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chaotic map with offset-boosting behavior

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Design and analysis of discrete fractional-order chaotic map with offset-boosting behavior

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Abstract This paper presents a 3D discrete fractional-order chaotic map (DFOCM) with offset-boosting behavior based on the memristor. Different from the integer-order chaotic map, the offset-boosting behavior of the DFOCM with parameter offset is associated with the initial value, primarily due to the unique memory properties of fractional-order differences. In addition, rich dynamical properties of the DFOCM are examined using the 0-1 test, maximal Lyapunov exponent, phase diagram, and bifurcation diagram. Numerical simulations show that this DFOCM exhibits more complex dynamical behavior than its integer-order one. Meanwhile, the multistability and conditional symmetry of the DFOCM are studied. The unpredictability and high pseudo-randomness of the chaotic sequence produced by the DFOCM are confirmed by the SE complexity. Finally, the suggested DFOCM is implemented on the DSP hardware platform, demonstrating the physical feasibility of the numerical simulation.

Keywords Discrete chaotic map · Fractional-order difference · Memristor · Offset-boosting · Multistability

1 Introduction

According to the actual engineering requirements, the dynamic behavior of chaotic map can be changed through offset-boosting in chaotic map to improve the controllability and stability of the map and provide greater flexibility for its application. In recent years, research on this issue has been gradually carried out and good progress has been made. Li et al. [1] constructed a two-dimensional discrete chaotic map based on the sinusoidal function and realized the offset-boosting of the map by changing the initial value. On the other hand, Huang et al. [2] achieved the offset-boosting behavior of the map by increasing the offset constant. Chaos map often requires a high degree of complexity in practical applications and can achieve it by coupling memristors. So far, continuous memristors have been practiced and applied in many fields [3–7]. Discrete memristor models are easier to implement with digital hardware platforms than continuous memristor models. In addition, discrete memristors show great potential in chaotic oscillation circuits, which attract more and more attention. Progressive work on constructing chaotic or hyperchaotic map based on discrete memristor models. Four discrete hyperchaotic maps were presented based on a unified discrete memristive model proposed in [8]. A novel discrete memristive hyperchaotic map was constructed in [9] and its achievability was confirmed through the use of digital circuits. Based on the trigonometric wave function, Huang et al. [10] designed a novel three-dimensional memristive chaotic map with a very broad range of hyperchaotic or chaotic states. All of the above studies are based on integer orders, and fewer studies have been done on fractional order offset-boosting and discrete memristors compared to integer orders.
Fractional calculus is not a very recent idea; it was first introduced in a letter to Leibniz sent by L’Hospital more than 300 years ago. However, due to the lack of specific application background and the large amount of computation, the field has been in a state of stagnation. Until the middle of the 20th century, it was found that many physical systems including colored noise, electromagnetic waves, and viscous systems exhibit fractional-order dynamic properties [11]. Fractional-order differential equations have a unique global memory effect when compared to differential equations of integer order, which is similar to that of memristors. In comparison with continuous fractional calculus, which boasts an extensive theoretical framework [12], discrete fractional differences are a novel topic for investigation. DFOCM has gained popularity recently due to reports on its use in real-world projects including parameter recognition [13], picture encryption [14], and chaos synchronization and control [15–17]. Therefore, the design of fractional-order discrete memristors chaotic map has attracted much attention. Peng et al. [18] constructed a discrete memristive model of proportionality and asymmetry based on the Caputo fractional difference, and numerical simulations showed that the map could exhibit complex behaviors such as chaos and coexistence attractors. Ma et al. [19] coupled a discrete memristor with the Rulkov neuron map to obtain a fractional-order neuron model, and analyzed the effect of order change, and implemented it through digital circuits. According to the content of the literature, like discrete integer order map, DFOCM is sensitive to initial conditions and parameter changes. At the same time, it is also very sensitive to changes in order. In the same discrete map, fractional-order map provide an additional parameter to describe the dynamic behavior of the map, so DFOCM can be better adapted to the modeling and describing needs of various complex systems, with higher flexibility and adaptability, and can better simulate the dynamic behavior of real systems. Additionally, like fractional-order differences and memristors, they have a memory effect, which allows for a more accurate description of neural networks, physics, biology, and other fields. Based on the above analysis, this paper proposes a DFOCM about parameters with offset-boosting behavior, which has higher complexity and larger chaotic range compared to that under integer order. The main contents of this paper are as follows. (1) A discrete memristor model is constructed, coupled with a two-dimensional discrete chaotic map as well as the introduction of fractional order difference to obtain a three-dimensional DFOCM. (2) The effect of order on the mapping is analyzed by studying the stability of the fixed point and the dynamical behavior. (3) The difference between the map at fractional order and integer order regarding the parameter offset-boosting behavior is compared and the reason is explained, which is rarely mentioned in other articles.

The rest of the paper is as follows: Section 2 gives the definition of fractional-order difference, which provides the theoretical basis for the subsequent analysis; section 3 constructs the DFOCM and analyzes its fixed point stability with respect to the order; section 4 provides a full analysis of the dynamical behavior of the map, including bifurcation diagram, MLE, 0-1 tests, phase diagram, multistability, and conditional symmetry; section 5 analyzes the map with respect to the parameters of the offset-boosting behavior; section 6 provides a performance analysis and hardware implementation of the chaotic sequences generated by the map. Section 7 summarizes the paper.

2 Discrete fractional-order difference

This paper adopts the Caputo definition [20–23] of fractional difference operators mainly because it provides sufficient theoretical analysis and does not require considerations of setting initial values, among other issues. First, build a function \( x(n) \) whose forward difference operator is represented by \( \Delta x(n) = x(n+1) - x(n) \), and define an independent time scalar \( N_a = \{a, a+1, a+2, \ldots\} \), where \( a \in \mathbb{R} \).

**Definition 1** [22]: There exists a relationship \( f : N_a \to \mathbb{R} \) with a fractional order \( \vartheta (\vartheta > 0) \), denoted as

\[
\Delta_{\vartheta}^a x(n) = \frac{1}{\Gamma(\vartheta)} \sum_{s=a}^{n-\vartheta} (t-s)^{(\vartheta-1)} x(s) \tag{1}
\]

where \( n \in N_{a+\vartheta} \), \( a \) is the starting point; \( \sigma(s) = s+1 \). \( \Gamma(\bullet) \) is the Gamma function, and \( \Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \). \( n^{(\vartheta)} \) is the descending factorial factor, defined as:

\[
n^{(\vartheta)} = \frac{\Gamma(n+1)}{\Gamma(n+1-\vartheta)} \tag{2}
\]

where \( n \in N_{a+m}, m = \lfloor \vartheta \rfloor \).

**Definition 2** [22]: The Caputo difference operator is defined when the fractional order is \( \vartheta > 0, \vartheta \notin \mathbb{N} \) as follows:

\[
C_{\vartheta} \Delta_{\vartheta}^a x(n) = \frac{\Gamma(n+1)}{\Gamma(m-\vartheta)} \sum_{s=a}^{n-(m-\vartheta)} (n-\sigma(s))^{(m-\vartheta-1)} \Delta_{\vartheta}^m x(s) \tag{3}
\]

**Theorem 1** [23]: The definition of the Caputo fractional difference equation is:

\[
C_{\vartheta} \Delta_{\vartheta}^a x(n) = f(n+\vartheta-1, x(n+\vartheta-1)) \tag{4}
\]
Equation (3) can be equivalently expressed as:

\[
x(n) = x_0(n) + \frac{1}{\Gamma(\vartheta)} \sum_{s=a+m-\vartheta}^{n-\vartheta} (n - \sigma(s))^{(\vartheta-1)} f(s + \vartheta - 1, x(a + \vartheta - 1))
\]  

(5)

Where \( n \in \mathbb{N}_{a+m} \), the initial values \( x_0(n) \) defined as:

\[
x_0(n) = \sum_{k=0}^{m-1} \frac{(n-a)^{(k)}}{k!} \Delta^k x(a)
\]

(6)

Where \( \Delta^k x(a) = c_k \), \( k = 0, 1, ..., m-1 \). In this paper, setting \( a = 0 \) and \( l = s + \vartheta \), where \( 0 < \vartheta \leq 1 \). With the global memory effect, the fractional-order difference equation (5) may be reduced to the following discrete form.

\[
x(n) = x(0) + \frac{1}{\Gamma(\vartheta)} \sum_{l=1}^{n} \Gamma(n-l+\vartheta) f(l+1, x(l-1))
\]

(7)

3 Three-Dimensional discrete fractional order chaotic Map

In this section, we first construct a discrete memristor model and observe its current-voltage characteristic curve by applying a sinusoidal current, demonstrating its compliance with the definition of a memristor [24]. Subsequently, we couple the memristor with a simple 2D discrete chaotic map. Then, we perform fractional differencing on the new mapping, constructing a novel 3D DFOCM and analyze the stability of its fixed points.

3.1 Discrete memristor model

A new discrete memristor (DM) model is proposed, which can be expressed as:

\[
\begin{aligned}
    v_n &= M(q_n)i_n, \\
    q_{n+1} &= q_n + gi_n \\
    M(q_n) &= \tanh[q_n]
\end{aligned}
\]

(8)

The variable \( v_n \) represents the discrete output voltage signal, \( M(q_n) \) is the memristor function, and its graph is shown in Fig. 1(a). \( q_n \) is the charge value at the \( n \)-th iteration, \( i_n \) represents the discrete input current signal, and \( m \) represents the iteration step size, which is fixed at 1 in this paper. To verify the properties of the discrete memristor, let \( i_n = A \sin(\omega n) \), where \( A \) is the amplitude of the input current, and \( \omega \) is the frequency. The voltage-current characteristic curve of the DM is shown in Fig. 1(b), (c), and (d).

![Fig. 1: memristor function and the pinched hysteresis loops of the memristor. a memristor function; b Fixed amplitude \( A = 0.02 \), initial value \( q_0 = 0 \), frequency \( \omega = 0.03, 0.05, 0.1, 1 \); c Fixed frequency \( \omega = 0.02 \), initial value \( q_0 = 0 \), amplitude \( A = 0.01, 0.04, 0.07, 1 \); d Fixed amplitude \( A = 0.02 \), frequency \( \omega = 0 \), initial value \( q_0 = -0.03, 0, 0.03, 0.05 \)](image)

For fixed \( \omega = 0.05 \) and \( q_0 = 0 \), from Fig. 1(b), it can be observed that the area of the tight hysteresis loop increases with the increase in amplitude. Fixing \( A = 0.02 \) and \( q_0 = 0 \), it can be seen from Fig. 1(c) that the sidelobe area decreases with increasing frequency. It is noteworthy that the current and voltage at the two ends of the memristor approximate a single-valued functional relationship when \( \omega = 3 \), thus demonstrating that the proposed memristor model conforms to the definition of a memristor. For \( A = 0.02 \) and \( \omega = 0.05 \), as depicted in Fig. 1(d), The tight hysteresis loop’s position and shape change as the memristor’s initial value \( q_0 \) is altered, demonstrating the reactivity of the memristor model to initial values.

3.2 Discrete fractional-order chaotic map

Due to their inherent nonlinear properties, memristors are often used to enhance the dynamic behavior of some basic chaotic maps. A novel three-dimensional memristor chaotic map is generated by combining the suggested memristor with a two-dimensional simple chaotic
\[ x(n+1) = x(n) + a \sin(y(n)) - k \tanh(|z(n)|)x(n) \]
\[ y(n+1) = bx(n) + c \]
\[ z(n+1) = x(n) + z(n) \]

Theorem 1

Where \( a, b, c \) is the control parameter, \( k \) is the coupling coefficient, and \( x(n), y(n), z(n) \) is the state variable. The map (9) can be expressed in the form of a first-order difference equation.

\[
\Delta x(n) = a \sin(y(n)) - k \tanh(|z(n)|)x(n) \\
\Delta y(n) = bx(n) - y(n) + c \\
\Delta z(n) = x(n)
\]

According to Caputo Definition and Theorem 1, the fractional-order form of the map (10) can be obtained.

\[
x(n) = x(0) + \frac{1}{\Gamma(\vartheta)} \sum_{j=1}^{n} \Gamma(n-j+1) \left( a \sin(y(j-1)) - k \tanh(|z(j-1)|)x(j-1) \right) \\
y(n) = y(0) + \frac{1}{\Gamma(\vartheta)} \sum_{j=1}^{n} \Gamma(n-j+1) \left( bx(j-1) - y(j-1) + c \right) \\
z(n) = z(0) + \frac{1}{\Gamma(\vartheta)} \sum_{j=1}^{n} \Gamma(n-j+1) x(j-1)
\]

The fixed point \((x^*, y^*, z^*)\) of the map can be obtained by setting the left side of (10) to zero.

\[
\begin{align*}
(a \sin(y^*) - k \tanh(|z^*|)x^*) &= 0 \\
bx^* - y^* + c &= 0 \\
x^* &= 0
\end{align*}
\]

The fixed point, obtained by computing equation (14), can be expressed as:

\[
(x^*, y^*, z^*) = (0, c, \eta)
\]

Where \( c = n\pi, n \in \mathbb{Z} \), and \( \eta \) is any real number.

At the fixed point, the Jacobian matrix is provided by:

\[
J = \begin{bmatrix}
1 - k \tanh((z^*)) & a \cos(y^*) \cosh((z^*)) & 0 \\
b & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

4 Dynamical behavior analysis

In order to describe the dynamical behavior of chaotic map more accurately, this section analyzes the map in terms of bifurcation diagrams, maximal Lyapunov exponents, 0-1 tests, and phase diagrams. The maximal Lyapunov exponent of a chaotic map is an important measure of the local exponential growth rate of a chaotic system. It represents the upper limit of the exponential growth rate between the local orbits of the system in phase space and the neighboring orbits. Its positivity or negativity is an important indicator and basis for determining whether the system can produce chaotic behavior. The larger the maximal Lyapunov exponent is, the higher the degree of chaos in the system, and the greater the uncertainty and complexity in the system. In this paper, the method of Jacobian matrix is used to find [26]. 0-1 test [27]: it can reveal the randomness and complexity of chaotic systems to a certain extent. Unlike the Lyapunov exponent, it acts directly on the time series, thus eliminating the need for phase reconstruction. When the asymptotic growth rate \( K \)
is close to 1, it means that the mapping is in a chaotic state. When the $p-q$ plane has a bounded regular variation, the mapping is in a periodic state; when it is an unbounded Brownian motion, the map is in a chaotic state.

4.1 Dynamical characteristics for different parameters

4.1.1 Influence of coupling strength $k$

DFOCM, compared to integer-order map, introduce the order $\vartheta$ as a system parameter. To study the dynamical behavior of the map at different orders, with fixed $a = 2$, $b = 1.7$, $c = 1$, and initial values $(x_0, y_0, z_0) = (1, 1, 1)$. Map (11) under order $\vartheta = 0.7, 0.9, 1$, with the bifurcation diagram, MLE, and 0-1 test as shown in Fig. 2 as the coupling strength $k$ increases. The MLE and 0-1 test graphs align well. From the MLE graph, it can be observed that, at integer orders, the map starts in a periodic state. As $k$ increases to 0.3, the map undergoes period doubling bifurcations into chaos, then, at $k = 1$, it undergoes reverse period doubling bifurcations back into a periodic state. Finally, at $k = 1.4$, it abruptly enters a chaotic state. When $\vartheta = 0.9$, unlike the integer-order case, the map exists only a periodic window at $k \in [0.13, 0.15]$, and it remains chaotic as $k$ increases to 1. At $k = 1.32$, it transitions from a periodic state to chaos. For $\vartheta = 0.7$, the map is entirely chaotic, and the range of the map is also increasing. Clearly, by changing the order $\vartheta$ while keeping other parameters fixed, one can effectively increase the chaotic region of the map. It is worth noting that, in Fig. 2(b), at $k = 1.1$, the bifurcation diagram suggests that the map might be in a chaotic state, but both the MLE and 0-1 test indicate a periodic or quasi-periodic state. Further validation is provided by Fig. 3, which shows the phase portrait and the $p-q$ phase trajectory plot. Fig. 3(a) indicates a limit cycle for the map trajectory, and Fig. 3(b) shows a regular bounded pattern in the $p-q$ phase plot, confirming that the map is in a quasi-periodic state.

4.1.2 Influence of order $\vartheta$

Setting $a = 1.5$, $b = 1.25$, $c = 1$, $k = 1$ and the initial values $(x_0, y_0, z_0) = (1, 1, 1)$, the range of variation for the order $\vartheta$ is from 0.5 to 1. The bifurcation diagram and MLE with respect to the order $\vartheta$ are shown in Fig. 4. When $\vartheta \in [0.51, 0.586]$, the map exhibits chaotic behavior. For $\vartheta \in [0.59, 0.63]$, chaos and periodicity alternate. As $\vartheta$ increases, except for a brief chaos near $\vartheta = 0.715$, periodicity and quasi-periodic alternate, and ultimately, when $\vartheta = 1$, the map is in a period-2 state. Thus, it is evident that the fluctuation of the order $\vartheta$ has a significant impact on the rich dynamical behavior of the fractional-order map (11). Fixing $\vartheta = 0.55$, the $x-y$ plane phase portrait and $p-q$ phase trajectory are shown in Fig. 4(c, d). Fig. 4(c) shows that the map trajectory is a chaotic attractor, and Fig. 4(d) trajectory is characterized by Brownian motion, so the map is chaotic at this point.

4.2 Multistability

Multistability refers to the phenomenon where, with other parameters of a nonlinear system fixed, changing the initial conditions produces multiple trajectories, leading to the coexistence of attractors with different characteristics. Formally, coexistence attractors can be divided into two categories: homogeneous coexistence attractors and heterogeneous coexistence attractors. Homoclinic attractors are those with different amplitude, frequency, and spatial location, but the same shape. Heterogeneous coexistence attractors are those with different topologies. An in-depth understanding of the multistability phenomenon of chaotic systems helps to reveal the complexity and dynamic properties of the system, which is of great significance for the modeling, control and application of the system.

Setting $b = 0.7$, $c = 0.2$, $k = 0.6$, $\vartheta = 0.9$, the bifurcation diagram and MLE of different initial values changing with parameters are shown in Fig. 5. The blue trajectory indicates that the initial value is $(1, 1, 1)$, and the red trajectory indicates that the initial value $(x_0, y_0, z_0) = (-3, -3, -3)$. When the initial value is $(1, 1, 1)$, when $a \in [2.925, 2.995] \cup [3.035, 3.33] \cup [3.435, 3.5]$, the map is chaotic; when the initial value $(x_0, y_0, z_0) = (-3, -3, -3)$, in the interval $a \in [3.105, 3.48]$, except for the periodic window near 3.2, the rest are chaotic. Obviously, different initial values lead to different mapped chaotic regions. Select three typical phase diagrams as shown in Fig. 6. By selecting different values of parameter $a$, we can observe the coexistence of chaotic attractors and periodic attractors, the coexistence of chaotic attractors and quasi-periodic attractors, and the coexistence of chaotic attractors with different topological structures coexist.

4.3 Conditional symmetry

The symmetry of a chaotic system refers to the property that the system remains unchanged under some transformation. Symmetry is of great significance in chaotic
Fig. 2: When $a = 2$, $b = 1.7$, $c = 1$ and $(x_0, y_0, z_0) = (1, 1, 1)$, the dynamical behavior with coupling strength $k$.
(a-c) Bifurcation diagrams at different orders $\vartheta = 1$; $b \vartheta = 0.9$; $c \vartheta = 0.7$; (d-f) MLE spectrum at different orders $d \vartheta = 1$; $e \vartheta = 0.9$; $f \vartheta = 0.7$; (g-i) 0-1 test diagrams at different orders $g \vartheta = 1$; $h \vartheta = 0.9$; $i \vartheta = 0.7$.

Fig. 3: a Phase diagram and b $p - q$ plane of (11) for $a = 2$, $b = 1.7$, $c = 1$, $k = 1.1$, $\vartheta = 0.9$ and $(x_0, y_0, z_0) = (1, 1, 1)$

systems, which is mainly reflected in the following aspects. Stability analysis: Symmetry can simplify the stability analysis of chaotic systems. By utilizing the symmetry of the system, the amount of calculation can be reduced and the stable point and stable orbit of the system can be found. Dynamical behavior: Symmetry can affect the dynamical behavior of chaotic systems. For example, the symmetry of a system may lead to the appearance of specific periodic orbits or different kinds of chaotic behavior. Control and regulation: Symmetry can also be used for the control and regulation of chaotic systems. By studying the symmetry of a system, more effective control strategies can be designed to achieve control and regulation of chaotic systems.
For map (11) with order $\vartheta$ for $a = 1.5$, $b = 1.25$, $c = 1$, $k = 1$ and $(x_0, y_0, z_0) = (1, 1, 1)$; a bifurcation diagram; b MLE; $\vartheta = 0.55$, c phase diagram; d $p-q$ plane.

5 On the offset-boosting behavior of map parameters

Offset-boosting is a phenomenon that the spatial position of the attractor is changed by changing the offset constant or initial value, while the topology of the attractor remains unchanged, resulting in the coexistence of attractors. In some practical systems, noise may negatively affect the performance of the system. By offset-boosting it helps the system to better resist external noise and improve the noise tolerance of the system. In the field of communication, chaotic systems are commonly used to encrypt and decrypt messages. The introduction of offset-boosting may enhance the security and robustness of information transmission, making the system better adaptable to external perturbations. Through offset-boosting, the mutual transformation of bipolar and unipolar signals can be realized, so that the chaotic system can be better adapted to the practical needs and the stability and controllability of the system can be improved. Firstly, the offset-boosting of integer order map (9) with parameter $c$ is proved.

$$c^* = c + 2j\pi \quad (18)$$

where $j$ is any integer, and $c$ meet the following requirements

$$-\pi < c < \pi \quad (19)$$

Bringing $c^*$ to (9) obtains

$$y^*(n+1) = bx(n) + c^* = y(n+1) + 2j\pi \quad (20)$$

Since the $z$ dimension does not contain $y$, only the $x$ dimension needs to be analyzed.

$$x^*(n+1) = x(n) + a\sin(y^*(n)) - k\tanh(|z(n)|)x(n) \quad (21)$$
Fig. 6: Map (11) different coexisting attractors when $a$ takes different values. (a) $a = 3.2$; (b) $a = 2.98$; (c) $a = 3.45$

Fig. 7: Dynamical behavior of map (17) at different initial values, blue: $(1, 1, 1)$, red: $(-1, -1, -1)$ a bifurcation diagram; b MLE; c $a = 2.3$, three-dimensional symmetric phase diagram

where

$$a \sin(y^*(n)) = a \sin(y(n) + 2j\pi) = a \sin(y(n)) \quad (22)$$

So $x^*(n + 1) = x(n + 1)$. The above analysis shows that when $c$ changes by $2j\pi$, the map (9) also changes by $2j\pi$ in the $y$ dimension, while the $x$ and $z$ dimensions remain unchanged, which fully proves that the map (9) has an offset-boosting behavior with a period of $2\pi$ as the parameter $c$ changes. Fixed $a = 4$, $b = 0.7$, initial condition is $(x_0, y_0, z_0) = (1, 1, 1)$. When parameter $c \in [-3\pi, 3\pi]$, the attraction basin of map (9) is shown in Fig. 8(a). It can be seen from the figure that the attraction basin changes with the parameter $c$ with a period of $2\pi$, which is consistent with the above analysis results. Set $a = 4$, $b = 0.7$, $k = 0.5$, the initial value is $(x_0, y_0, z_0) = (1, 1, 1)$, the bifurcation diagram and MLE of map (9) dynamically changing with $c$ are shown in Fig. 9(a,b). It can be clearly seen from the figure that the bifurcation diagram and MLE exhibit the offset-boosting behavior with a period of $2\pi$ as the parameter $c$ changes. Based on the above parameters, set $c = 1.78 + 2j\pi$, where $j = 0, \pm 1, \pm 2$. The phase trajectory diagram map (9) on the $x - y$ plane is shown in Fig. 9(c). It can be seen that the five attractors are chaotic attractors with unified topological structure, equal size and different spatial positions. It is worth noting that the above five attractors are generated under the same initial value conditions. It shows that the translation of the attractor on the y-axis can be achieved by controlling the size of parameter $c$.

Fig. 8: The basin of attraction with $c$ and coupling strength $k$ a map(9); b map(11), $\vartheta = 0.7$

Although the fractional map (11) is obtained by differencing and fractionalizing the integer order map (9), its offset boosting behavior with mapping parameters is different from that of the integer order due to the global
memory property of the fractional difference equation. This is explained by the following analysis: The $y$ dimension of the fractional map (23)

$$y(n) = y(0) + \sum_{j=1}^{n} \frac{\Gamma(n-j+\vartheta)}{\Gamma(1+\vartheta)}(bx(j-1) - y(j-1) + c)$$

(23) can be seen from the above formula that the value of $y$ at time $n$ is related to the first $n-1$ items. Substituting equation (18) into equation (20), we get

$$y^*(1) = bx(0) + c + 2j\pi$$

(24)

$$y^*(2) = y(0) + \sum_{j=1}^{n} \frac{\Gamma(n-j+\vartheta)}{\Gamma(1+\vartheta)}(bx(0) - y(0) + c + 2j\pi + \Gamma(\vartheta)((bx(1) - y^*(1) + c)))$$

(25) Obviously $\frac{1}{\Gamma(\vartheta)} \times \frac{\Gamma(1+\vartheta)}{\Gamma(2)} \neq 1$, so $y^* \neq y(2) + 2j\pi$.

Due to the global memory of fractional-order difference equations, as $n$ increases, the deviation between $y^*(n)$ and $y(n) + 2j\pi$ will become larger and larger, thus affecting the values of other dimensions. Here, put $y^*(0) = y(0) + 2j\pi$ and equation (18) into equation (20).

$$y^*(n) = y^*(0) + \sum_{j=1}^{n} \frac{\Gamma(n-j+\vartheta)}{\Gamma(1+\vartheta)}(bx(j-1) - y^*(j-1) + c^*)$$

(26)

Likewise, for the $x$ dimension:

$$x^*(n) = x(0) + \sum_{j=1}^{n} \frac{\Gamma(n-j+\vartheta)}{\Gamma(1+\vartheta)}(ax(j-1) - k \tanh(|z^*(j-1)|x^*(j-1))$$

(27) When $n = 1$, $y(1)^* = y(1) + 2j\pi, x^*(1) = x(1)$, assuming that when $n = h$, $y^*(h) = y(h) + 2j\pi$ and $x^*(h) = x(h)$
is satisfied. When \( n = h + 1 \),
\[
y^*(h + 1) = y^*(0) + \frac{1}{\Gamma(\vartheta)} \sum_{j=1}^{h+1} \Gamma(n - j + \vartheta) (b x^*(j - 1) - y^*(j - 1) + c^*)
\]
\[
= y(0) + 2j\pi + \frac{1}{\Gamma(\vartheta)} \sum_{j=1}^{h+1} \Gamma(n - j + \vartheta) (b x(j - 1) - y(j - 1) + c)
\]
\[
= y(h + 1) + 2j\pi
\]
(28)

It can be proved by mathematical induction that when \( c^* = c + 2j\pi \), \( y^*(0) = y(0) + 2j\pi \), the offset enhancement of map (11) can be achieved. Setting the map parameters the same as integer order, order \( \vartheta = 0.7 \), initial value \( (x_0, y_0, z_0) = (1, 1 + 2j\pi, 1) \) where \( j \) is \(|c| + \pi \) to round \( 2\pi \). The attraction basin of map (11) is shown in Fig. 8(b). Consistent with the preceding analysis results, the attraction basin varies with the parameter \( c \) and the initial value \( y_0 \) over a time of \( 2\pi \), much like in the case of integer order. Similarly, when \( k = 0.5 \) is fixed, the bifurcation diagram and MLE of the fractional-order map (11) dynamically changing with \( c \) and the initial value \( y_0 \) are shown in Fig. 10(a,b). In comparison with the integer order, its chaotic zone has grown, and its other characteristics are comparable. Set \( c = 1.78 + 2j\pi \), \( y_0 = 1 + 2j\pi \), where \( j = 0, \pm 1, \pm 2 \). Its phase trajectory in the \( x-y \) plane is shown in Fig. 10(c). It can be seen that the generated attractor only produces a displacement of \( 2\pi \) on the \( y \)-axis, and its size and topology do not change. The specific reason why fractional order difference equations differ from integer order is because of their unique memory effect, which allows for a more accurate description of real processes.

6 Performance analysis and DSP implementation

6.1 SE complexity

Spectral entropy complexity (SE) [28] is often used to study the information content of the signal, the dynamical properties, and the complexity of the system behavior. For chaotic systems, their dynamical properties may cause the spectrum to vary over time. SE complexity can be used to analyze such dynamical variations and provide information about the behavior of the system. The larger the value of SE complexity, the closer the generated chaotic sequence is to a random sequence. Fixed parameters \( a = 4, b = 0.7, c = 0.2 \), Fig. 11 plots the SE complexity map of parameters \( k \) and \( \vartheta \) in the range. The figure shows that the map SE complexity is generally over 0.8 and sometimes surpasses 0.9 when the system order is \( \vartheta \in [0.2, 0.6] \). Thus, by altering the system’s order, the complexity of the mapping can be increased.

6.2 DSP implementation

The completion of hardware circuit design is essential for the application of chaotic systems. This map is implemented using the DSP hardware platform with chip type TMS320F28335 in this article. The physical diagram of the hardware is shown in Fig. 12(a). The parameters are set to \( a = 1.5, b = 1.25, c = 1, k = 1, \vartheta = 0.55 \) and the initial value \( (x_0, y_0, z_0) = (1, 1, 1) \). The chaotic attractor generated by map (11) is observed with an oscilloscope as shown in Fig. 12(b). It is evident from comparing Fig. 4(c) that the chaotic signal produced using the DSP hardware platform agrees with the outcomes of the simulation.

7 Conclusion

This paper first constructs a new discrete memristor model. The proposed discrete memristor meets the defi-
Design and analysis of discrete fractional-order chaotic map with offset-boosting behavior

Fig. 12: DSP implementation: a Hardware diagram; b Chaotic attractor

ition of a memristor by observing its volt-ampe characterisic curve. Afterwards, it is coupled with a simple two-dimensional map to obtain a three-dimensional discrete map. By introducing the Caputo difference operator, the DFOCM is obtained. By analyzing the fixed point, it is shown that the stability at the fixed point is closely related to the order. Through the bifurcation diagram, the maximal Lyapunov exponential spectrum and the 0-1 test diagram, it is found that the chaotic region of DFOCM is closely related to the system order $\theta$. Therefore, compared with integer-order map, fractional-order map can adjust the chaotic range more flexibly and describe the physical system more accurately. Meantime, by observing the phase diagram, it is found that the map has multistability. Furthermore, by analyzing the parameter $c$, a series of interesting dynamic behaviors were found. When $c = 0$ is fixed, the map has a symmetric behavior about the center of the origin. Moreover, there are differences in the behavior of parameter $c$ to generate offset-boosting between map in integer order and fractional order. When map to fractional order, the initial value $y_0$ needs to be added or subtracted by $2j\pi$. The amplitude control ability of chaos sequences has been enhanced. Afterwards, it was verified that the chaotic sequence generated by the map has high randomness through SE complexity. Finally, DSP is used as the hardware platform, which lays the foundation for its subsequent applications in secure communications and other fields. In the future, we expect to be able to use it in real projects.

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Data availability statement

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflict of interest.

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