

Global existence of mild solutions to non-autonomous stochastic evolution equations

Huani Zhao*

School of Mathematics and Information Engineering,
Longdong University, Qingyang 745000, China

Abstract

We investigate the global existence of mild solutions for Cauchy problem to a class of non-autonomous stochastic evolution equations with additive noise, where the operators in linear part depend on time t and generate compact evolution family. Finally, we provide an illustrative example of practical significance to show the practical usefulness of the theoretical results that are obtained in this paper. The results obtained in this paper are supplement to the existing literature and essentially extend some existing results in this area.

Keywords: Non-autonomous stochastic evolution equations; Parabolicity condition; Wiener process; Evolution family

Mathematics Subject Classification(2010): 37B55; 37L55; 47J35; 60H15

1 Introduction

The aim of this paper is to investigate the global existence of mild solutions for Cauchy problem to the following non-autonomous stochastic evolution equations with additive noise

$$\begin{cases} du(t) - A(t)u(t)dt = f(t, u(t))dt + F(t, u(t))d\mathbb{W}(t), & t \geq 0, \\ u(0) = u_0 \end{cases} \quad (1.1)$$

*E-mail address: huanizhao@126.com.

in the real separable Hilbert space $\mathbb{H}$, where $A(t)$ is a family of linear operators depending on time and having the domains $D(A(t))$ for every $t \in [0, +\infty)$, and $\{A(t) : t \geq 0\}$ generates a compact evolution family $\{\mathcal{U}(t, s) : t \geq s \geq 0\}$, the state $u(\cdot)$ takes values in the real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot, \cdot)_{\mathbb{K}}$ and norm $\|\cdot\|_{\mathbb{K}}$. Assume that $\{\mathbb{W}(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$ -valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We are also employing the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(\mathbb{K}, \mathbb{H})$, which denotes the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. We denote by $\mathcal{L}(\mathbb{H}) = \mathcal{L}(\mathbb{H}, \mathbb{H})$. $f : [0, +\infty) \times \mathbb{H} \rightarrow \mathbb{H}$ and $F : [0, +\infty) \times \mathbb{H} \rightarrow \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous nonlinear functions, $u_0 \in L^2(\Omega, \mathbb{H})$.

The theory of stochastic differential and integro-differential equations have attracted great interest in recent years because of their practical applications in many areas such as physics, chemistry, economics, social sciences, finance and other areas of science and engineering. For more details about stochastic differential equations we refer to the books by Da Prato and Zabczyk [6], Grecksch and Tudor [7], Liu [9], Mao [10] and Sobczyk [14]. One of the branches of stochastic differential equations is the theory of stochastic evolution equations. Since semilinear stochastic evolution equations are abstract formulations for many problems arising in the domain of engineering technology, biology and economic system etc., stochastic evolution equations have attracted increasing attention in recent years and the existence, uniqueness and asymptotic behavior of mild solutions to stochastic evolution equations have been considered by many authors. In 2002, Taniguchi, Liu and Truman [15] discussed the existence, uniqueness, p -th moment and almost sure Lyapunov exponents of mild solutions to a class of autonomous stochastic partial functional differential equations with finite delays by using semigroup methods. By introducing a suitable metric between the transition probability functions of mild solutions, Bao, Hou and Yuan [2] obtained the exponential stability of mild solutions to a class of autonomous stochastic partial differential equations in 2010. In 2011, Ren, Zhou and Chen [12] established the existence, uniqueness and stability of mild solutions for a class of time-dependent autonomous stochastic evolution equations with poisson jumps and infinite delay under non-Lipschitz condition. In 2016, Sakthivel, Ren, Debbouche and Mahmudov [13] investigated the approximate controllability of fractional autonomous stochastic differential inclusions with nonlocal conditions with the help of the fixed point theorem for multi-valued operators and fractional calculus. Later, by establishing a sufficient condition for judging the relative compactness of a class of abstract continuous family of functions on infinite intervals, Chen, Abdelmonem and

Li [3] obtained the global existence, uniqueness and asymptotic stability of mild solutions for a class of nonlinear autonomous evolution equations with nonlocal initial conditions on infinite interval by using stochastic analysis theory, analytic semigroup theory, relevant fixed point theory and the well known Gronwall-Bellman type inequality. Recently, Zhang et al. [4] and [16] considered the existence of mild solutions for Cauchy problem to stochastic non-autonomous evolution equations governed by noncompact evolution family by using fixed point theory.

However, we observed that most of the existing articles are only devoted to investigating the well-posedness for autonomous stochastic evolution equations, up until now the existence of saturated mild solutions and global mild solutions for non-autonomous stochastic evolution equations in Hilbert spaces have not been considered in the literature. In addition, it is very significant and interesting to investigate non-autonomous stochastic evolution equations, i.e., the differential operators in the linear parts of the considered problems are dependent about time variable t . Therefore, we will close this gap and continue to investigate the existence of saturated mild solutions and global mild solutions for Cauchy problem (1.1) to non-autonomous stochastic evolution equations with additive noise and compact evolution family.

2 Preliminaries

Throughout the paper, we assume that $\{A(t) : t \geq 0\}$ is a family of closed and densely defined operator on Hilbert space $\mathbb{H}$, which satisfies the following well-known "Acquistapace-Terreni" conditions, which can be find in Acquistapace and Terreni [1].

- (I) There exist constants $\lambda_0 \geq 0$, $\alpha \in (\frac{\pi}{2}, \pi)$ and $M_1 \geq 0$ such that $\Sigma_\alpha \cup \{0\} \subset \rho(A(t) - \lambda_0)$ and for all $\lambda \in \Sigma_\alpha \cup \{0\}$ and $t \geq 0$,

$$\|\mathcal{R}(\lambda, A(t) - \lambda_0)\|_{\mathcal{L}(\mathbb{H})} \leq \frac{M_1}{1 + |\lambda|};$$

- (II) There exist constants $M_2 > 0$ and $\theta_1, \theta_2 \in (0, 1]$ with $\theta_1 + \theta_2 > 1$ such that for all $\lambda \in \Sigma_\alpha$ and $t \geq s \geq 0$,

$$\|(A(t) - \lambda_0)\mathcal{R}(\lambda, A(t) - \lambda_0)[\mathcal{R}(\lambda_0, A(t)) - \mathcal{R}(\lambda_0, A(s))]\| \leq \frac{M_2|t - s|^{\theta_1}}{|\lambda|^{\theta_2}},$$

where

$$\Sigma_\alpha = \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \leq \alpha\}.$$

Conditions (I) and (II), which are initiated by Acquistapace and Terreni [1] for $\lambda_0 = 0$, are well understood and widely used in the literature. Under the above conditions (I) and (II), the family $\{A(t) : t \geq 0\}$ generates a unique linear evolution system, or called linear evolution family, $\{\mathcal{U}(t, s) : t \geq s \geq 0\}$.

Lemma 2.1. ([1]) *The family of the linear operator $\{\mathcal{U}(t, s) : t \geq s \geq 0\}$ satisfies the following properties:*

- (1) $\mathcal{U}(t, r)\mathcal{U}(r, s) = \mathcal{U}(t, s)$, $\mathcal{U}(t, t) = I$ for $t \geq r \geq s \geq 0$;
- (2) The map $(t, s) \mapsto \mathcal{U}(t, s)x$ is continuous for all $x \in \mathbb{H}$ and $t \geq s \geq 0$;
- (3) $\mathcal{U}(\cdot, s) \in C^1((s, \infty), \mathcal{B}(\mathbb{H}))$, $\frac{\partial \mathcal{U}(t, s)}{\partial t} = A(t)\mathcal{U}(t, s)$ for $t > s$, and $\|A^k(t)\mathcal{U}(t, s)\| \leq M(t-s)^{-k}$ for $0 < t-s \leq 1$ and $k = 0, 1$;
- (4) $\frac{\partial \mathcal{U}(t, s)x}{\partial s} = -\mathcal{U}(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

From the property (3) we know that

$$\|\mathcal{U}(t, s)\|_{\mathcal{B}(\mathbb{H})} \leq M \quad \text{for } t \geq s \geq 0, \quad M > 0 \text{ is a constant.} \quad (2.1)$$

Let $\mathbb{H}$ and $\mathbb{K}$ be two real separable Hilbert spaces, we denote by $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\mathbb{K}}$ their inner products, and by $\|\cdot\|$ and $\|\cdot\|_{\mathbb{K}}$ their vector norms, respectively. We use θ to present the zero element in $\mathbb{H}$. We denote by $\mathcal{B}(\mathbb{H})$ the Banach space of all linear and bounded operators on $\mathbb{H}$ endowed with the topology defined by operator norm. In this paper, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete filtered probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets of $\mathcal{F}$. Let $\{e_k, k \in \mathbb{N}\}$ be a complete orthonormal basis of $\mathbb{K}$. Suppose that $\{\mathbb{W}(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$ -valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a finite trace nuclear covariance operator $Q \geq 0$, denote $\text{Tr}(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty$, which satisfies that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$. Let $\{\mathbb{W}_k(t), k \in \mathbb{N}\}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that

$$\mathbb{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \mathbb{W}_k(t) e_k.$$

We further assume that $\mathcal{F}_t = \sigma\{\mathbb{W}(s), 0 \leq s \leq t\}$ is the σ -algebra generated by $\mathbb{W}$ and $\mathcal{F}_b = \mathcal{F}$. For $\varphi, \psi \in \mathcal{B}(\mathbb{K}, \mathbb{H})$, we define $(\varphi, \psi) = \text{Tr}(\varphi Q \psi^*)$, where ψ^* is the adjoint of the operator ψ . Clearly, for any bounded operator $\psi \in \mathcal{B}(\mathbb{K}, \mathbb{H})$,

$$\|\psi\|_Q^2 = \text{Tr}(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k} \psi e_k\|.$$

If $\|\psi\|_Q^2 < \infty$, then ψ is called a Q -Hilbert-Schmidt operator. The collection of all strongly-measurable, square-integrable $\mathbb{H}$ -valued random variables, denoted $L^2(\Omega, \mathbb{H})$, which is a Banach space equipped with the norm

$$\|u(\cdot)\|_{L^2} = (\mathbb{E}\|u(\cdot, \mathbb{W})\|^2)^{\frac{1}{2}},$$

where the expectation $\mathbb{E}$ is defined by

$$\mathbb{E}u = \int_{\Omega} u(\mathbb{W}) d\mathbb{P}.$$

An important subspace of $L^2(\Omega, \mathbb{H})$ is given by

$$L_0^2(\Omega, \mathbb{H}) = \{u \in L^2(\Omega, \mathbb{H}) : u \text{ is } \mathcal{F}_0\text{-measurable}\}.$$

In what follows, we denote by I the closed and bounded subinterval of $[0, +\infty)$ and by $C(I, L^2(\Omega, \mathbb{H}))$ the space of all continuous $\mathcal{F}_t$ -adapted measurable processes from I to $L^2(\Omega, \mathbb{H})$ satisfying $\sup_{t \in I} \mathbb{E}\|u(t)\|^2 < \infty$. Then it is easy to see that $C(I, L^2(\Omega, \mathbb{H}))$ is a Banach space endowed with the supnorm

$$\|u\|_C = \left(\sup_{t \in I} \mathbb{E}\|u(t)\|^2 \right)^{\frac{1}{2}}.$$

Lemma 2.2. ([5]) *If $F : I \times \mathbb{H} \rightarrow \mathcal{B}(\mathbb{K}, \mathbb{H})$ is continuous and $u \in C(I, L^2(\Omega, \mathbb{H}))$, then*

$$\mathbb{E} \left\| \int_I F(t, u(t)) d\mathbb{W}(t) \right\|^2 \leq \text{Tr}(Q) \int_I \mathbb{E} \|F(t, u(t))\|^2 dt.$$

Definition 2.1. *An $\mathcal{F}_t$ -adapted stochastic process $u : [0, +\infty) \rightarrow \mathbb{H}$ is called a mild solution of Cauchy problem (1.1) if $u(t) \in \mathbb{H}$ has càdlàg paths on $t \in [0, +\infty)$ almost surely and for each $t \in [0, +\infty)$, $u(t)$ $\mathbb{P}$ -almost surely satisfies the integral equation*

$$u(t) = \mathcal{U}(t, 0)u_0 + \int_0^t \mathcal{U}(t, s)f(s, u(s))ds + \int_0^t \mathcal{U}(t, s)F(s, u(s))d\mathbb{W}(s).$$

3 Main results

Theorem 3.1. *Assume that the evolution family $\{\mathcal{U}(t, s) : t \geq s \geq 0\}$ generated by $\{A(t) : t \geq 0\}$ is compact and the following assumption*

(H1) *The function $f : [0, +\infty) \times \mathbb{H} \rightarrow \mathbb{H}$ and $F : [0, +\infty) \times \mathbb{H} \rightarrow \mathcal{B}(\mathbb{K}, \mathbb{H})$ are continuous and the sets $\{\mathbb{E}\|f(t, u(t))\|^2 : t \in I\}$ and $\{\mathbb{E}\|F(t, u(t))\|^2 : t \in I\}$ are bounded for any bounded $\{\mathbb{E}\|u(t)\|^2\}$,*

is satisfied. Then Cauchy problem (1.1) has at least one saturated mild solution u on a maximal interval of existence $[0, T_m)$. If $T_m < +\infty$, then $\lim_{t \rightarrow T_m^-} \mathbb{E}\|u(t)\|^2 = +\infty$.

Proof. We firstly prove the local existence of mild solutions for the Cauchy problem to the following non-autonomous stochastic evolution equations of parabolic type with arbitrary initial time $t_0 \in [0, \infty)$ and arbitrary initial value $x_0 \in L^2(\Omega, \mathbb{H})$

$$\begin{cases} du(t) - A(t)u(t)dt = f(t, u(t))dt + F(t, u(t))d\mathbb{W}(t), & t \geq t_0, \\ u(t_0) = x_0 \end{cases} \quad (3.1)$$

on interval $[t_0, t_0 + h]$, where $h = h(t_0, \mathbb{E}\|x_0\|^2) > 0$ is a constant will be specified later. For $t \in [t_0, t_0 + h]$, consider the operator $\mathbb{Q} : C([t_0, t_0 + h], L^2(\Omega, \mathbb{H})) \rightarrow C([t_0, t_0 + h], L^2(\Omega, \mathbb{H}))$ defined by

$$(\mathbb{Q}u)(t) = \mathcal{U}(t, t_0)x_0 + \int_{t_0}^t \mathcal{U}(t, s)f(s, u(s))ds + \int_{t_0}^t \mathcal{U}(t, s)F(s, u(s))d\mathbb{W}(s). \quad (3.2)$$

From Definition 2.1, it is easy to see that the mild solution of Cauchy problem (3.1) on $[t_0, t_0 + h]$ is equivalent to the fixed point of the operator $\mathbb{F}$ defined by (3.2). Furthermore, from the continuity of nonlinear functions f and F one can easily see that the operator $\mathbb{Q} : C([t_0, t_0 + h], L^2(\Omega, \mathbb{H})) \rightarrow C([t_0, t_0 + h], L^2(\Omega, \mathbb{H}))$ is continuous. In what follows, we will prove that the operator $\mathbb{Q}$ has at least one fixed point by applying Schauder's fixed point theorem. Denote

$$\begin{aligned} R(t_0) &= 3M^2(2\mathbb{E}\|x_0\|^2 + 1), \\ M_f(t_0) &= \sup_{t_0 \leq t \leq t_0+1} \{\mathbb{E}\|f(t, u(t))\|^2 : \mathbb{E}\|u(t)\|^2 \leq R(t_0)\}, \\ M_F(t_0) &= \sup_{t_0 \leq t \leq t_0+1} \{\mathbb{E}\|F(t, u(t))\|^2 : \mathbb{E}\|u(t)\|^2 \leq R(t_0)\}. \end{aligned}$$

We firstly prove that the operator $\mathbb{Q}$ defined by (3.2) maps the bounded closed convex set

$$B_{R(t_0)} = \{u \in C([t_0, t_0 + h], L^2(\Omega, \mathbb{H})) : \mathbb{E}\|u(t)\|^2 \leq R(t_0), t \in [t_0, t_0 + h]\}$$

into itself. Let

$$h = h(t_0, \mathbb{E}\|x_0\|^2) := \min \left\{ 1, \frac{\mathbb{E}\|x_0\|^2 + 1}{M_f(t_0) + \text{Tr}(Q)M_F(t_0)} \right\}.$$

For any $u \in B_{R(t_0)}$ and $t \in [t_0, t_0 + h]$, by (2.1), (3.2), the assumption (H1), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} \mathbb{E}\|(\mathbb{Q}u)(t)\|^2 &\leq 3\mathbb{E}\|\mathcal{U}(t, t_0)x_0\|^2 + 3\mathbb{E}\left\|\int_{t_0}^t \mathcal{U}(t, s)f(s, u(s))ds\right\|^2 \\ &\quad + 3\mathbb{E}\left\|\int_{t_0}^t \mathcal{U}(t, s)F(s, u(s))d\mathbb{W}(s)\right\|^2 \\ &\leq 3M^2\mathbb{E}\|x_0\|^2 + 3M^2 \int_{t_0}^t \mathbb{E}\|f(s, u(s))\|^2 ds \\ &\quad + 3\text{Tr}(Q)M^2 \int_{t_0}^t \mathbb{E}\|F(s, u(s))\|^2 ds \\ &\leq 3M^2\mathbb{E}\|x_0\|^2 + 3M^2[M_f(t_0) + \text{Tr}(Q)M_F(t_0)]h \\ &\leq R(t_0). \end{aligned}$$

Therefore, $\mathbb{Q}u \in B_{R(t_0)}$. Hence, we proved that $\mathbb{Q} : B_{R(t_0)} \rightarrow B_{R(t_0)}$ is a continuous operator.

In what follows, we demonstrate that the operator $\mathbb{Q} : B_{R(t_0)} \rightarrow B_{R(t_0)}$ is compact. To prove this, we first show that $\{(\mathbb{Q}u)(t) : u \in B_{R(t_0)}\}$ is relatively compact in $\mathbb{H}$ for every $t \in [t_0, t_0 + h]$. It is easy to see that the set $\{(\mathbb{Q}u)(t_0) : u \in B_{R(t_0)}\} = \{x_0\}$ is relatively compact in $\mathbb{H}$. Let $t_0 < t \leq t_0 + h$ be given, $0 < \epsilon < t$ and $u \in B_{R(t_0)}$, we define the operator $\mathbb{Q}^\epsilon$ by

$$\begin{aligned} (\mathbb{Q}^\epsilon u)(t) &= \mathcal{U}(t, t_0)x_0 + \mathcal{U}(t, t - \frac{\epsilon}{2})\mathcal{U}(t - \frac{\epsilon}{2}, t - \epsilon) \int_{t_0}^{t-\epsilon} \mathcal{U}(t - \epsilon, s)f(s, u(s))ds \\ &\quad + \mathcal{U}(t, t - \frac{\epsilon}{2})\mathcal{U}(t - \frac{\epsilon}{2}, t - \epsilon) \int_{t_0}^{t-\epsilon} \mathcal{U}(t - \epsilon, s)F(s, u(s))d\mathbb{W}(s). \end{aligned} \quad (3.3)$$

Since $\mathcal{U}(t, t - \frac{\epsilon}{2}) \in \mathcal{B}(\mathbb{H})$ and $\mathcal{U}(t - \frac{\epsilon}{2}, t - \epsilon)$ is compact in $\mathbb{H}$, the set $\{(\mathbb{Q}_0^\epsilon u)(t) : u \in B_{R(t_0)}\}$ is relatively compact in $\mathbb{H}$ for every $\epsilon \in (0, t)$ and $t_0 < t \leq t_0 + h$. Moreover,

for every $u \in B_{R(t_0)}$ and $t_0 < t \leq t_0 + h$, by (2.1), (3.2), (3.3) the assumption (H1), Lemma 2.1 and Lemma 2.2, we get that

$$\begin{aligned}
& \mathbb{E} \|(\mathbb{Q}^\epsilon u)(t) - (\mathbb{Q}u)(t)\|^2 \\
& \leq 2\mathbb{E} \left\| \mathcal{U}(t, t - \frac{\epsilon}{2}) \mathcal{U}(t - \frac{\epsilon}{2}, t - \epsilon) \int_{t_0}^{t-\epsilon} \mathcal{U}(t - \epsilon, s) f(s, u(s)) ds \right. \\
& \quad \left. - \int_{t_0}^t \mathcal{U}(t, s) f(s, u(s)) ds \right\|^2 \\
& \quad + 2\mathbb{E} \left\| \mathcal{U}(t, t - \frac{\epsilon}{2}) \mathcal{U}(t - \frac{\epsilon}{2}, t - \epsilon) \int_{t_0}^{t-\epsilon} \mathcal{U}(t - \epsilon, s) F(s, u(s)) d\mathbb{W}(s) \right. \\
& \quad \left. - \int_{t_0}^t \mathcal{U}(t, s) F(s, u(s)) d\mathbb{W}(s) \right\|^2 \\
& \leq 2M^2 [M_f(t_0) + \text{Tr}(Q)M_F(t_0)]\epsilon \\
& \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.
\end{aligned}$$

Hence, we have proved that there are relatively compact set $\{(\mathbb{Q}^\epsilon u)(t) : u \in B_{R(t_0)}\}$ arbitrarily close to the set $\{(\mathbb{Q}u)(t) : u \in B_{R(t_0)}\}$ for $t_0 < t \leq t_0 + h$, this means that the set $\{(\mathbb{Q}u)(t) : u \in B_{R(t_0)}\}$ is relatively compact in $\mathbb{H}$ for $t_0 < t \leq t_0 + h$. Therefore, $\{(\mathbb{Q}u)(t) : u \in B_{R(t_0)}\}$ is relatively compact in $\mathbb{H}$ for $t_0 \leq t \leq t_0 + h$.

At least, we show that $\{\mathbb{Q}u : u \in B_{R(t_0)}\}$ is a family of equicontinuous functions in $C([t_0, t_0 + h], L^2(\Omega, \mathbb{H}))$. For any $u \in B_{R(t_0)}$ and $t_0 \leq t' < t'' \leq t_0 + h$, by (2.1), (3.2), the assumption (H1), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
& \mathbb{E} \|(\mathbb{Q}u)(t'') - (\mathbb{Q}u)(t')\|^2 \leq 5\mathbb{E} \| [U(t'', t_0) - U(t', t_0)]x_0 \|^2 \\
& + 5\mathbb{E} \left\| \int_{t_0}^{t'} [\mathcal{U}(t'', s) - \mathcal{U}(t', s)] f(s, u(s)) ds \right\|^2 + 5\mathbb{E} \left\| \int_{t'}^{t''} \mathcal{U}(t'', s) f(s, u(s)) ds \right\|^2 \\
& + 5\mathbb{E} \left\| \int_{t_0}^{t'} [\mathcal{U}(t'', s) - \mathcal{U}(t', s)] F(s, u(s)) d\mathbb{W}(s) \right\|^2 \\
& + 5\mathbb{E} \left\| \int_{t'}^{t''} \mathcal{U}(t'', s) F(s, u(s)) d\mathbb{W}(s) \right\|^2 \\
& \leq 5\mathbb{E} \| [\mathcal{U}(t'', t_0) - \mathcal{U}(t', t_0)]x_0 \|^2 \\
& + 5 \int_{t_0}^{t'} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|^2 \mathbb{E} \|f(s, u(s))\|^2 ds + 5M^2 \int_{t'}^{t''} \mathbb{E} \|f(s, u(s))\|^2 ds
\end{aligned}$$

$$\begin{aligned}
& +5\text{Tr}(Q) \int_{t_0}^{t'} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|^2 \mathbb{E}\|F(s, u(s))\|^2 ds \\
& +5\text{Tr}(Q)M^2 \int_{t'}^{t''} \mathbb{E}\|F(s, u(s))\|^2 ds \\
& := J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

In order to prove that $\mathbb{E}\|(\mathbb{Q}u)(t'') - (\mathbb{Q}u)(t')\|^2 \rightarrow 0$ as $t'' - t' \rightarrow 0$, we only need to check $J_i \rightarrow 0$ independently of $u \in B_{R(t_0)}$ when $t'' - t' \rightarrow 0$ for $i = 1, 2, \dots, 5$.

For J_1 , by the compactness of the evolution family $\mathcal{U}(t, s)$ one can easily get that $J_1 \rightarrow 0$ as $t'' - t' \rightarrow 0$.

For $t' = 0$, $0 < t'' \leq t_0 + h$, it is easy to see that $J_2 = J_4 = 0$. For $0 < t' < t_0 + h$ and arbitrary $0 < \delta < t'$, by (2.1), the compactness of the evolution family $\mathcal{U}(t, s)$, the assumption (H1) and the arbitrariness of δ , we get that

$$\begin{aligned}
J_2 & \leq 5 \int_0^{t'-\delta} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|^2 M_f(t_0) ds \\
& + 5 \int_{t'-\delta}^{t'} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|^2 M_f(t_0) ds \\
& \leq \sup_{s \in [0, t'-\delta]} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|_{\mathcal{B}(\mathbb{H})}^2 \cdot 5M_f(t_0)(t' - \delta) + 10M^2 M_f(t_0)\delta \\
& \rightarrow 0 \quad \text{as} \quad t'' - t' \rightarrow 0 \quad \text{and} \quad \delta \rightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
J_4 & \leq 5\text{Tr}(Q) \int_0^{t'-\delta} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|^2 M_F(t_0) ds \\
& + 5\text{Tr}(Q) \int_{t'-\delta}^{t'} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|^2 M_F(t_0) ds \\
& \leq \sup_{s \in [0, t'-\delta]} \|\mathcal{U}(t'', s) - \mathcal{U}(t', s)\|_{\mathcal{B}(\mathbb{H})}^2 \cdot 5\text{Tr}(Q)M_F(t_0)(t' - \delta) \\
& + 10\text{Tr}(Q)M^2 M_F(t_0)\delta \\
& \rightarrow 0 \quad \text{as} \quad t'' - t' \rightarrow 0 \quad \text{and} \quad \delta \rightarrow 0.
\end{aligned}$$

For J_3 and J_5 , by the assumption (H1), we get that

$$J_3 = 5M^2 M_f(t_0)(t'' - t') \rightarrow 0 \quad \text{as} \quad t'' - t' \rightarrow 0$$

and

$$J_5 = 5\text{Tr}(Q)M^2M_F(t_0)(t'' - t') \rightarrow 0 \quad \text{as} \quad t'' - t' \rightarrow 0.$$

As a result, we have proved that $\mathbb{E}\|(\mathbb{Q}u)(t'') - (\mathbb{Q}u)(t')\|^2 \rightarrow 0$ independently of $u \in B_{R(t_0)}$ as $t'' - t' \rightarrow 0$, which means that the operator $\mathbb{Q} : B_{R(t_0)} \rightarrow B_{R(t_0)}$ is equicontinuous. Hence, combined with the Arzela-Ascoli theorem one gets that $\mathbb{Q} : B_{R(t_0)} \rightarrow B_{R(t_0)}$ is a compact operator. Therefore, by Schauder's fixed point theorem we know that $\mathbb{Q}$ has at least one fixed point $u \in B_{R(t_0)}$, which is in turn a mild solution of Cauchy problem (3.1) on $[t_0, t_0 + h]$.

Therefore, by the local existence of mild solutions we have just proved, we know that there exists a constant $0 < h_0 \leq 1$ such that Cauchy problem (1.1) exists a mild solution $u \in C([0, h_0], L^2(\Omega, \mathbb{H}))$ on interval $[0, h_0]$. Moreover, u can be extended to a large interval $[0, h_0 + h_1]$ with $0 < h_1 \leq 1$ by defining $u(t) = v(t)$ on $[h_0, h_0 + h_1]$, where $v(t)$ is the mild solution of the following Cauchy problem to non-autonomous stochastic evolution equations of parabolic type

$$\begin{cases} dv(t) - A(t)v(t)dt = f(t, v(t))dt + F(t, v(t))d\mathbb{W}(t), & h_0 \leq t \leq h_0 + h_1, \\ v(h_0) = u(h_0), \end{cases}$$

where h_1 depends only on $\mathbb{E}\|u(h_0)\|^2$, $M_f(h_0)$, $M_F(h_0)$ and $R(h_0)$. Hence, repeating the above procedure and using the methods of steps, we can prove that u can be extended to a maximal existence interval $[0, T_m)$, namely $u \in C([0, T_m), L^2(\Omega, \mathbb{H}))$ is a saturated mild solution of Cauchy problem (1.1).

At last, using a completely similar method with which used in the proof of [16, Theorem 4.1], one can prove that the saturated mild solution u of Cauchy problem (1.1) blows up in finite time, i.e., if $T_m < +\infty$, then $\lim_{t \rightarrow T_m^-} \mathbb{E}\|u(t)\|^2 = +\infty$. This completes the proof of Theorem 3.1. $\square$

Next, we investigate the global existence of mild solutions to Cauchy problem (1.1).

Theorem 3.2. *Assume that the evolution family $\{\mathcal{U}(t, s) : t \geq s \geq 0\}$ generated by $\{A(t) : t \geq 0\}$ is compact and the following assumption*

(H2) *The function $f : [0, +\infty) \times \mathbb{H} \rightarrow \mathbb{H}$ and $F : [0, +\infty) \times \mathbb{H} \rightarrow \mathcal{B}(\mathbb{K}, \mathbb{H})$ are continuous and there exist functions $\alpha_f, \beta_f, \alpha_F, \beta_F \in C([0, +\infty), [0, +\infty))$ such that*

$$\mathbb{E}\|f(t, u(t))\|^2 \leq \alpha_f(t)\mathbb{E}\|u(t)\|^2 + \beta_f(t) \quad \text{for } t \geq 0 \text{ and } u \in \mathbb{H},$$

$$\mathbb{E}\|F(t, u(t))\|^2 \leq \alpha_F(t)\mathbb{E}\|u(t)\|^2 + \beta_F(t) \quad \text{for } t \geq 0 \text{ and } u \in \mathbb{H},$$

is satisfied. Then Cauchy problem (1.1) has at least one global mild solution $u \in C([0, +\infty), L^2(\Omega, \mathbb{H}))$.

Proof. From the continuity of the functions α_f , β_f , α_F and β_F one can easily see that the assumption (H2) implies (H1). Therefore, by Theorem 3.1 we know that Cauchy problem (1.1) has at least one saturated mild solution u on a maximal interval of existence $[0, T_m)$. If $T_m = +\infty$, then the result has been proved. If $T_m < +\infty$, by the proof process of Theorem 3.1 we know that Cauchy problem (1.1) has a global mild solution if $\mathbb{E}\|u(t)\|^2 < +\infty$ for every t in the interval of existence of mild solution u . Hence, in order to prove Theorem 3.2 we only need to prove $\mathbb{E}\|u(t)\|^2 < +\infty$ for every $t \in [0, T_m)$ with $T_m < +\infty$. Denote

$$\begin{aligned} \alpha &= \max_{t \in [0, T_m]} \alpha_f(t), & \beta &= \max_{t \in [0, T_m]} \beta_f(t), \\ \bar{\alpha} &= \max_{t \in [0, T_m]} \alpha_F(t), & \bar{\beta} &= \max_{t \in [0, T_m]} \beta_F(t). \end{aligned}$$

By (2.1), Lemma 2.1, Lemma 2.2, Definition 2.1 and the assumption (H2) one gets that for any $t \in [0, T_m)$,

$$\begin{aligned} \mathbb{E}\|u(t)\|^2 &\leq 3\mathbb{E}\|\mathcal{U}(t, 0)u_0\|^2 + 3\mathbb{E}\left\|\int_0^t \mathcal{U}(t, s)f(s, u(s))ds\right\|^2 \\ &\quad + 3\mathbb{E}\left\|\int_0^t \mathcal{U}(t, s)F(s, u(s))d\mathbb{W}(s)\right\|^2 \\ &\leq 3M^2\mathbb{E}\|u_0\|^2 + 3M^2 \int_0^t \mathbb{E}\|f(s, u(s))\|^2 ds \\ &\quad + 3\text{Tr}(Q)M^2 \int_0^t \mathbb{E}\|F(s, u(s))\|^2 ds \\ &\leq 3M^2[\mathbb{E}\|u_0\|^2 + (\beta + \text{Tr}(Q)\bar{\beta})T_m] \\ &\quad + 3M^2(\alpha + \text{Tr}(Q)\bar{\alpha}) \int_0^t \mathbb{E}\|u(s)\|^2 ds. \end{aligned}$$

Therefore, the above inequality combined with the well-known Gronwall-Bellman type inequality we know that

$$\mathbb{E}\|u(t)\|^2 \leq \left[3M^2[\mathbb{E}\|u_0\|^2 + (\beta + \text{Tr}(Q)\bar{\beta})T_m]\right] e^{3M^2(\alpha + \text{Tr}(Q)\bar{\alpha})T_m} < +\infty$$

for $t \in [0, T_m)$. This completes the proof of Theorem 3.2. $\square$

4 An example

In this section, we provide a concrete example to illustrate the practical usefulness of our theoretical results established in the preceding sections.

Example 4.1. *Consider the following one dimensional non-autonomous stochastic partial differential equation of parabolic type*

$$\begin{cases} \frac{\partial}{\partial t}u(x, t) - \frac{\partial^2}{\partial^2 x}u(x, t) - 7 \sin t \cdot u(x, t) = f(x, t, u(x, t)) \\ \quad + F(x, t, u(x, t)) \frac{d\mathbb{W}(t)}{dt}, \quad x \in (0, 1), \\ u(0, t) = u(1, t) = 0, \quad t \in [0, +\infty), \\ u(x, 0) = u_0(x), \quad x \in (0, 1), \end{cases} \quad (4.1)$$

where $\mathbb{W}(t)$ denotes a one-dimensional standard cylindrical Wiener process defined on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $u_0 \in L^2(\Omega, L^2(0, 1; \mathbb{R}))$,

$$f(x, t, u(x, t)) = \cos(\pi t)u(x, t), \quad F(x, t, u(x, t)) = \frac{|u(x, t)|}{4 + |u(x, t)|}.$$

Let $\mathbb{H} = L^2(0, 1; \mathbb{R})$ be a real Hilbert space with the L^2 -norm $\|\cdot\|_2$ defined by

$$\|u\|_2 = \left(\int_0^1 |u(x)|^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in L^2(0, 1; \mathbb{R})$$

and inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx, \quad \forall u, v \in L^2(0, 1; \mathbb{R}).$$

Consider the operator B on $\mathbb{H}$ defined by

$$Bu = \frac{\partial^2}{\partial^2 x}u \quad (4.2)$$

with domain

$$D(B) = \{u \in L^2(0, 1; \mathbb{R}), u'' \in L^2(0, 1; \mathbb{R}) \text{ and } u(0) = u(1) = 0\}.$$

It is well known from Pazy [11] that B generates a compact and analytic C_0 -semigroup in $L^2(0, 1; \mathbb{R})$, B has a discrete spectrum, and its eigenvalues are $-n^2\pi^2$, $n \in \mathbb{N}^+$ with the corresponding normalized eigenvectors $v_n(x) = \sqrt{2} \sin(n\pi x)$.

Define the operator $A(t)$ on $\mathbb{H}$ by

$$A(t)u = Bu + 7 \sin t \cdot u \quad (4.3)$$

with domain

$$D(A(t)) = D(B), \quad t \in [0, +\infty).$$

It follows from [11, Lemma 6.1 in Chapter 7] that there are constants $\alpha \in (\frac{\pi}{2}, \pi)$ and $M_1 \geq 0$ such that $A(t)$ satisfy the Acquistapace and Terreni condition (I). Furthermore, by again [11, Lemma 6.1 in Chapter 7] together with continuously differentiable of coefficient $7 \sin t$ one know that there exist constants $M_2 > 0$ and $\theta_1, \theta_2 \in (0, 1]$ with $\theta_1 + \theta_2 > 1$ such that for all $\lambda \in \Sigma_\alpha$ and $t \geq s \geq 0$, the Acquistapace and Terreni condition (II) is satisfied. Therefore, the family $\{A(t) : t \geq 0\}$ generates an strongly continuous evolution family $\{\mathcal{U}(t, s) : t \geq s \geq 0\}$ defined by

$$\mathcal{U}(t, s)u = \sum_{n=1}^{\infty} e^{-n^2\pi^2(t-s) + \int_s^t 7 \sin \tau d\tau} \langle u, v_n \rangle v_n, \quad t \geq s \geq 0, \quad u \in \mathbb{H}.$$

A direct calculation gives

$$\|\mathcal{U}(t, s)\|_{\mathcal{B}(\mathbb{H})} \leq e^{-(\pi^2-7)(t-s)}, \quad t \geq s \geq 0,$$

which means that

$$M := \sup_{t \geq s \geq 0} \|\mathcal{U}(t, s)\|_{\mathcal{B}(\mathbb{H})} = 1. \quad (4.4)$$

Note also from [8] that, for each $t, s \in [0, +\infty)$ with $t > s$, the evolution family $\mathcal{U}(t, s)$ is a nuclear operator, which implies the compactness of $\mathcal{U}(t, s)$ for $t > s$.

Let $u(t) = u(\cdot, t)$, $f(t, u(t)) = \cos(\pi t)u(\cdot, t)$, $F(t, u(t)) = \frac{|u(\cdot, t)|}{4+|u(\cdot, t)|}$, $u_0 = u_0(\cdot)$. Then the one dimensional non-autonomous stochastic partial differential equation of parabolic type (4.1) can be rewritten into the abstract form of Cauchy problem (1.1) in $L^2(0, 1; \mathbb{R})$.

Theorem 4.1. *The one dimensional non-autonomous stochastic partial differential equation of parabolic type (4.1) has at least one global mild solution $u \in C([0, 1] \times [0, +\infty))$.*

Proof. By the definition of nonlinear functions f and F , we can easily to verify that the assumption (H2) hold with

$$\alpha_f(t) = \cos^2(\pi t), \quad \alpha_F(t) = \frac{1}{16}, \quad \beta_f(t) = \beta_F(t) = 0.$$

Therefore, all the assumptions of Theorem 3.2 are satisfied. Hence, the one dimensional non-autonomous stochastic partial differential equation of parabolic type (4.1) has at least one global mild solution $u \in C([0, 1] \times [0, +\infty))$ due to Theorem 3.2. This completes the proof of Theorem 4.1. $\square$

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions to improve the quality of the paper.

Funding

Research supported by Education Technology Innovation Project of Gansu Province (No. 2022A-133).

Data availability

In this research no data was used.

Declarations

Competing interests

The authors declare no competing interests

Author contributions

Only one author for this paper. The author read and approved the final manuscript.

References

- [1] P. Acquistapace, B. Terreni, A unified approach to abstract linear parabolic equations, Rend. Semin. Mat. Univ. Padova 78 (1987) 47-107.
- [2] J. Bao, Z. Hou, C. Yuan, Stability in distribution of mild solutions to stochastic partial differential equations, Proc. Amer. Math. Soci. 138 (2010) 2169-2180.

- [3] P. Chen, A. Abdelmonem, Y. Li, Global existence and asymptotic stability of mild solutions for stochastic evolution equations with nonlocal initial conditions, *J. Integral Equations Appl.* 29 (2017) 325-348.
- [4] P. Chen, Y. Li, X. Zhang, Cauchy problem for stochastic non-autonomous evolution equations governed by noncompact evolution families, *Discrete Contin. Dyn. Syst. Ser. B* 26 (2021) 1531-1547.
- [5] R.F. Curtain, P.L. Falb, Stochastic differential equations in Hilbert space, *J. Differential Equations* 10 (1971) 412-430.
- [6] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.
- [7] W. Grecksch, C. Tudor, *Stochastic Evolution Equations: A Hilbert Space Approach*, Akademik Verlag, Berlin, 1995.
- [8] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, New York, 1981.
- [9] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, Chapman and Hall, London, 2006.
- [10] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Ltd., Chichester, 1997.
- [11] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
- [12] Y. Ren, Q. Zhou, L. Chen, Existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with poisson jumps and infinite delay, *J. Optim. Theory Appl.* 149 (2011) 315-331.
- [13] R. Sakthivel, Y. Ren, A. Debbouche, N.I. Mahmudov, Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions, *Appl. Anal.* 95 (2016) 2361-2382.
- [14] K. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic Publishers, London, 1991.
- [15] T. Taniguchi, K. Liu, A. Truman, Existence, uniqueness and asymptotic behavior of mild solutions to stochastic functional differential equations in Hilbert spaces, *J. Differential Equations* 181 (2002) 72-91.
- [16] X. Zhang, Z. Feng, Well-posedness and stability for non-autonomous stochastic evolution equations of parabolic-type with additive noise, *Discrete Contin. Dyn. Syst. Ser. B* 29 (2024) 2435-2452.