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In[*]:= L = Min[a x3 + a (1 - ε) x4 + x1, 1] ;
x1n = x1 
$$\frac{1}{x1 (1 - (1 - L + s L) d) + L x2 g}$$


$$\left( (1 - L + s L) (1 - d) + L (1 - s) \frac{x1 + \frac{x3}{2} a + \frac{x4}{2} a (1 - \epsilon)}{x1 + x3 a + x4 a (1 - \epsilon)} \right); (* \text{ Hermaphrodite } *)$$

x2n = x2 
$$\frac{L g}{x1 (1 - (1 - L + s L) d) + L x2 g} \frac{x1 + \frac{x3}{2} a + \frac{x4}{2} a (1 - \epsilon)}{x1 + x3 a + x4 a (1 - \epsilon)}; (* \text{ Female } *)$$

x3n = x1 
$$\frac{L (1 - s)}{x1 (1 - (1 - L + s L) d) + L x2 g} \frac{\frac{x3}{2} a + \frac{x4}{2} a (1 - \epsilon)}{x1 + x3 a + x4 a (1 - \epsilon)}; (* \text{ Male } *)$$

x4n = x2 
$$\frac{L g}{x1 (1 - (1 - L + s L) d) + L x2 g} \frac{\frac{x3}{2} a + \frac{x4}{2} a (1 - \epsilon)}{x1 + x3 a + x4 a (1 - \epsilon)}; (* \text{ Male with CMS } *)$$


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In[*]:= J = {{D[x1n, x1], D[x1n, x2], D[x1n, x3], D[x1n, x4]},
             {D[x2n, x1], D[x2n, x2], D[x2n, x3], D[x2n, x4]},
             {D[x3n, x1], D[x3n, x2], D[x3n, x3], D[x3n, x4]},
             {D[x4n, x1], D[x4n, x2], D[x4n, x3], D[x4n, x4]}}; (* Jacobian *)

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In[*]:= eig = Eigenvalues[J]; (* λ is the leading eigenvalue,
that is the largest numerically *)
λ = eig[[4]]; (* We need -1 < λ < 1 for the considered
point to be stable (i.e., resistant to invasion) *)

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In[*]:= eig /. x1 → 0.25 /. x2 → 0.25 /. x3 → 0.25 /. x4 → 0.25 /. a → 4 /. s → 0.3 /. d → 0.1 /.
g → 4 /. ε → 0.2

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Out[*]:= {0, 0., 0.239176, 0.666072}

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In[*]:= Assuming[a > 0 && 0 < d < 1 && 0 < s < 1 && 0 < ε < 1 && 0 < g ,
eig[[4]] /. x1 → 1 /. x3 → 0 /. x2 → 0 /. x4 → 0 // Simplify]
(* Condition for males and CMS
not to invade hermaphroditism is this term < 1 *)

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Out[*]:= 
$$\frac{1}{2 (-1 + d s)^2} \text{Root}\left[\left(2 a g - 2 a g s - 4 a d g s + 4 a d g s^2 + 2 a d^2 g s^2 - 2 a d^2 g s^3\right) \#1 + \right. \\ \left. (-a - 2 g + a s + a d s + 2 d g s - a d s^2) \#1^2 + \#1^3 \&, 3\right]$$


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In[*]:= Reduce[
$$\frac{1}{2 (-1 + d s)^2} \text{Root}\left[\left(2 a g - 2 a g s - 4 a d g s + 4 a d g s^2 + 2 a d^2 g s^2 - 2 a d^2 g s^3\right) \#1 + \right. \\ \left. (-a - 2 g + a s + a d s + 2 d g s - a d s^2) \#1^2 + \#1^3 \&, 3\right] < \\ 1 \&\& 0 < d < 1 \&\& 0 < s < 1 \&\& a > 0 \&\& 0 < g ]$$


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Out[*]:= 
$$0 < s < 1 \&\& 0 < d < 1 \&\& 0 < g < 1 - d s \&\& 0 < a < \frac{-2 + 2 d s}{-1 + s}$$


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(* Condition for CMS not to invade hermaphroditism g+d s<1 ,
Condition for males not to invade hermaphroditism a+2 d s<2+a s *)

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In[*]:= Assuming[a >  $\frac{2(-1+ds)}{-1+s}$  && 0 < d < 1 && 0 < s < 1 && 0 < e < 1 && 0 < g,
  eig[[4]] /. x1 -> 0 /. x3 -> 0 /. x2 -> 1/2 /. x4 -> 1/2 // Simplify]
(* Condition for CMS fixation is this term < 1 *)
Out[*]:= 
$$\left( 8 \sqrt[4]{\left( -2 a^2 g \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right] + 2 a^2 d g \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right] + \right.} \right. \\
4 a^2 g e \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right] - 4 a^2 d g e \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right] - \\
2 a^2 g e^2 \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right] + 2 a^2 d g e^2 \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right] + \\
a^2 g \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 - 2 a^2 d g \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 - \\
a^2 g s \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 + 2 a^2 d g s \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 - \\
2 a^2 g e \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 + 4 a^2 d g e \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 + \\
2 a^2 g s e \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 - 4 a^2 d g s e \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 + \\
a^2 g e^2 \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 - 2 a^2 d g e^2 \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 - \\
\left. \left. a^2 g s e^2 \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 + 2 a^2 d g s e^2 \operatorname{Min}\left[1, \frac{1}{2} a (1-e)\right]^2 \right) \mp 1^2 + 8 \mp 1^3 \&, \right. \\
\left. 3 \right) \Bigg/ \left( a^2 g^2 (-1+e)^2 \operatorname{Min}\left[1, -\frac{1}{2} a (-1+e)\right]^2 \right)$$


```

Reduce[

$$\begin{aligned}
& \left( 8 \operatorname{Root} \left[ \left( -2 a^2 g \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right] + 2 a^2 d g \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right] + 4 a^2 g \epsilon \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right] - 4 a^2 d g \epsilon \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right] - 2 a^2 g \epsilon^2 \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right] + \right. \right. \\
& \quad 2 a^2 d g \epsilon^2 \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right] + a^2 g \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 - \\
& \quad 2 a^2 d g \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 - a^2 g s \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 + \\
& \quad 2 a^2 d g s \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 - 2 a^2 g \epsilon \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 + \\
& \quad 4 a^2 d g \epsilon \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 + 2 a^2 g s \epsilon \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 - \\
& \quad 4 a^2 d g s \epsilon \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 + a^2 g \epsilon^2 \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 - \\
& \quad 2 a^2 d g \epsilon^2 \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 - a^2 g s \epsilon^2 \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 + \\
& \quad \left. \left. 2 a^2 d g s \epsilon^2 \operatorname{Min} \left[ 1, \frac{1}{2} a (1 - \epsilon) \right]^2 \right) \#1^2 + 8 \#1^3 \&, 3 \right] \right) / \\
& \left( a^2 g^2 (-1 + \epsilon)^2 \operatorname{Min} \left[ 1, -\frac{1}{2} a (-1 + \epsilon) \right]^2 \right) < 1 \& \& 0 < d <
\end{aligned}$$

$$1 \& \& 0 < s < 1 \& \&$$

$$a >$$

$$\frac{2 (-1 + d s)}{-1 + s} \& \& 0 <$$

$$\epsilon <$$

$$1 \& \& 0 <$$

$$g \& \&$$

$$\frac{1}{2} a (1 - \epsilon) < 1 \& \&$$

$$0 <$$

$$\epsilon <$$

$$1, g] // \text{Simplify}$$

$$\text{Out[8]= } 0 < d < 1 \& \& 0 < s < 1 \& \& a > \frac{2 (-1 + d s)}{-1 + s} \& \& \frac{-2 + a}{a} < \epsilon < 1 \& \& 1 + g + 2 d (-1 + s) > s + \frac{4 - 4 d}{a - a \epsilon}$$

(\* Condition for CMS fixation given there is pollen limitation at dioecious population, i.e.,  $\frac{1}{2} a (1 - \epsilon) < 1$ , is  $1 + g + 2 d (-1 + s) > s + \frac{4 - 4 d}{a - a \epsilon}$  \*)

```

In[*]:= Reduce[
  (8 Root[(-2 a^2 g Min[1, 1/2 a (1 - ε)] + 2 a^2 d g Min[1, 1/2 a (1 - ε)] + 4 a^2 g ε Min[1, 1/2 a
    (1 - ε)] - 4 a^2 d g ε Min[1, 1/2 a (1 - ε)] - 2 a^2 g ε^2 Min[1, 1/2 a (1 - ε)] +
    2 a^2 d g ε^2 Min[1, 1/2 a (1 - ε)] + a^2 g Min[1, 1/2 a (1 - ε)]^2 -
    2 a^2 d g Min[1, 1/2 a (1 - ε)]^2 - a^2 g s Min[1, 1/2 a (1 - ε)]^2 +
    2 a^2 d g s Min[1, 1/2 a (1 - ε)]^2 - 2 a^2 g ε Min[1, 1/2 a (1 - ε)]^2 +
    4 a^2 d g ε Min[1, 1/2 a (1 - ε)]^2 + 2 a^2 g s ε Min[1, 1/2 a (1 - ε)]^2 -
    4 a^2 d g s ε Min[1, 1/2 a (1 - ε)]^2 + a^2 g ε^2 Min[1, 1/2 a (1 - ε)]^2 -
    2 a^2 d g ε^2 Min[1, 1/2 a (1 - ε)]^2 - a^2 g s ε^2 Min[1, 1/2 a (1 - ε)]^2 +
    2 a^2 d g s ε^2 Min[1, 1/2 a (1 - ε)]^2) #1^2 + 8 #1^3 &, 3]) /
  (a^2 g^2 (-1 + ε)^2 Min[1, -1/2 a (-1 + ε)]^2) < 1 && 0 < d <
  1 && 0 < s < 1 &&
  a >
  2 (-1 + d s)
  -1 + s
  && 0 <
  ε <
  1 &&
  1 - s d < g &&
  1
  2
  a
  (1 - ε) >
  1 && 0 < ε < 1, g] // Simplify

```

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Out[*]:= 0 < d < 1 && 0 < s < 1 && a > 2 (-1 + d s)
-1 + s
&& 0 < ε < -2 + a
a
&& g > 1 + s - 2 d s

```

(\* Condition for CMS fixation given there is pollen limitation at dioecious population, i.e.,  $\frac{1}{2} a (1-\epsilon) > 1$ , is  $g > 1 + s - 2 d s$  \*)

(\* We do not analyse the condition for CMS not to invade androdioecy for 2 reasons. (1) It is the same as in the case of no pollen limitation - model 1, because no pollen limitation should arise in our model if females did not invade and all plants could produce pollen. Hence, this condition is already analysed in the script of model 1. (2) The analysis of the condition for CMS to invade androdioecy takes too much time to run because estimating equilibrium frequencies is too complicated for this model \*)