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Unified interpolative of Reich–Rus–Ćirić type contraction in relational metric space with an application

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Abstract. In this paper, we introduce the notion of unified interpolative contractions of the Reich–Rus–Ćirić type and give some results about the fixed points for these mappings in the framework of relational metric spaces. We present examples where the results of some previous research are not relevant. Also, we apply our results to solving problems related to nonlinear matrix equations, emphasizing their practical importance.

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1. Introduction

The Banach contraction principle, a cornerstone of metric fixed point theory, has found extensive applications across various disciplines, including physics, chemistry, economics, computer science, and biology. Consequently, the exploration and generalization of this principle have become focal points of research within nonlinear analysis [1–4].

The mappings that satisfy the Banach contraction principle are continuous. This prompts a natural question:

Can a discontinuous map in a complete metric space while satisfying analogous contractive conditions, possess a fixed point?

This intriguing question spurred investigation within the field, leading to an affirmative answer by Kannan [5]. Through the introduction of a novel form of contraction, Kannan [5] illuminated the possibility of fixed points within the realm of discontinuous maps and expanded the domain of inquiry within nonlinear analysis.

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In the year 1972, Reich [6] extended the principles introduced by Banach and Kannan. For instance, a self-mapping $S : X \to X$ is referred to as a Reich-contraction mapping if there exist values $\alpha, \beta, \gamma \in [0, 1)$ where $\alpha + \beta + \gamma < 1$, such that

$$\partial(S\nu, S\mu) \leq \alpha \partial(\nu, S\nu) + \beta \partial(\mu, S\mu) + \gamma \partial(\nu, \mu),$$

(1.1)

for all $\nu, \mu \in X$.

Additional significant variations of the Banach contraction principle were explored independently by Ćirić-Reich and Rus [7–9]. A collective outcome attributed to their work is presented below, recognized as the Ćirić-Reich-Rus contraction if there exists $\lambda \in \left[0, \frac{1}{3}\right)$ such that

$$\partial(S\nu, S\mu) \leq \lambda [\partial(\nu, \mu) + \partial(\nu, S\nu) + \partial(\mu, S\mu)],$$

(1.2)

for all $\nu, \mu \in X$.

In 2018, Karapınar [10] employed the interpolative method and converted the fundamental contraction concept of Kannan [5] into an interpolative form. Karapinar et al. [11] detected a deficiency in the analysis conducted by [10] concerning the assumption that the fixed point is unique. They accomplished this by presenting a counter-example and formulated an amended version, while also introducing the notions of interpolative Reich-Rus and Ćirić type contractions, e.g., a mapping $S : X \to X$ is called an interpolative Reich–Rus–Ćirić type contraction, if there are constants $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that

$$\partial(S\nu, S\mu) \leq \lambda \left(\partial(\nu, \mu)^{\alpha} \cdot \partial(\nu, S\nu)^{\beta} \cdot \partial(\mu, S\mu)^{1-\alpha-\beta}\right),$$

(1.3)

for all $\nu, \mu \in X \setminus F(S)$. And proved that in the framework of partial metric space $(X, \partial)$, a mapping $S$, characterized as an interpolative Reich–Rus–Ćirić type contraction, possesses a fixed point. Additionally, noteworthy contributions have been made by several authors [12–16], further enriching this area of study.

In our current study, we introduce a broader idea called unified interpolative Reich-Rus-Ćirić type contraction. This concept encompasses many existing findings, including those presented by [7–11, 14, 17]. We demonstrate several fixed point results for such contractions within relational metric spaces.

It is important to note that in relational metric spaces, we often deal with weaker properties like $\mathcal{R}$-continuity (not necessarily implying continuity), $\mathcal{R}$-completeness (not necessarily implying completeness), and so on. In this context, we have more flexibility since the contraction condition is not required for every element but only for related ones. Importantly, these contraction conditions return to their usual forms when considering the universal relation.

2. Preliminaries

Before presenting our main results, it is important to introduce formal notations that will be used throughout this paper.

Let $X$ be a non-empty set, with a binary relation $\mathcal{R}$. In this context, the pair $(X, \mathcal{R})$ is acknowledged as a relational set. Similarly, within the framework of a
metric space \((X, \partial)\), we designate the triplet \((X, \partial, \mathcal{R})\) constitutes a relational metric space (RMS, for brevity). The collection of fixed points of the self-mapping \(S\) is indicated by \(F(S)\), and let \(X_\mathcal{R}\) denotes the set defined by,

\[
X_\mathcal{R} = \{(\nu, \mu) \in X^2 : (\nu, \mu) \in \mathcal{R} \text{ and } \nu, \mu \notin F(S)\}
\]

Furthermore, \(X(\mathcal{S}, \mathcal{R})\) is a subset of \(X\), containing elements \(\nu\) such that \((\nu, S\nu) \in \mathcal{R}\). These formalized notations ensure precision and consistency throughout our subsequent analyses and discussions.

**Definition 2.1.** [17] In the context of a relational set \((X, \mathcal{R})\), and a selfmap \(S\) defined on \(X\),

(i) any two elements \(\nu, \mu \in X\) are considered \(\mathcal{R}\)-comparative if \((\nu, \mu) \in \mathcal{R}\) or \((\mu, \nu) \in \mathcal{R}\). This relationship is symbolically represented as \([\nu, \mu] \in \mathcal{R}\),

(ii) a sequence \(\{\nu_k\} \subset X\) satisfies the condition \((\nu_k, \nu_{k+1}) \in \mathcal{R}\) for all \(k \in \mathbb{N}_0\), is referred to as an \(\mathcal{R}\)-preserving sequence.

(iii) \(\mathcal{R}\) is designated as \(\mathcal{S}\)-closed when it satisfies the condition that if \((\nu, \mu)\) belongs to \(\mathcal{R}\), then \((S\nu, S\mu)\) also belongs to \(\mathcal{R}\), for any \(\nu, \mu \in X\).

(iv) \(\mathcal{R}\) is referred to as \(\partial\)-self-closed under the condition that whenever there exists a \(\mathcal{R}\)-preserving sequence \(\{\nu_k\}\) such that \(\nu_k \rightharpoonup \nu\), we can always find a subsequence \(\{\nu_{k_n}\}\) of \(\{\nu_k\}\) such that \([\nu_{k_n}, \nu]\) belongs to \(\mathcal{R}\) for all \(n \in \mathbb{N}_0\).

**Definition 2.2.** [18] \((X, \partial, \mathcal{R})\) is considered \(\mathcal{R}\)-complete if every sequence in \(X\), which is both \(\mathcal{R}\)-preserving and Cauchy, converges.

**Definition 2.3.** [18] A self-map \(S\) defined on \(X\) is termed \(\mathcal{R}\)-continuous at \(\nu \in X\), if any \(\mathcal{R}\)-preserving sequence \(\nu_k \rightharpoonup \nu\), implies \(S\nu_k \rightharpoonup S\nu\). Furthermore, if \(S\) exhibits this behavior at every point in \(X\), it is simply categorized as \(\mathcal{R}\)-continuous.

**Definition 2.4.** [19] Consider a self-mapping \(S\) defined on \(X\). If for every \(\mathcal{R}\)-preserving sequence \(\{\nu_n\} \subset S(X)\), with a range denoted as \(E = \{\nu_n : n \in \mathbb{N}\}\), \(\mathcal{R}|_E\) is transitive, then \(S\) is designated as locally \(S\)-transitive.

Samet et al. [20] introduced the concept of \(\alpha\)-admissible mappings, which has been applied by various authors in numerous fixed-point theorems.

**Definition 2.5.** [20] Suppose \(S\) is a self-map on \(X\), and \(\alpha : X \times X \to \mathbb{R}^+\) is a function. Then, \(S\) is considered \(\alpha\)-admissible if \(\alpha(\nu, \mu) \geq 1 \Rightarrow \alpha(S\nu, S\mu) \geq 1\) for all \(\nu, \mu \in X\).

In the following definition, we generalize this concept by incorporating certain relational metrical notions.

**Definition 2.6.** Let \((X, \mathcal{R})\) be a relational set. A self-map \(S\) defined on \(X\) is termed \(\mathcal{R}\)-admissible if there exists a function \(\vartheta : X \times X \to [0, +\infty)\), satisfying the following conditions:

\((r_1)\) \(\vartheta(\nu, \mu) \geq 1\) for all \((\nu, \mu) \in \mathcal{R}\),

\((r_2)\) \(\mathcal{R}\) is \(S\)-closed.
Remark 2.7. From the above two definitions, we can observe that if $S$ is $\alpha$-admissible, it also holds that $S$ is $\mathcal{R}$-admissible when considering
$$\mathcal{R} = \{(\nu, \mu) \in X^2 : \vartheta(\nu, \mu) \geq 1\}.$$  
However, it should be noted that the converse is not necessarily true, as illustrated in the following example.

Example 2.8. Let $X = \{0, 1, 2, 3\}$, $\vartheta : X \times X \to \mathbb{R}^+$ be
$$\vartheta(\nu, \mu) = \begin{cases} 
2, & (\nu, \mu) \in \{(0, 1), (1, 2), (2, 3)\} \\
1, & (\nu, \mu) \in \{(0, 2), (1, 1), (2, 1), (2, 2)\} \\
\frac{2}{\nu + 5}, & \text{otherwise.}
\end{cases}$$

Let $S : X \to X$ be defined by $S0 = 0, S1 = 2, S2 = 1, \text{ and } S3 = 3$.
In this example, it is evident that $\vartheta(2, 3) \geq 1$, but $\vartheta(S2, S3) = \vartheta(1, 3) \not\geq 1$, indicating that $S$ is not $\vartheta$-admissible. Now, let’s consider the binary relation $\mathcal{R}$ defined as,
$$\mathcal{R} = \{(0, 1), (0, 2), (1, 2), (2, 1), (1, 1), (2, 2)\}.$$  
It is straightforward to observe that $\mathcal{R}$ is $S$-closed, and for all $\nu, \mu \in X$ with $(\nu, \mu) \in \mathcal{R}$, $\vartheta(\nu, \mu) \geq 1$. Therefore, $S$ is $\mathcal{R}$-admissible.

Let $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be two functions. Then we consider the following conditions:

(C$_1$) $\phi$ is u.s.c. with $\phi(0) = 0$,
(C$_2$) $\psi$ is l.s.c.,
(C$_3$) $\psi, \phi$ are non-decreasing,
(C$_4$) $\psi(t) > \phi(t)$, for all $t > 0$,
(C$_5$) $\limsup_{t \to e^+} \phi(t) < \psi(e^+)$, for all $e > 0$,
(C$_6$) $\limsup_{t \to e^+} \phi(t) \leq \liminf_{t \to e^+} \psi(t)$, for any $e > 0$.

In the next section, we will introduce a novel concept termed as the unified interpolative Reich–Rus–Ćirić type contraction condition and establish several fixed-point results for such contractions.

3. Main results

First, we give a definition of unified interpolative Reich-Rus-Ćirić type contraction.

Definition 3.1. Let $(X, \vartheta, \mathcal{R})$ be an RMS. A self-mapping $S$ defined on $X$ is a unified interpolative Reich-Rus-Ćirić type contraction (UIRCC), if there exist the functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$, and exists a function $\nu : X \times X \to \mathbb{R}^+$, along with the parameters $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that
$$\vartheta(\nu, \mu)\psi(\vartheta(S\nu, S\mu)) \leq \phi \left( \Omega(\vartheta(\nu, \mu), \vartheta(\nu, S\nu), \vartheta(\mu, S\mu)) \right),$$
for all $\nu, \mu \in X_R$, where $\Omega : \mathbb{R}^3 \to \mathbb{R}$ be a mapping such that
$$\Omega(u, v, w) \leq \max \{u, v, w^\alpha v^\beta w^{1-\alpha-\beta} \}.$$
Remark 3.2. By giving the precise definitions of the functions $\psi$, $\phi$, and $\nu$, along with the non-negative constant $\Omega$, it becomes evident that we can draw the following conclusions, underscoring the extensive applicability and versatility of Definition 3.1.

(i) When we consider $\partial(\nu, \mu) = 1$, and $\Omega(u, v, w) = u^\alpha \cdot v^\beta \cdot w^{1-\alpha-\beta}$, where $\alpha + \beta < 1$ in equation (3.1), also consider the binary relation $R$ as

$$R = \{(\nu, \mu) \in X^2 : \nu \perp \mu\},$$

we obtain the $(\psi, \phi)$-orthogonal interpolative Ćirić Reich-Rus type contraction [14],

$$\psi(\partial(S\nu, S\mu)) \leq \phi(\partial(\nu, \mu)^\alpha \cdot \partial(\nu, S\nu)^\beta \cdot \partial(\mu, S\mu)^{1-\alpha-\beta}),$$

(3.2)

for all $\nu, \mu \in X_R$.

(ii) By taking $\alpha = 0$ in (3.2) we obtain $(\psi, \phi)$-orthogonal Kannan contraction [14],

$$\psi(\partial(S\nu, S\mu)) \leq \phi(\partial(\nu, S\nu)^\beta \cdot \partial(\mu, S\mu)^{1-\beta}),$$

(3.3)

for all $\nu, \mu \in X_R$.

(iii) By taking $\psi(t) = t$, and $\psi(t) = \lambda t$, $\lambda < 1$, and considering $R$ as a universal relation in (3.2) and (3.3) we obtain the interpolative Reich-Rus-Ćirić type contraction [11] and interpolative Kannan contraction [10], respectively.

(iv) By considering $\psi(t) = t$, $\phi(t) = \lambda t$, $\lambda < 1$, and $\Omega(u, v, w) = \frac{u + v + w}{3}$, we obtain the combined result of Ćirić, Reich and Rus [7–9]

$$\partial(S\nu, S\mu) \leq \lambda (\partial(\nu, \mu) + \partial(\nu, S\nu) + \partial(\mu, S\mu)),$$

(3.4)

for all $\nu, \mu \in X_R$.

(v) By considering $\psi(t) = t$ and $\phi(t) = \lambda t$, $\lambda < 1$, and $\Omega(u, v, w) = u$, then we obtain the relational theoretic version of the famous Banach contraction which is introduced by Aftab Alam and Mohammad Imdad [17].

(vi) By considering $\psi(t) = t$, $\phi(t) = \lambda t$ and $\Omega(u, v, w) = \frac{v + w}{2}$, we obtain

Kannan contraction with the constant $\lambda \in \left[0, \frac{1}{2}\right]$,

$$\partial(S\nu, S\mu) \leq \lambda (\partial(\nu, S\nu) + \partial(\mu, S\mu)),$$

(3.5)

for all $\nu, \mu \in X_R$.

Now, we will proceed to establish our main results concerning the unified interpolative Reich-Rus-Ćirić contraction maps.

Theorem 3.3. Consider the RMS $(X, \partial, R)$ where $R$ is a locally $S$-transitive binary relation. Suppose that $S$ is an UIRRCC, and there exist functions $\psi, \phi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying conditions $C_i$, $(i = 1, 2, 3, 4)$. Under the following conditions:

(D1) $S$ is $R$-admissible,

(D2) there exists $Y \subseteq X$ with $S(X) \subseteq Y$, such that $(Y, \partial, R)$ is $R$-complete,

(D3) $X(S, R)$ is non-empty,

(D4) either $S$ is $R|_Y$-continuous or $R$ is $\partial$-self-closed.
there exists at least one $\gamma \in X$ such that $\gamma \in F(\mathcal{S})$.

**Proof.** Under the assumption $(D_3)$, suppose that $\nu_0 \in X(\mathcal{S}, \mathcal{R})$. Define the sequence $\{\nu_n\}$ of Picard iterates with initial point $\nu_0$, i.e. $\nu_n = S^n \nu_0$ for all $n \in \mathbb{N}_0$. As $(\nu_0, S\nu_0) \in \mathcal{R}$ and $\mathcal{S}$ is $\mathcal{R}$-admissible, using $(r_1)$ it follows that $(S^n \nu_0, S^{n+1} \nu_0) \in \mathcal{R}$. Consequently, $(\nu_n, \nu_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$, and this yields that the sequence $\{\nu_n\}$ is $\mathcal{R}$-preserving and from $(r_2)$ we have $\partial(\nu_n, \nu_{n+1}) \geq 1$. Let $\partial_n = \partial(\nu_n, \nu_{n+1})$, and applying contractive condition (3.1), we obtain that,

$$
\psi(\partial_n) \leq \partial(\nu_{n-1}, \nu_n)\psi(\partial(S\nu_{n-1}, S\nu_n)) \\
\leq \phi(\Omega(\partial(\nu_{n-1}, \nu_n), \partial(\nu_{n-1}, S\nu_{n-1}), \partial(\nu_n, S\nu_n))) \\
\leq \phi \left( \max \left\{ \partial_{n-1}, \partial_n, \partial_n^\alpha \cdot \partial_n^{1-\alpha-\beta} \right\} \right) \\
< \psi \left( \max \left\{ \partial_{n-1}, \partial_n, \partial_n^{\alpha+\beta} \cdot \partial_n^{1-(\alpha+\beta)} \right\} \right). \quad (3.6)
$$

By monotonicity of the function $\psi$ we obtain,

$$
\partial_n < \max \left\{ \partial_{n-1}, \partial_n, \partial_n^{\alpha+\beta} \cdot \partial_n^{1-(\alpha+\beta)} \right\}. \quad (3.7)
$$

Now suppose there exists $n \in \mathbb{N}$ for which $\partial_{n-1} \leq \partial_n$, then from (3.7) we obtain that $\partial_n < \partial_n$, a contradiction. Therefore $\partial_n \leq \partial_{n-1}$, now we can conclude that $\{\nu_n\}$ is a non-increasing sequence and thus a non-negative constant $C$ exists such that, $\lim_{n \to +\infty} \partial_n = C^+$. Suppose if possible $C > 0$, then from (3.6), it can be deduced that

$$
\psi(C^+) \leq \liminf \psi(\partial_n) \leq \limsup \phi(\partial_{n-1}) \leq \phi(C^+),
$$

but, form $(C_4)$ we have $\psi(\nu) > \phi(\nu)$ for all $\nu > 0$, therefore $C$ must be $0$, i.e. $\lim_{n \to +\infty} \partial_n = 0$. Our next objective is to establish that the sequence $\{\nu_n\}$ is Cauchy.

For the sake of contradiction, suppose it is not, then there exists a positive real number $\epsilon > 0$ along with sub-sequences $\{\nu_{n_k}\}$ and $\{\nu_{m_k}\}$ of $\{\nu_n\}$, with $n_k > m_k \geq k$, such that

$$
\partial(\nu_{m_k}, \nu_{n_k}) \geq \epsilon, \quad \text{for all } k \in \mathbb{N}. \quad (3.8)
$$

Selecting $n_k$ as the smallest integer exceeding $m_k$ such that (3.8) holds, we deduce that

$$
\partial(\nu_{m_k}, \nu_{n_k-1}) < \epsilon. \quad (3.9)
$$

Using triangular inequality and (3.8), (3.9) we obtain that

$$
\epsilon \leq \partial(\nu_{m_k}, \nu_{n_k}) \leq \partial(\nu_{m_k}, \nu_{n_k-1}) + \partial(\nu_{n_k-1}, \nu_{n_k}) \\
< \epsilon + \partial(\nu_{n_k-1}, \nu_{n_k}).
$$

On taking the limit $k \to +\infty$ and utilizing the fact that $\lim_{n \to +\infty} \partial_n = 0$, we obtain

$$
\lim_{k \to +\infty} \partial(\nu_{m_k}, \nu_{n_k}) = \epsilon + . \quad (3.10)
$$

By using triangular inequality, we obtain that

$$
|\partial(\nu_{m_k+1}, \nu_{n_k+1}) - \partial(\nu_{m_k}, \nu_{n_k})| \leq \partial\nu_{m_k} + \partial\nu_{n_k},
$$
It is important to note that in equation (3.10), we obtain the following,
\[ \lim_{k \to +\infty} \partial(\nu_{m_k+1}, \nu_{n_k+1}) = \lim_{k \to +\infty} \partial(\nu_{m_k}, \nu_{n_k}) = \epsilon. \] (3.11)
Since \( \{\nu_n\} \subset S(X) \) and \{\nu_n\} is \( \mathcal{R} \)-preserving, the local \( S \)-transitivity of \( \mathcal{R} \) leads to the implication that \( \{\nu_{m_k}, \nu_{n_k}\} \subset \mathcal{R} \). Thus, we can deduce
\[
\psi(\partial(\nu_{m_k+1}, \nu_{n_k+1})) \leq \partial(\nu_{m_k}, \nu_{n_k})\psi(\partial(S\nu_{m_k}, S\nu_{n_k})) \\
\leq \phi(\Omega(\partial(\nu_{m_k}, \nu_{n_k}), \partial(\nu_{m_k}, S\nu_{m_k}), \partial(\nu_{n_k}, S\nu_{n_k}))) \\
\leq \phi(\max\{\partial(\nu_{m_k}, \nu_{n_k}), \partial_{m_k, \nu_{n_k}}, \partial(\nu_{m_k}, \nu_{n_k})^\alpha \cdot \partial_{m_k, \nu_{n_k}} \gamma \}) .
\]
On taking the limit as \( k \to +\infty \) in the aforementioned inequality, leads to the contradiction of (C4). Hence, \{\nu_n\} is the \( \mathcal{R} \)-preserving Cauchy sequence in \( Y \).

The \( \mathcal{R} \)-completeness of the metric space \( (Y, \partial, \mathcal{R}) \) now guarantees the existence of a point \( \gamma \in Y \) such that \( \lim_{n \to +\infty} \nu_n = \gamma \).

First, we assumed that \( S \) is \( \mathcal{R} \)-continuous, we can deduce that \( \lim_{n \to +\infty} \nu_{n+1} = S\nu_n = S\gamma \). Applying the uniqueness of the limit, we consequently establish that \( S\gamma = \gamma \), indicating that \( \gamma \in F(S) \).

Alternatively, let \( \mathcal{R}|_{Y} \) is \( \partial \)-self-closed. We again utilize the fact that \{\nu_n\} is \( \mathcal{R} \)-preserving and \{\nu_n\} \( \to \gamma \). This implies the existence of a sub-sequence \{\nu_{n_k}\} of \{\nu_n\} \[ [\nu_{n_k}, \gamma] \in \mathcal{R}, \text{ for all } k \in \mathbb{N}_0. \] If \( (\nu_{n_k}, \gamma) \in \mathcal{R} \), then since \( S \) is a unified interpolative Reich-Rus-Ćirić contraction, we have
\[
\psi(\partial(S\nu_{n_k}, S\gamma)) \leq \partial(\nu_{n_k}, \gamma)\psi(\partial(S\nu_{n_k}, S\gamma)) \\
\leq \phi(\Omega(\partial(\nu_{n_k}, \gamma), \partial(\nu_{n_k}, S\nu_{n_k}), \partial(\gamma, S\gamma))) \\
\leq \phi(\max\{\partial(\nu_{n_k}, \gamma), \partial_{n_k, \nu_{n_k}}, \partial(\gamma, S\gamma), \partial(\nu_{n_k}, \gamma)^\alpha \cdot \partial_{n_k, \nu_{n_k}} \gamma \}) ,
\]
(3.12)
on taking the limit \( k \to +\infty \), in (3.12), we obtain
\[
\psi(\partial(\gamma, S\gamma)) \leq \phi(\partial(\gamma, S\gamma)) .
\] (3.13)
It is important to note that in equation (3.13), if \( \partial(\gamma, S\gamma) \neq 0 \), then we face a contradiction from (C4). Similarly, if \( (\gamma, \nu_{n_k}) \in \mathcal{R} \), then by utilizing the symmetry of \( \partial \), we once again encounter a contradiction with (C4). Therefore, \( \partial(\gamma, S\gamma) = 0 \), implying \( \gamma \in F(S) \).

**Theorem 3.4.** Consider the RMS \((X, \partial, \mathcal{R})\) where \( \mathcal{R} \) is a locally \( S \)-transitive binary relation. Suppose that \( S \) is an UIRRCC and there exist functions \( \psi, \phi : [0, +\infty) \to [0, +\infty) \) satisfying conditions \( C_i \), \( (i = 3, 4, 5, 6) \) and \( D_j \), \( (j = 1, 2, 3, 4) \) holds. Then, there exists at least one \( \gamma \in X \) such that \( \gamma \in F(S) \).

**Proof.** Following the steps of the previous theorem we can obtain an \( \mathcal{R} \)-preserving and non-increasing sequence \{\nu_n\} such that there exists some \( C \geq 0 \) and \( \partial_n \) converges to \( C+ \) as \( n \to +\infty \). Suppose if \( C > 0 \), then (3.6) implies that
\[
\psi(C+) = \lim_{n \to +\infty} \psi(\partial_n) \leq \lim_{n \to +\infty} \sup_{k \to C+} \phi(\max\{\partial_{n-1}, \partial_{n-1}, \partial_n, \partial_n^{\alpha} \cdot \partial_{n-1}^{\beta} \cdot \partial_{n-1}^{1-\alpha-\beta}\}) \\
\leq \lim_{k \to C+} \sup_{k \to C+} \phi(k) ,
\]
a contradiction from \((C_5)\), thus \(C = 0\) i.e. \(\lim_{n \to +\infty} \partial_n = 0\). Now, to establish that the sequence \(\{\nu_n\}\) is Cauchy, we make a counter assumption. Suppose it is not Cauchy, then following the steps outlined in the previous theorem, there exists a positive real number \(\epsilon > 0\), along with sub-sequences \(\{\nu_{n_k}\}\) and \(\{\nu_{m_k}\}\) of \(\{\nu_n\}\), where \(n_k > m_k \geq k\), satisfying condition (3.11). Since \(\{\nu_n\} \subset \mathcal{S}(X)\) and \(\{\nu_n\}\) is \(\mathcal{R}\)-preserving, the local \(\mathcal{S}\)-transitivity of \(\mathcal{R}\) leads to the implication that \((\nu_{m_k}, \nu_{n_k}) \in \mathcal{R}\). Thus, we can deduce
\[
\psi(\partial(\nu_{m_{k+1}}, \nu_{n_{k+1}})) \leq \psi(\partial(\nu_{m_k}, \nu_{n_k})) \psi(\partial(S\nu_{m_k}, S\nu_{n_k}))
\leq \phi(\max \{\partial(\nu_{m_k}, \nu_{n_k}), \partial_{\nu_{m_k}}, \partial(\nu_{n_k}, \nu_{n_k})^\alpha \cdot \partial_{\nu_{m_k}}^\beta \cdot \partial_{\nu_{n_k}}^{-1-\alpha-\beta}\}),
\]
on taking the limit \(k \to +\infty\) in the above equation, it implies that
\[
\liminf_{a \to +\infty} \psi(a) \leq \liminf_{k \to +\infty} \psi(\partial(\nu_{m_{k+1}}, \nu_{n_{k+1}}))
\leq \limsup_{k \to +\infty} \phi(\max \{\partial(\nu_{m_k}, \nu_{n_k}), \partial_{\nu_{m_k}}, \partial(\nu_{n_k}, \nu_{n_k})^\alpha \cdot \partial_{\nu_{m_k}}^\beta \cdot \partial_{\nu_{n_k}}^{-1-\alpha-\beta}\})
\leq \limsup_{a \to +\infty} \phi(a).
\]
This results a contradiction of \((C_6)\), thus establishing that the \(\{\nu_n\}\) is an \(\mathcal{R}\)-preserving Cauchy sequence is in \(Y\). Given that \((Y, \partial, \mathcal{R})\) is an \(\mathcal{R}\)-complete metric space, there exists \(\gamma \in Y\) such that \(\lim_{n \to +\infty} \nu_n = \gamma\). If the self-mapping \(\mathcal{S}\) is \(\mathcal{R}\)-continuous, we can derive the desired conclusion, as demonstrated in the previous theorem.

Alternatively, let \(\mathcal{R}|_Y\) be \(\partial\)-self-closed then utilizing the fact that \(\{\nu_n\}\) is \(\mathcal{R}\)-preserving and \(\{\nu_n\} \to \gamma\). This implies the existence of a sub-sequence \(\{\nu_{n_k}\}\) of \(\{\nu_n\}\) with \([\nu_{n_k}, \gamma ] \in \mathcal{R}\), for all \(k \in \mathbb{N}_0\). We claim that \(\partial(\gamma, S\gamma) = 0\). Let us assume that \(\partial(\gamma, S\gamma) > 0\), if \((\nu_{n_k}, \gamma) \in \mathcal{R}\), then since \(\mathcal{S}\) is a unified interpolative Reich-Rus-Ćirić contraction, we have
\[
\psi(\partial(\nu_{n_{k+1}}, S\gamma)) \leq \psi(\partial(\nu_{n_k}, \gamma)) \psi(\partial(S\nu_{n_k}, S\gamma))
\leq \phi(\Omega(\partial(\nu_{n_k}, \gamma), \partial(\nu_{n_k}, S\nu_{n_k}), \partial(\gamma, S\gamma)))
\leq \phi(\max \{\partial(\nu_{n_k}, \gamma), \partial_{\nu_{n_k}}, \partial(\gamma, S\gamma), \partial(\nu_{n_k}, \gamma)^\alpha \cdot \partial_{\nu_{n_k}}^\beta \cdot \partial(\gamma, S\gamma)^{-1-\alpha-\beta}\})
\leq \psi(\max \{\partial(\nu_{n_k}, \gamma), \partial_{\nu_{n_k}}, \partial(\gamma, S\gamma), \partial(\nu_{n_k}, \gamma)^\alpha \cdot \partial_{\nu_{n_k}}^\beta \cdot \partial(\gamma, S\gamma)^{-1-\alpha-\beta}\}),
\]
by using \((C_3)\) and taking the limit as \(k \to +\infty\), we deduce
\[
\partial(\gamma, S\gamma) < \partial(\gamma, S\gamma), \tag{3.14}
\]
which leads to a contradiction. Furthermore, if \((\gamma, \nu_{n_k}) \in \mathcal{R}\), then by utilizing the symmetry of \(\partial\), we encounter again a contradiction. Hence, \(\partial(\gamma, S\gamma) = 0\), implying \(\gamma \in F(\mathcal{S})\).

**Theorem 3.5.** Consider the RMS \((X, \partial, \mathcal{R})\), where \(\mathcal{R}\) is a locally \(\mathcal{S}\)-transitive and \(S\)-closed. Suppose conditions \(D_j, (j = 1, 2, 3, 4)\) holds and there exist functions \(\psi, \phi : [0, +\infty) \to [0, +\infty)\) satisfying conditions \(C_i\), \((i = 1, 2, 3, 4)\) or \((i = 3, 4, 5, 6)\), such that,
\[
\psi(\partial(\mathcal{S}\nu, \mathcal{S}\mu)) \leq \phi(\Omega(\partial(\nu, \mathcal{S}\nu), \partial(\mu, \mathcal{S}\mu))), \quad \text{for all } \nu, \mu \in X_\mathcal{R}. \tag{3.15}
\]
Then there exists at least one $\gamma \in X$ such that $\gamma \in F(S)$.

**Example 3.6.** Let $(X, d)$ be a metric space with $X = [0, +\infty)$ and $\partial$ is the usual metric, define the self-map $S$ on $X$ by,

$$S\nu = \begin{cases} 
\frac{5\nu^2}{6}, & \text{if } \nu \leq 1, \\
\frac{1}{2}, & \text{if } \nu > 1.
\end{cases}$$

Then, it is important to note that $S$ is not a Ćirić, Reich Rus type contraction [7–9]. This is evident that when considering $\nu = 1$ and $\mu = \frac{1}{2}$, there does not exist any constant $\lambda \in \left(0, \frac{1}{3}\right]$ for which condition (1.2) holds. Additionally, for the same values of $\nu = 1$ and $\mu = \frac{1}{2}$, there is no pair of $\lambda \in [0, 1)$ and $\alpha, \beta \in [0, 1)$ satisfying $\alpha + \beta < 1$ for which (3.2) holds. Consequently, $S$ is not an interpolative Reich-Rus-Ćirić type contraction [10]. Now, let us define the binary relation $\mathcal{R}$ on $X$ as,

$$\mathcal{R} = \left\{ (\nu, \mu) \in X^2 : \max\{\nu, \mu\} \leq \frac{1}{2} \right\}.$$ 

This relation $\mathcal{R}$ exhibits the property of being locally $S$-transitive, and $S$ is $\mathcal{R}$-continuous. It can also be observed that $\mathcal{R}$ is $S$-closed. Moreover, the set $X(S, \mathcal{R})$ is nonempty, and there exists a subset $Y = [0, 1]$ of $X$ such that $S(X) \subseteq Y$ and $(Y, \partial)$, is $\mathcal{R}$-complete.

Observing the definition of $\mathcal{R}$, it is evident that $\mathcal{R}$ is not an orthogonal relation. It is important to recall that a binary relation $\mathcal{R}$ is considered as an orthogonal relation if for any element $\nu_0 \in X$, either (for all $\mu, (\nu_0, \mu) \in \mathcal{R}$) or (for all $\mu, (\mu, \nu_0) \in \mathcal{R}$). As a consequence, the function $S$ is not a $(\psi, \phi)$-orthogonal interpolative Reich-Rus-Ćirić type contraction [14]. However, we will now demonstrate that $S$ is indeed a unified interpolative Reich-Rus-Ćirić type contraction. Con-
sider $\vartheta : X \times X \to [0, +\infty)$ defined by

$$\vartheta(\nu, \mu) = \begin{cases} 1, & \text{if } \nu, \mu \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Observing that $\vartheta(\nu, \mu) \geq 1$ for all $\nu, \mu \in X$ with $(\nu, \mu) \in \mathcal{R}$, and that $(\nu, \mu) \in \mathcal{R}$ implies $(S\nu, S\mu) \in \mathcal{R}$, it follows that $S$ is $\mathcal{R}$-admissible. Suppose there exist functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$ defined by $\phi(t) = \frac{t}{7}$,

and, $\psi(t) = \begin{cases} 2t, & \text{if } t \leq 1, \\ \frac{4}{5}t^2, & \text{if } t > 1. \end{cases}$

In Figure 1, the red (dashed) line represents $\phi(t)$, while the blue line denotes $\psi(t)$. It is evident that $\phi$ is a u.s.c., $\phi(0) = 0$, and $\psi$ is l.s.c., such that $\psi(t) > \phi(t)$ and both $\psi, \phi$ are nondecreasing.

We now aim to show that $S$ satisfies (3.1). Consider the function $\Omega : X \times X \to [0, +\infty)$ defined as $\Omega(u, v, w) = \frac{4}{5} \max \{u, v, w, u^{\alpha}v^{\beta}w^{1-\alpha-\beta}\}$. Then, for every $\nu, \mu \in X_\mathcal{R}$, we can observed that,

$$\vartheta(\nu, \mu)\psi(S\nu, S\mu) = \frac{5|\nu^2 - \mu^2|}{33}, \quad (3.16)$$
and,
\[
\phi(\Omega(d(\nu, \mu), d(\nu, T\nu), d(\mu, T\mu))) = \frac{4}{35} \left( \max \left\{ |\nu - \mu|, \frac{|6\nu - 5\mu^2|}{6} \right\} \right)
\]

In Figure 2, for each point \(\nu, \mu \in X\) such that \(\nu, \mu \in \mathcal{R}\), corresponds to a three-dimensional representation of equation (3.16) (illustrated by the red plane) and equation (3.17) (depicted by the blue plane), with the given parameters \(\alpha = 0.1\) and \(\beta = 0.2\). It is evident from the observation that the red plane remains consistently below or coincident with the blue plane. Consequently, we can deduce that equation (3.16), representing the left-hand side of (3.1), consistently maintains a value that is less than or equal to equation (3.17), representing the right-hand side of (3.1). Hence, it follows that Equation (3.1) holds true for all real \(x, y \in \mathcal{R}\).

Consequently, we deduce that \(S\) is a unified interpolative Reich-Rus-Ćirić contraction.

4. An application

In this section, we have applied our research findings to derive a result concerning the existence of solutions for a nonlinear matrix equation. In this context, let the set denoted as \(\mathcal{M}(n)\) encompasses all square matrices with dimensions of \(n \times n\), while \(\mathcal{H}(n), \mathcal{P}(n),\) and \(\mathcal{K}(n)\), respectively represent the sets of Hermitian matrices, positive definite positive, and semi-definite matrices. When we have a matrix \(C\) from \(\mathcal{H}(n)\), we use the notation \(\|C\|_{tr}\) to refer to its trace norm, which is the sum of all its singular values. If we have matrices \(P\) and \(Q\) from \(\mathcal{H}(n)\), the notation \(P \succeq Q\) signifies that the matrix \(P - Q\) is an element of the set \(\mathcal{K}(n)\), while \(P \succ Q\) indicates that \(P - Q\) belongs to the set \(\mathcal{P}(n)\). The upcoming discussion relies on the significance of the following lemmas.

**Lemma 4.1.** [21] If \(X \in \mathcal{H}(n)\) satisfies \(X \prec I_n\), then \(\|X\| < 1\).

**Lemma 4.2.** [21] For \(n \times n\) matrices \(X \succeq O\) and \(Y \succeq O\), the following inequalities hold:
\[
0 \leq tr(XY) \leq \|X\|tr(Y).
\]

We shall now examine the following nonlinear matrix equation,
\[
X = A + \sum_{i=1}^{u} \sum_{k=1}^{v} C^*_j \Upsilon_k(X)C_j
\]
(4.1)

In the above equation, \(A\) is defined as a Hermitian and positive definite matrix. Additionally, the notation \(C^*_j\) refers to the conjugate transpose of a square matrix \(C_j\) of size \(n \times n\). Furthermore, \(\Upsilon_k\) represents continuous functions that preserve order, mapping from \(\mathcal{H}(n)\) to \(\mathcal{P}(n)\). It is noteworthy that \(\Upsilon(O) = O\), where \(O\) represents a zero matrix.
Theorem 4.3. Consider the nonlinear matrix equation expressed in (4.1) and assume the following:

\((H_1)\) there exists \(A \in \mathcal{P}(n)\) with \(\sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(A)C_j \succ 0;\)

\((H_2)\) for every \(X, Y \in \mathcal{P}(n), X \preceq Y\) implies

\[
\sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(X)C_j \preceq \sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(Y)C_j;
\]

\((H_3)\) \(\sum_{j=1}^{u} C_j C_j^* \prec N \mathcal{I}_n,\) for some positive number \(N,\) and for all \(X, Y \in \mathcal{P}(n)\) with \(X \preceq Y,\) the following inequality holds

\[
\max_k (\text{tr}(\Upsilon_k(Y) - \Upsilon_k(X))) \leq \frac{1}{2Nv} \max \left\{ \text{tr}(Y - X), \text{tr}(X - A - \sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(X)C_j), \text{tr}(Y - A - \sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(Y)C_j) \right\};
\]

Then, there exists at least one solution of the nonlinear matrix equation (4.1). Moreover, the iteration

\[
X_r = A + \sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(X_{r-1})C_j,
\]

where \(X_0 \in \mathcal{P}(n)\) satisfies \(X_0 \preceq A + \sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(X_0)C_j,\) Convergence towards the solution of the matrix equation, in the context of trace norm \(\| \cdot \|_{tr}.\)

Proof. Let \(\mathfrak{T} : \mathcal{P}(n) \to \mathcal{P}(n)\) be a mapping defined by,

\[
\mathfrak{T}(X) = A + \sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(X)C_j, \quad \text{for all } X \in \mathcal{P}(n).
\]

Consider \(\mathcal{R} = \{(X, Y) \in \mathcal{P}(n) \times \mathcal{P}(n) : X \preceq Y\}.\) Consequently, the fixed point of \(\mathfrak{T}\) serves as a solution to the nonlinear matrix equation (4.1). It is pertinent to mention that \(\mathcal{R}\) is \(\mathfrak{T}\)-closed and \(\mathfrak{T}\) is well-defined as well as \(\mathcal{R}\)-continuous. Form condition \((H_1)\) we have \(\sum_{j=1}^{u} \sum_{k=1}^{v} C_j^* \Upsilon_k(X)C_j \succ 0\) for some \(X \in \mathcal{P}(n),\) thus \((X, \mathfrak{T}(X)) \in \mathcal{R}\) and consequently \(\mathcal{P}(n)(\mathfrak{T}, \mathcal{R})\) is non-empty. Define \(\partial : \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{R}^+\) by

\[
\partial(X, Y) = \|X - Y\|_{tr}, \quad \text{for all } X, Y \in \mathcal{P}(n).
\]
Then $\mathcal{P}(n)$ is $\mathcal{R}$-complete RMS. Then
\[
\|\tilde{\Sigma}(Y) - \tilde{\Sigma}(X)\|_{tr} = tr(\tilde{\Sigma}(Y) - \tilde{\Sigma}(X))
\]
\[
= tr \left( \sum_{j=1}^u \sum_{k=1}^v \mathcal{C}_j^*(\mathcal{Y}_k(Y) - \mathcal{Y}_k(X)) \mathcal{C}_j \right)
\]  
\[
= \sum_{j=1}^u \sum_{k=1}^v tr(\mathcal{C}_j \mathcal{C}_j^*(\mathcal{Y}_k(Y) - \mathcal{Y}_k(X)))
\]
\[
= tr \left( \left( \sum_{j=1}^u \mathcal{C}_j \mathcal{C}_j^* \right) \left( \sum_{k=1}^v (\mathcal{Y}_k(Y) - \mathcal{Y}_k(X)) \right) \right)
\]  
\[
\leq \left\| \sum_{j=1}^u \mathcal{C}_j \mathcal{C}_j^* \right\| \times v \times \max \| (\mathcal{Y}_k(Y) - \mathcal{Y}_k(X)) \|_{tr}
\]  
\[
\leq \frac{1}{2} \times \max \left\{ \| X - Y \|_{tr}, \| X - \tilde{\Sigma} X \|_{tr}, \| Y - \tilde{\Sigma} Y \|_{tr}, \| X - \tilde{\Sigma} X \|_{tr}^{\frac{1}{2}}, \| Y - \tilde{\Sigma} Y \|_{tr}^{\frac{1}{2}} \right\}
\]  
\[
= \frac{1}{2} \left( \Omega(\| X - Y \|_{tr}, \| X - \tilde{\Sigma} X \|_{tr}, \| Y - \tilde{\Sigma} Y \|_{tr}) \right) 
\]  
(4.3)

Now, we consider $\psi(\nu) = \nu, \phi(\nu) = \frac{\nu}{2}, \alpha = 0, \beta = \frac{1}{2}$ then equation (4.3) becomes,
\[
\psi(\partial(\tilde{\Sigma} X, \tilde{\Sigma} Y)) \leq \phi \left( \Omega(\partial(X, Y), \partial(X, \tilde{\Sigma} X), \partial(Y, \tilde{\Sigma} Y)) \right).
\]

Consequently, upon fulfilling all the hypotheses stated in Theorem 3.3, it can be deduced that there exists an element $X^* \in \mathcal{P}(n)$ for which $\tilde{\Sigma}(X^*) = X^*$ holds good. As a result, the matrix equation (4.1) is guaranteed to possess a solution within the set $\mathcal{P}(n)$. \quad \Box

Example 4.4. Consider the nonlinear matrix equation (4.1) for $u = v = 2$, and $n = 7$, with $\mathcal{Y}_1(X) = X^{\frac{1}{2}}, \mathcal{Y}_2(X) = X^{\frac{1}{2}}, \text{i.e.,}$
\[
X = A + C_1 X^{\frac{1}{2}} C_1 + C_1^* X^{\frac{1}{2}} C_1 + C_2^* X^{\frac{1}{2}} C_2 + C_2^* X^{\frac{1}{2}} C_2 \quad (4.4)
\]
\[\text{where} \]
\[
A = \begin{bmatrix} 0.177855454222667 & 0.001123654123643 & 0.14456321456439 \\ 0.001123654123643 & 0.177856213654400 & 0.133214521452362 \\ 0.144562121365390 & 0.133214526352116 & 0.266521364125960 \end{bmatrix}
\]
\[
C_1 = \begin{bmatrix} 0.21358808080307819 & 0.166601444550095 & 0.127622658649550 \\ 0.116601444550695 & 0.113891601170827 & 0.022954463850304 \\ 0.127622658649550 & 0.122954463850304 & 0.307677316314136 \end{bmatrix}
\]
\[
C_2 = \begin{bmatrix} 1.835353913428885 & 0.533419165306540 & 0.639329778947828 \\ 0.533419165306540 & 0.334906218729761 & 0.379215073620121 \\ 0.639329778947828 & 0.379215073620121 & 1.70520335236594 \end{bmatrix}
\]
By taking $N = \frac{3}{5}$, the conditions specified in Theorem 4.3 can be validated numerically by evaluating various specific values for the matrices involved. For example, they can be tested (and verified to be true) for

$$X = \begin{bmatrix}
0.601344857294582 & 0.894489764709381 & 0.981555295771015 \\
0.894489764709381 & 2.488267073906088 & 0.767182917713691 \\
0.981555295771015 & 0.967182917713691 & 0.424257724862306
\end{bmatrix}$$

$$Y = \begin{bmatrix}
1.000171251644134 & 0.123565455662234 & 0.231452114522554 \\
0.234512141422554 & 1.213180056807297 & 0.365455111122332 \\
0.551221455112244 & 0.231452334558489 & 1.113265841608538
\end{bmatrix}$$

To ascertain the convergence of $\{X_n\}$ defined in (4.2), we commence with three distinct initial values.

$$U_0 = \begin{bmatrix}
\frac{1}{20} & 0 & 0 \\
0 & \frac{1}{15} & 0 \\
0 & 0 & \frac{1}{15}
\end{bmatrix}$$

$$V_0 = \begin{bmatrix}
1.380006158840432 & 0.729620753048356 & 0.597565069778261 \\
0.729620753048356 & 0.547945757472219 & 0.551317326877231 \\
0.597565069778261 & 0.551317326877231 & 1.3012791227936167
\end{bmatrix}$$

$$W_0 = \begin{bmatrix}
2.592408887372435 & 1.027321364808460 & 0.873755458971548 \\
1.027321364808460 & 0.593069924137297 & 0.762603684965625 \\
0.873755458971548 & 0.762603684965625 & 1.252077566327681
\end{bmatrix}$$
After conducting 15 iterations, the subsequent approximation of the positive definite solution for the system presented in (4.1) is as follows,

\[
\hat{U} \approx U_{15} = \begin{bmatrix}
17.163329497461348 & 6.253639655399002 & 12.195915736525205 \\
6.253639507101289 & 2.692272944078430 & 5.374711308897728 \\
12.195914405376412 & 5.3747112649647395 & 14.480006392095785
\end{bmatrix}
\]

with error \(1.24906 \times 10^{-7}\),

\[
\hat{V} \approx V_{15} = \begin{bmatrix}
17.163329508247507 & 6.253639659448254 & 12.195915744696777 \\
6.253639511150539 & 2.692272945689973 & 5.374711312279821 \\
12.195914413547982 & 5.374711268346832 & 14.480006400501384
\end{bmatrix}
\]

with error \(5.28502 \times 10^{-8}\),

\[
\hat{W} \approx W_{15} = \begin{bmatrix}
17.16332950926931 & 6.253639659834532 & 12.195915745492652 \\
6.253639511536818 & 2.692272945842136 & 5.374711312591148 \\
12.19591441434386 & 5.374711268658160 & 14.480006401183488
\end{bmatrix}
\]

with error \(4.64279 \times 10^{-8}\).

In Figure 3, we present a graphical depiction illustrating the convergence phenomenon.
Conclusion

In our current study, we introduce a broader idea called unified interpolative Reich-Rus-Ćirić type contraction. This concept encompasses many existing findings, including those presented by [7–11, 14, 17]. We demonstrate several fixed point results for such contractions within relational metric spaces.

It is important to note that in relational metric spaces, we often deal with weaker properties like $\mathcal{R}$-continuity (not necessarily implying continuity), $\mathcal{R}$-completeness (not necessarily implying completeness), and so on. In this context, we have more flexibility since the contraction condition is not required for every element but only for related ones. Importantly, these contraction conditions return to their usual forms when considering the universal relation.

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Conflict of interests

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