiferromagnetic interaction, where weights b_i for $q < 0$ are shown in (c) while weights B_i for $q > 0$ are shown in (d).

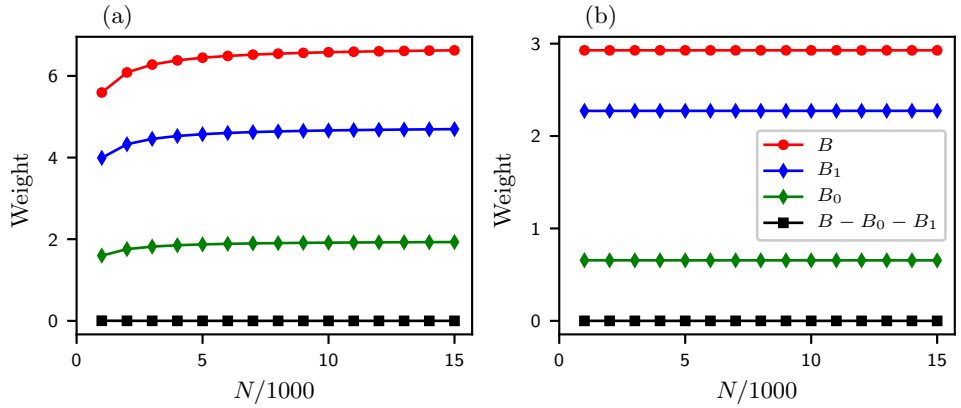

FIG. 2. Weights of ground state twisted vector in the ground and low-lying excited states as functions of system size N at $q = 2.1$ for ferromagnetic interaction (a) and at $q = 0.2$ for antiferromagnetic interaction (b).

time evolution and Fourier transform of $F(t)$ are shown in Fig. 4, calculated for $N = 500$ at a temperature of $\beta \equiv 1/T = 1$. The Fourier transform is performed for $\mathrm{Re}(F)$ over $t = [0, 1000]$ with the zero frequency (DC) component subtracted or for $\mathrm{Im}(F)$. The upper (lower) panel corresponds to ferromagnetic (antiferromagnetic) interaction at $q = 3$ ($q = 2$). For ferromagnetic interaction, two distinct frequency components are clearly identified for $q = 3$, associated with the two different energy level gaps. Thus, the gTC phase remains at finite temperature. Moreover, we also find a generalized time quasi-crystalline order phase assuming the two frequencies are incommensurate, by fine tuning their corresponding energy gaps such that the relation $\Delta_1/\Delta_2 = m_1/m_2$ with m_1 and m_2 being co-primes is not satisfied. The gTC phase at finite temperature here is robust which is in contrast to the melting behavior of Continuous Time Crystal (CTC) shown in Ref. [7].

Finally, we hope to address the critical question about how could this time order, sort of a perpetual time dependence, can be observed. We note the bulk two-time auto-correlation function introduced $F(t) =$

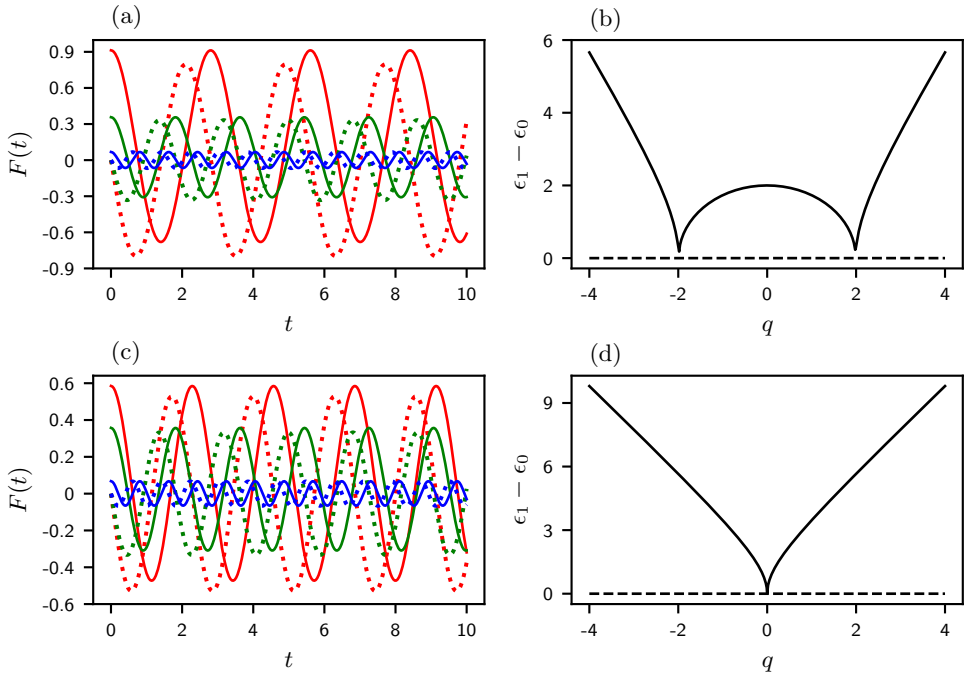

FIG. 3. $F(t)$ for different q as a function of time t. The solid and dotted lines correspond to Re(F) and Im(F) respectively. The red, green, and blue lines correspond to $q = 2.5$, $q = 3$, and $q = 5$ respectively for ferromagnetic interaction (a). The red, green, and blue lines correspond to $q = 0.7$, $q = 1$, and $q = 3$ respectively for antiferromagnetic interaction (c). The energy gap between ground and the first excited state $\epsilon_1 - \epsilon_0$ as a function of q for ferromagnetic (b) and antiferromagnetic interactions (d), at system size $N = 5000$.

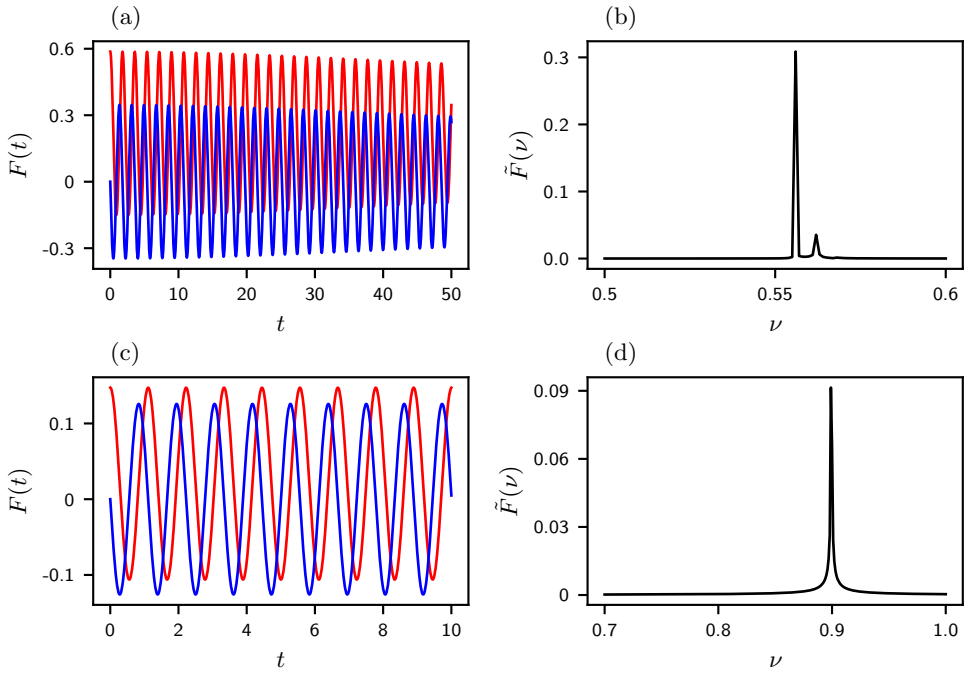

FIG. 4. $F(t)$ as a function of time t at $q = 3$ for ferromagnetic interaction (a) and $q = 2$ for antiferromagnetic interaction (c). The red and blue solid lines respectively correspond to Re(F) and Im(F). The Fourier transform spectrum $\tilde{F}(\nu)$ of Re(F) or Im(F) with $\nu = 1/T$ the frequency, T the period, for ferromagnetic (b) and antiferromagnetic (d) interactions, at temperature $\beta = 1$ and system size $N = 500$.

$\lim_{N\to\infty}\left\langle \hat{N}_{\rm sum}(t)\hat{N}_{\rm sum}(0)\right\rangle$ denotes nothing but the ground state (averaged) conditional outcome of measuring $N_{\rm sum}(t)$ at t after starting with $N_{\rm sum}(0)$ initially. The dynamics of $F(t)$ follows that of $N_{\rm sum}(t)$ as in quantum regression theorem. Given the system is well controlled, highly reproducible, one can simply detect $F(t)$ by measuring $N_{\rm sum}(t)$, although for each measurement at an instant t, a condensate is destroyed, and a follow up one will have to be prepared as closely as possible in every respects (through selection and post-selection) and be measured at a different $t' > t$. Thus, a plausible way to detecting the ground state time dependence will require reconstructing the time dependence of $F(t)/N_{\rm sum}(0)$. As along as the oscillation amplitude is more than a few percent, it will be easily observable with not too much difficulty, although such a reconstruction will still be difficult as $N_{\rm sum}(0)$ can be rather small compared to $N_0 \sim N$ in the polar state. Alternatively, one can perhaps start from a twin-Fock state, i.e., by preparing an initial state with $N_{\rm sum}(0) \sim N$.

In Figure 5(a) we show the behavior of oscillation amplitude for $F(t)/N_{\rm sum}(0)$. The time dependence of $F(t)/N_{\rm sum}(0)$ at $q = 2.5$ for ferromagnetic interaction is shown in Fig. 5(b).

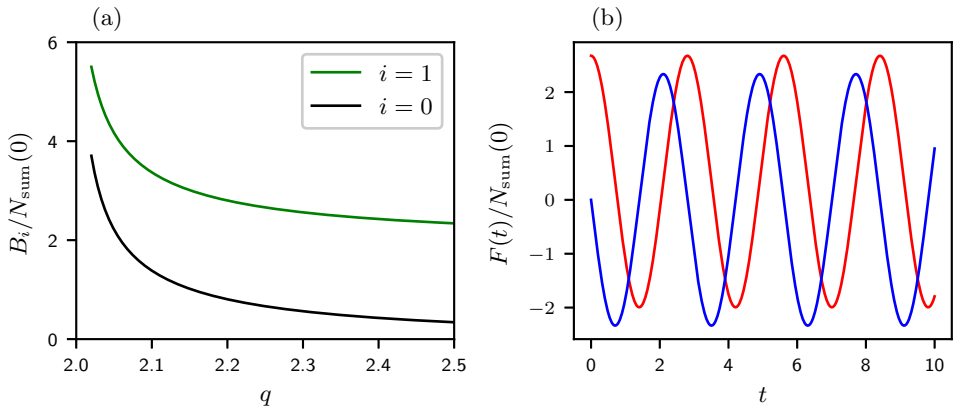

FIG. 5. (a) $B_i/N_{\rm sum}(0)$ as a function of q for ferromagnetic interaction. (b) $F(t)/N_{\rm sum}(0)$ as a function of time t at $q = 2.5$ for ferromagnetic interaction. The red and blue solid lines correspond to the real and imaginary part of $F(t)/N_{\rm sum}(0)$ respectively.

III. A VARIATIONAL POLAR STATE FOR FERROMAGNETIC SPIN-1 BEC

One might naively expect that nothing particularly interesting could happen in the polar phase of a ferromagnetic spin-1 BEC, where essentially all atoms reside in the single particle state $|1, 0\rangle$. Nevertheless, due to the competition between spin exchange interaction c_2 and quadratic Zeeman shift q, the ground state of our system differs from $|N_1 = 0, N_0 = N, N_{-1} = 0\rangle$, which can be affirmed based on a simple variational analytical calculation given in this section.

We use the number state basis $|N_1, N_0, N_{-1}\rangle \equiv |[N], M, k\rangle$, where N_{m_F} denotes occupation number of the m_F magnetic state, $M \equiv N_1 - N_{-1}$, and $k \equiv N_{-1}$. We take the following ground state variational ansatz $|\psi_0\rangle = \frac{1}{\sqrt{1+|a|^2}}(|0, N, 0\rangle + a|1, N-2, 1\rangle)$ for the polar state of ferromagnetic spin-1 BEC, where $a = re^{i\phi}$ is a (complex) variational parameter with r and ϕ real. From Eq. (3) and (assumed) $p = 0$, the ground state energy follows from

$$\begin{aligned} E &= \frac{\langle\psi_0|H|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} \\ &= \frac{1}{1+a^*a}\Big[\big(\frac{c_2(2N-5)}{N} + 2q\big)a^*a + c_2\sqrt{\frac{N-1}{N}}(a^* + a)\Big] \\ &= \frac{1}{1+r^2}\Big[\big(\frac{c_2(2N-5)}{N} + 2q\big)r^2 + 2c_2\sqrt{\frac{N-1}{N}}\, r\cos(\phi)\Big]. \end{aligned} \tag{6}$$

We see the extreme value (the minimum) of E is reached when $\cos(\phi) = \pm 1$, i.e., for a real variational parameter a, which will be assumed from now on. This gives

$$E = \frac{x_1 a^2 + x_2 a}{1 + a^2}, \tag{7}$$

with $x_1 = \frac{c_2(2N-5)}{N} + 2q$ and $x_2 = 2c_2\sqrt{\frac{N-1}{N}}$. The derivative of the energy function $E(a)$ is

$$E'(a) = \frac{-x_2a^2 + 2x_1a + x_2}{(1+a^2)^2}, \tag{8}$$

which determines the locations for the extreme values

$$a_\pm = \frac{1}{2c_2\sqrt{N(N-1)}}\Big[c_2(2N-5) + 2Nq \pm N\sqrt{\frac{c_2^2(8N^2-24N+25)}{N^2} + \frac{4c_2q(2N-5)}{N} + 4q^2}\Big], \tag{9}$$

and the corresponding extreme values are

$$E_\pm = c_2 + q \pm \frac{1}{2}\sqrt{4q^2 + 8c_2q + 8c_2^2 + \frac{-24c_2^2 - 10c_2q}{N} + \frac{25c_2^2}{N^2}} - \frac{5c_2}{2N}. \tag{10}$$

In the thermodynamic limit $N \to \infty$, they reduce respectively to $a_\pm = 1 + \frac{q}{c_2} \pm \frac{\sqrt{2c_2^2+2c_2q+q^2}}{c_2}$ and $E_\pm = c_2 + q \pm \sqrt{2c_2^2 + 2c_2q + q^2}$. The left and right asymptotic value for the energy function $E(a)$ is therefore

$$E(a) = c_2(2 - \frac{5}{N}) + 2q, \quad (\text{when}\, a \to \pm\infty). \tag{11}$$

For ferromagnetic interaction ($c_2 < 0$), E_- assumes the minimum, which corresponds to the ground state $|\psi_0\rangle = \frac{1}{\sqrt{1+a_-^2}}(|0, N, 0\rangle + a_-|1, N-2, 1\rangle)$ with $N_{\text{sum}} = 2a_-^2/(1+a_-^2)$, and $a_- = 1 + \frac{q}{c_2} - \frac{\sqrt{2c_2^2+2c_2q+q^2}}{c_2}$ in the thermodynamic limit $N \to \infty$.

Despite of the vanishing order parameter n_{sum} in the polar phase (here the gTC order phase from the time order perspective), the enlarged quantity N_{sum} retains a finite value. Hence, the physics we present here clearly belongs to the realm of quantum effects, beyond the reach of mean-field theory.

IV. THE NON-HERMITIAN SPIN MODEL WITH MULTI-BODY INTERACTION

The non-Hermitian quantum many body model Hamiltonian is

$$\hat{H} = \hat{H}_0 + (\lambda + \gamma\mathrm{i})\hat{H}_1, \tag{12}$$

with

$$\begin{aligned} \hat{H}_0 &= -\sum_{j=1}^{N} \sigma_j^z\sigma_{j+1}^z, \\ \hat{H}_1 &= \sigma_1^x\sigma_2^x\cdots\sigma_{[N/2]}^x - \sigma_{[N/2]+1}^x\cdots\sigma_N^x, \end{aligned} \tag{13}$$

where $[\cdot]$ denotes the integral part, $\boldsymbol{\sigma}_{N+1} \equiv \boldsymbol{\sigma}_1$, λ and γ are spin-string interaction strength and dissipation strength respectively. λ and γ are both real numbers. i is the imaginary unit. $\sigma^{x,y,z}$ are Pauli operators. N is the qubit number of the system. The Hamiltonian has the $[(N+1)/2]-$ body interaction term and supports the GHZ state $|G_+\rangle$ as a non-degenerate excited state.

Firstly, denote the Greenberger-Horner-Zeilinger (GHZ) states as

$$|G_\pm\rangle = \frac{1}{2}(|0\rangle^{\otimes N} \pm |1\rangle^{\otimes N}). \tag{14}$$

Denote

$$|\tilde{G}_{-,\mathcal{I}}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_1\cdots|0\rangle_{[N/2]}|1\rangle_{[N/2]+1}\cdots|1\rangle_N - |1\rangle_1\cdots|1\rangle_{[N/2]}|0\rangle_{[N/2]+1}\cdots|0\rangle_N\right), \tag{15}$$

where $\mathcal{I} = ([N/2]+1, [N/2]+2, \cdots, N)$ is a multi-index.

We immediately know that $|G_\pm\rangle$ are the degenerate ground state of the ferromagnetic Ising Hamiltonian $\hat{H}_0$ with

eigen-energy $E_{(0)} = -N$, $|\tilde{G}_{-,\mathcal{I}}\rangle$ is the excited state of $\hat{H}_0$ with eigen-energy $E_{(1)} = -N + 4$.

The action of $\hat{H}_1$ on $|G_-\rangle$ ($|\tilde{G}_{-,\mathcal{I}}\rangle$) gives $|\tilde{G}_{-,\mathcal{I}}\rangle$ ($|G_-\rangle$) with a multiplicative factor -2.

$$\begin{aligned} \hat{H}_1|G_-\rangle &= -2|\tilde{G}_{-,\mathcal{I}}\rangle, \\ \hat{H}_1|\tilde{G}_{-,\mathcal{I}}\rangle &= -2|G_-\rangle. \end{aligned} \tag{16}$$

Then we know the two eigen-states of $\hat{H}$ is a superpositon of $|G_-\rangle$ and $|\tilde{G}_{-,\mathcal{I}}\rangle$ and can be written as

$$|\Psi\rangle = \alpha_1|G_-\rangle + \alpha_2|\tilde{G}_{-,\mathcal{I}}\rangle, \tag{17}$$

where $\alpha_{1,2}$ are the undetermined coefficients. Substitute into the Schrödinger equation $\hat{H}\Psi\rangle = \epsilon|\Psi\rangle$, we get

$$\epsilon^2 - \left(E_{(0)} + E_{(1)}\right)\epsilon + \left(E_{(0)}E_{(1)} - 4(\lambda + \gamma\mathrm{i})^2\right) = 0, \tag{18}$$

$$2(\lambda + \gamma\mathrm{i})\alpha_2 = (E_{(0)} - \epsilon)\alpha_1. \tag{19}$$

The eigen-energy $\epsilon^{(\pm)} = \frac{1}{2}[E_{(0)} + E_{(1)} \pm \sqrt{(E_{(1)} - E_{(0)})^2 + 16(\lambda + \gamma\mathrm{i})^2}]$. Choose $\alpha_1 = 1$, we have $\alpha_2 = \frac{E_{(0)} - \epsilon}{2(\lambda + \gamma\mathrm{i})}$. Imposing the normalization condition, we have

$$|\Psi^{(\pm)}\rangle = \frac{\alpha_1}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}|G_-\rangle + \frac{\alpha_2}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}|\tilde{G}_{-,\mathcal{I}}\rangle. \tag{20}$$

If $\gamma = 0$, the Hamiltonian is Hermitian and we have the ground state $|\Psi_0\rangle \equiv |\Psi^{(-)}\rangle$ with energy $\epsilon_0 \equiv \epsilon^{(-)} = -N - 2(\sqrt{1 + \lambda^2} - 1)$. See more details about the Hermitian version of the system in Ref. [8]. Here, we choose the eigen-state from $\{|\Psi^{(+)}\rangle, |\Psi^{(-)}\rangle\}$ as the ground state $|\Psi_0\rangle$ of our generalized non-Hermitian system, for it deforms into the ground state of the Hermitian case when γ approaches zero. If ϵ is real, then the ground state energy ϵ_0 corresponds to the smaller one from $\epsilon^{(\pm)}$. However, the ground state energy ϵ_0 corresponds to the one with the larger imaginary part when ϵ is a complex number. The ground state $|\Psi_0\rangle$ are obtained straightforwardly.

For the GHZ state $|G_+\rangle$, we can know it's a non-degenerate excited state with energy $\epsilon_+ = -N$, for

$$\hat{H}_1|G_+\rangle = 0. \tag{21}$$

[1] T.-L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
[2] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998).
[3] M. Koashi and M. Ueda, Phys. Rev. Lett. **84**, 1066 (2000).
[4] D. M. Stamper-Kurn and M. Ueda, Rev. Mod. Phys. **85**, 1191 (2013).
[5] H. Pu, C. K. Law, S. Raghavan, J. H. Eberly, and N. P. Bigelow, Phys. Rev. A **60**, 1463 (1999).
[6] X.-Y. Luo, Y.-Q. Zou, L.-N. Wu, Q. Liu, M.-F. Han, M. K. Tey, and L. You, Science **355**, 620 (2017).
[7] V. K. Kozin and O. Kyriienko, Phys. Rev. Lett. **123**, 210602 (2019).
[8] P. Facchi, G. Florio, S. Pascazio, and F. V. Pepe, Phys. Rev. Lett. **107**, 260502 (2011).