Supplementary Information

Quantum Geometry in Hexagonal Circuit QED Lattice with Triple-leg Stripline Resonators

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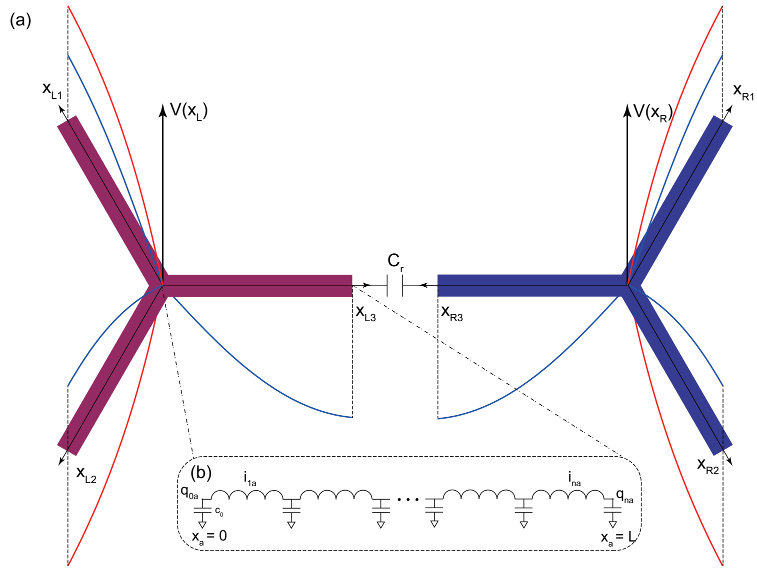

Figure S1. (a) Two TSRs coupled by capacitor with capacitance C_r: The blue solid curves on the TSR represent the s-mode with eigenvector $\left(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}\right)$, while the red solid curves denote the p-mode with eigenvector $\left(1/\sqrt{2}, -1/\sqrt{2}, 0\right)$. (b) A lumped circuit model for the left TSR: $i_{m\alpha}$ indicates the current through the m-th inductor with inductance l at the α-th leg and $q_{m\alpha}$ denotes the charge on the m-th capacitor with capacitance c at the α-th leg.

S1. DERIVATION OF THE TIGHT BINDING HAMILTONIAN FOR THE PHOTONIC HEXAGONAL LATTICE WITH TSRS

Distinctly from conventional linear stripline resonator in circuit quantum electrodynamics (QED) system [1], a triple-leg stripline resonator (TSR) features a three-legged design as illustrated Fig. S1. Using the lumped element circuit model, the Lagrangian of the system can be written by

$$\mathcal{L}\left[\{q_{n\alpha}\}\right] = \sum_{\alpha=1}^{3}\sum_{n=1}^{N}\left[\frac{l}{2}i_{n\alpha}^2 - \frac{1}{2c}q_{n\alpha}^2\right] - \frac{1}{2c_0}q_0^2$$

where $q_{n\alpha}$ represents the charge on the capacitor at position n of the α-th leg, $i_{n\alpha}$ denotes the current on the inductor at the position n of the α-th leg, and q_0 is the charge on the central capacitor with capacitance c_0 [2]. Applying the Kirchhoff rules, we introduce the nonlocal variable $\vartheta_{m\alpha} = \sum_{n=m}^{N} q_\alpha$. By using the variable $\vartheta_{m\alpha}$, one can express the Lagrangian of a single TSR as follows

$$\mathcal{L}\left[\{\vartheta_{m\alpha}\}\right] = \sum_{\alpha=1}^{3}\sum_{n=1}^{N}\left[\frac{l}{2}\dot{\vartheta}_{n\alpha}^2 - \frac{1}{2c}\left(\vartheta_{n\alpha} - \vartheta_{n+1\alpha}\right)^2\right] - \frac{1}{2c_0}\vartheta_{1\alpha}^2. \tag{S1}$$

By solving the Euler-Lagrange equation in the continuum limit and imposing the charge neutrality condition for $x \neq 0$, the spatial part of $\vartheta_\alpha(x,t)$ can be obtained to be $\chi_\alpha(x) = A_\alpha \sin k(x-L)$ with $k = \omega\sqrt{lc}$ by imposing boundary condition that $\vartheta_\alpha(x=L) = 0$. The equation of motion for χ_α at $x = 0$ yields the following equations

$$\left(\frac{k}{c}\cos kL - l\omega^2 \sin kL\right) A_\alpha + \frac{1}{c_0}\sin kL \sum_{\beta=1}^{3} A_\beta = 0. \tag{S2}$$

The above equations yield the following two solutions: A single symmetric mode and two-fold degenerate modes. The symmetric spatial mode is described by $A_\alpha = 1/\sqrt{3}$ for all α with $k \cong n\pi/L$. The two-fold degenerate mode satisfies the following condition $\sum_\alpha^3 A_\alpha = 0$ with $k \cong (n-1/2)\pi/L$. Hence the ground states are two-fold degenerate states. Here we have chosen the following two orthonormal eigenvectors: $\mathbf{A}_s = \left(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}\right)$ and $\mathbf{A}_p = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ as demonstrated on the left TSR in Fig. S1.

Next, we place a TSR to the right, which is coupled by capacitor located at end of the third leg as shown in Fig. S1. The Lagrangian of two coupled TSR system in terms of the variable $\vartheta(x,t)$ can be written by

$$\begin{aligned}\mathcal{L} = &\sum_{\alpha=1}^{3}\int_0^L dx_\alpha \left[\frac{l}{2}\dot{\vartheta}_{\alpha l} - \frac{1}{2c}\left(\frac{\partial \vartheta_{\alpha l}}{\partial x_\alpha}\right)^2\right] - \frac{1}{2c_0}\left(\sum_{\alpha=1}^{3}\vartheta_{\alpha l}(0)\right)^2 \\ &+ \sum_{\alpha=1}^{3}\int_0^L dy_\alpha \left[\frac{l}{2}\dot{\vartheta}_{\alpha r} - \frac{1}{2c}\left(\frac{\partial \vartheta_{\alpha r}}{\partial y_\alpha}\right)^2\right] - \frac{1}{2c_0}\left(\sum_{\alpha=1}^{3}\vartheta_{\alpha r}(0)\right)^2 \\ &- \frac{C_r}{2c^2}\left(\frac{\partial \vartheta_{3l}(L)}{\partial x_3} - \frac{\partial \vartheta_{3r}(L)}{\partial y_3}\right)^2\end{aligned} \tag{S3}$$

where x_α and y_α denote coordinates on the α-th leg of the left and right TSR and α indicates the legs of the two TSRs. $\vartheta_{\alpha l}, \vartheta_{\alpha r}$ represent the nonlocal variable for the left and right TSR respectively. In the Lagrangian S3, the first and second terms describe the Lagrangian of the left TSR, the third and fourth terms describe the right TSR, and the last term represents the capacitative coupling of two TSRs. Assuming a weak coupling between two TSRs with $C_r \ll 3Lc$, the frequency shift of each TSR can be considered to be negligible. The voltage operator at the end of the third leg of the left TSR can be written by

$$\hat{V}(L) = \frac{1}{c}\frac{\partial \vartheta_{3l}(L)}{\partial x_3} \cong \sum_{\mu=s,p} A_{3\mu}\sqrt{\frac{\hbar\omega}{3Lc}}\left(\hat{a}_\mu + \hat{a}_\mu^\dagger\right). \tag{S4}$$

Hence the Hamiltonian for the two coupled TSR system can be written by

$$\hat{H}_2 = \sum_{\mu=s,p}\hbar\omega \hat{a}_\mu^\dagger \hat{a}_\mu + \sum_{\nu=s',p'}\hbar\omega \hat{a}_\nu^\dagger \hat{a}_\nu + \sum_{\mu,\nu} A_{l\mu}A_{r\nu}\frac{\hbar\omega C_r}{3LC}\hat{a}_\mu^\dagger \hat{a}_\nu + h.c. \tag{S5}$$

Hence the hopping strength $t_{\mu\nu}$ can be written by

$$t_{\mu\nu} = \hbar\omega \frac{C_r}{C_g} A_{l\mu} A_{r\nu}$$

where $C_g(= 3Lc)$ represents the capacitance to the ground plane of a TSR. The hopping strength depends on the coupling between two specific spatial modes of each TSR, which is proportional to the multiple of two amplitudes.

The Hamiltonian for the photonic hexagonal lattice system with TSRs can be written by

$$\hat{H} = \sum_{i,\mu} \hbar\omega \hat{a}^{\dagger}_{i,\mu}\hat{a}_{i,\mu} + \sum_{j,\nu} \hbar\omega \hat{a}^{\dagger}_{j,\nu}\hat{a}_{j,\nu} + \sum_{\langle i,j\rangle}\sum_{\mu,\nu} t_{\mu\nu}\hat{a}^{\dagger}_{i,\mu}\hat{a}_{j,\nu} + h.c., \tag{S6}$$

where the indices $i(j)$ denotes the sites of sublattice $A(B)$ as defined in the main text. Due to the two-fold degeneracy, any two orthonormal bases can be chosen to represent the Hamiltonian of the system. By taking $\hat{\psi}(\mathbf{k}) = (\hat{a}_{As}(\mathbf{k}), \hat{a}_{Ap}(\mathbf{k}), \hat{a}_{Bs}(\mathbf{k}), \hat{a}_{Bp}(\mathbf{k}))^T$, one can describe the Hamiltonian of the system as follows

$$H(\mathbf{k}) = \begin{pmatrix} 0 & 0 & T_{ss'} & T_{sp'} \\ 0 & 0 & T_{ps'} & T_{pp'} \\ T^*_{ss'} & T^*_{ps'} & 0 & 0 \\ T^*_{sp'} & T^*_{pp'} & 0 & 0 \end{pmatrix} \tag{S7}$$

where $T_{ss'} = t_{ss2} + t_{ss1}(e^{ik_1} + e^{ik_2})$, $T_{sp'} = T_{ps'} = t_{sp}(-e^{ik_1} + e^{ik_2})$, $T_{pp'} = t_{pp}(e^{ik_1} + e^{ik_2})$ with the coefficients $t_{ss2} = 4t_{pp}/3$, $t_{ss1} = t_{pp}/3$, $t_{sp} = t_{pp}/\sqrt{3}$. The momenta k_1 and k_2 is given by $\left(k_x + \sqrt{3}k_y\right)/2$ and $\left(-k_x + \sqrt{3}k_y\right)/2$, respectively. Hence, the energy eigenvalues can be obtained by

$$E = \pm 2t_{pp}, \pm\frac{2t_{pp}}{3}\sqrt{3 + 2\left(\cos k_1 + \cos k_2 + \cos(k_1 - k_2)\right)} \tag{S8}$$

which exhibits the top and bottom flat bands and the dispersive Dirac nodes at $\mathbf{k} = (k_x, k_y) = (\pm 4\pi/3, 0)$. One can notice that the dispersive band quadratically touches to the top and bottom flat bands.

S2. EFFECT OF LENGTH VARIATIONS OF THREE LEGS IN TSR

We now demonstrate that variation in leg length of TSR does not disrupt the degeneracy of the system significantly. Suppose that the length of three legs in a TSR is slightly different from each other as shown in Fig. S2. We will take the length difference of the α-th leg from L to be dL_α

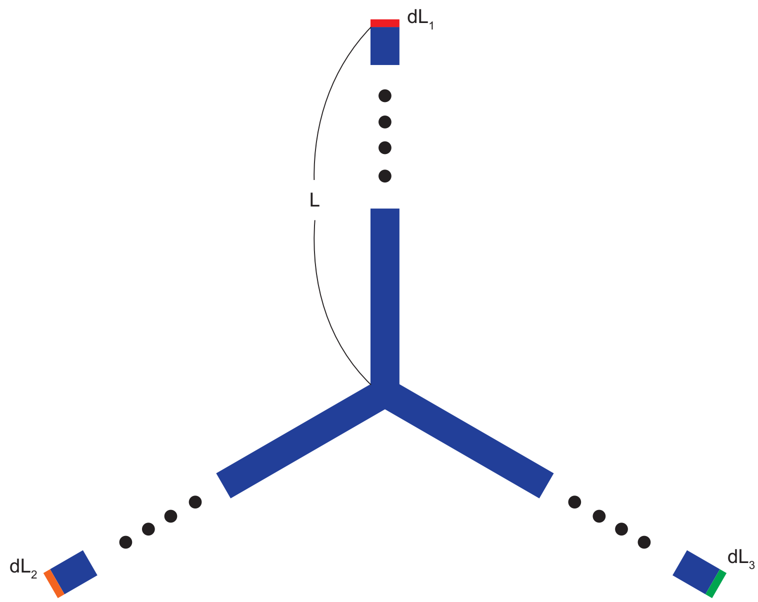

Figure S2. An illustrative schematic depicting length variations within a TSR, where each leg undergoes a slight change in length, denoted as dL_α for $\alpha = 1, 2, 3$.

with $\alpha = 1, 2, 3$ and assume that $dL_\alpha \ll L$. From Eq. S1, the equation of motion of the TSR with leg length L_α can be written by

$$\left(\frac{a_0}{c}k\cos kL_\alpha - l\omega^2 \sin kL_\alpha\right) A_\alpha + \frac{1}{c_0}\sin kL_\alpha \sum_{\beta=1}^{3} A_\beta = 0$$

where L_α is the length of the α-th leg defined to be $L_\alpha = L + dL_\alpha$. By introducing the average length of three legs $\widetilde{L} = L + \sum_\alpha dL_\alpha/3$, one can rewrite the length as $L_\alpha = \widetilde{L} + d\widetilde{L}_\alpha$, which satisfies $\sum_\alpha d\widetilde{L}_\alpha = 0$. Then the eigenvalue equations for the spatial modes can be written by

$$\begin{pmatrix} d_1(k) & h_2(k)+s(k) & h_3(k)+s(k) \\ h_1(k)+s(k) & d_2(k) & h_3(k)+s(k) \\ h_1(k)+s(k) & h_2(k)+s(k) & d_3(k) \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0 \tag{S9}$$

where $d_\alpha(k) = f(k) + g_\alpha(k) + h_\alpha(k) + s(k)$, $f(k) = (k/c)\cos k\widetilde{L} + (k^2/c)\sin k\widetilde{L}$, $g_\alpha(k) = (d\widetilde{L}_\alpha/3)((k/2\widetilde{L})\cos k\widetilde{L} - k^2 \sin k\widetilde{L} + (k^2/2\widetilde{L})\sin k\widetilde{L} - k^3\cos k\widetilde{L})$, $h_\alpha(k) = (d\widetilde{L}_\alpha/c_0)((1/2\widetilde{L})\sin k\widetilde{L} - k\cos k\widetilde{L}))$, $s(k) = (1/c_0)\sin k\widetilde{L}$. By neglecting the higher order terms than $\mathcal{O}(d\widetilde{L}_\alpha/\widetilde{L})$, the determinant of the matrix can be simplified to $f^2(k)(f(k) + 3s(k))$. For $f(k) = 0$, the relation $1/k = -\tan k\widetilde{L}$ holds, which implies that $k \cong (n - 1/2)\pi/\widetilde{L}$. This manifests that the system supports a two-fold degenerate mode up to order $\mathcal{O}(d\widetilde{L}_\alpha/\widetilde{L})^2$. For $f(k) = -3s(k)$, the relation

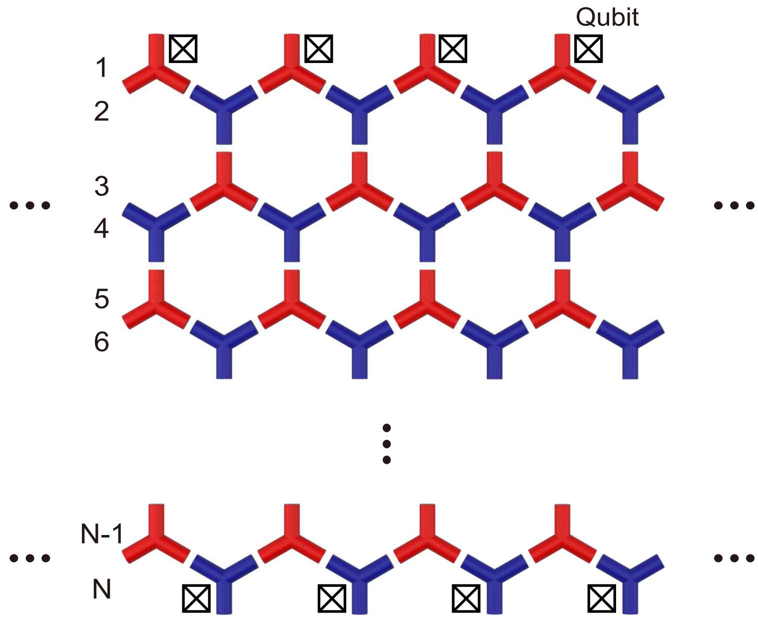

Figure S3. Schematic representation of zigzag TSR lattice coupled to superconducting qubits laterally along the top and bottom zigzag edges of the lattice.

$\tan k\widetilde{L} = k/[(k)^2 - 3c/c_0]$ leads to $k \cong n\pi\widetilde{L}$ exhibiting the non-degenerate symmetric mode. One can notice that the degeneracy of the ground state persists quite strongly against length variations of TSR.

S3. DERIVATION OF EDGE MODE CONTROLLED BY COUPLING TO THE SUPERCONDUCTING QUBIT

As shown in Fig. S3, superconducting qubit arrays are placed laterally to the outermost legs of TSRs. The lateral coupling of qubits to the resonator influences the energy levels of the edge states depending on the qubit states. According to [1], in a disperisve regime, an effective Hamiltonian of single TSR coupled to a qubit can be written by

$$\hat{H}_{\text{eff}} = \hbar\omega_0 \left(\hat{a}_s^\dagger \hat{a}_s + \hat{a}_p^\dagger \hat{a}_p\right) + \chi \hat{a}_s^\dagger \hat{a}_s \sigma_z + \frac{1}{2}\left(E + \chi\right)\sigma_z \tag{S10}$$

where E denotes the resonance frequency of a qubit and g_s represents the coupling strength between TSR and a qubit with $\chi = g_s^2/\Delta$ and $\Delta = E - \hbar\omega_0$. One can see that only s-mode couples to superconducting qubits and the frequency of a s-mode is shifted by $\chi\sigma_z/\hbar$. We will first consider the zigzag nanoribbon made of TSR lattice without qubit coupling. By solving the Heisenberg equation of motion, we derive the recurrence relations to obtain the eigenfunction for the edge state, which can be written by

$$\begin{aligned}
\epsilon\psi_{sm} &= t_{ss2}\psi_{s'm+1} + 2t_{ss1}\cos\frac{k}{2}\psi_{s'm} - 2it_{sp}\sin\frac{k}{2}\psi_{p'm} \\
\epsilon\psi_{pm} &= -2it_{sp}\sin\frac{k}{2}\psi_{s'm} + 2t_{pp}\cos\frac{k}{2}\psi_{p'm} \\
\epsilon\psi_{s'm} &= t_{ss2}\psi_{sm+1} + 2t_{ss1}\cos\frac{k}{2}\psi_{sm} + 2it_{sp}\sin\frac{k}{2}\psi_{pm} \\
\epsilon\psi_{p'm} &= 2it_{sp}\sin\frac{k}{2}\psi_{sm} + 2t_{pp}\cos\frac{k}{2}\psi_{pm}
\end{aligned}$$

where ϵ is the energy eigenvalue and $\psi_{\alpha m}$ refers to the α-th component of the eigenvector at position $m(=1,\cdots,N)$ of TSR zigzag nanoribbon. First, we will consider the flat edge modes at $\epsilon = 0$. The recurrence relations divide into two parts at each site as follows. $\psi_{sm+1} = -(1/2)\sec(k/2)\psi_{sm}$ for site A and $\psi_{s'm+1} = -(1/2)\sec(k/2)\psi_{s'm}$ for site B. One can obtain the following two edge-localized states, which can be written by

$$\begin{aligned}
\psi_{\text{edge}} &= |\{\psi_s\},\{\psi_p\},\{\psi_{s'}\},\{\psi_{p'}\}\rangle \\
&= \begin{cases} \zeta\,|\{1,\mu,\ldots,\mu^{N-1}\},\{\beta,\beta\mu,\ldots,\beta\mu^{N-1}\},\{0,\ldots,0\},\{0,\ldots,0\}\rangle \\ \zeta\,|\{0,\ldots,0\},\{0,\ldots,0\},\{\mu^{N-1},\ldots,\mu,1\},\{\beta\mu^{N-1},\ldots,\beta\mu,\beta\}\rangle \end{cases}
\end{aligned}$$

where $\mu = -(1/2)\sec(k/2)$, $\beta = -(i/\sqrt{3})\tan(k/2)$, and the normalization factor $\zeta = [(1+|\beta|^2)(1-\mu^{2N})/(1-\mu^2)]^{-1/2}$. The two edge-localized states ψ_{edge} represent the top and bottom localized states respectively. Now we will consider the lateral coupling to qubit arrays. We will assume that the qubit states are uniformly controlled by input microwave pulse. Since only the s-mode couples to the outermost qubits and the overlap between two edge states localized to the opposite edges is negligible, the energy shift due to the coupling can be easily obtained by calculating $\langle\psi_{edge}|\hat{V}|\psi_{edge}\rangle = |\zeta|^2$ for individual edge state. Here $\hat{V}$ represents the coupling Hamiltonian between the outermost legs of TSRs and qubits, which can be written by

$$\hat{V} = \begin{pmatrix} \{\chi\sigma_z,0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0\} \\ \{0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0\} \\ \{0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0,\chi\sigma_z\} & \{0,\ldots,0\} \\ \{0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0\} & \{0,\ldots,0\} \end{pmatrix}.$$

At the most localized point ($k=0$), $\mu \cong -1/2$, $\beta \cong 0$ and hence the energy shift is given by $3\chi\sigma_z/4$. At the vicinity of the Dirac point ($k=\pm 2\pi/3$), $\mu \to 1-(\sqrt{3}/2)\Delta k$ with $\Delta k = k \mp 2\pi/3$.

For semi-infinite zigzag nanoribbon with $N \to \infty$, $\zeta \to ((1+|\beta|^2)/(1-\mu^2))^{-1/2}$ and hence the slope of the energy dispersion near Dirac point is given by $\sqrt{3}\chi\sigma_z/2$. Next, we investigate the dispersive bands, which touch the top or bottom flat bands at $\mathbf{k}=0$. Since the edge state localized at one edge remains unaffected by coupling to the opposite edge for TSR zigzag nanoribbon, one can take $\psi_s = 0$ (or $\psi_{s'} = 0$). From the recurrence relations, the eigen-energies of the dispersive edge states can be obtained to be $\epsilon = \pm\sqrt{(2/3)\sin^2(k/2) + 4\cos^2(k/2)}$ in unit of t_{pp} and the wavefunction of the dispersive edge states are given by

$$\begin{aligned}\psi_{\text{edge}} &= |\{\psi_s\},\{\psi_p\},\{\psi_{s'}\},\{\psi_{p'}\}\rangle \\ &= \begin{cases} \xi\,|\{1,\nu,\ldots,\nu^{N-1}\},\{\beta,\beta\nu,\ldots,\beta\nu^{N-1}\},\{0,\ldots,0\},\{\gamma,\gamma\nu,\ldots,\gamma\nu^{N-1}\}\rangle \\ \xi\,|\{0,\ldots,0\},\{\gamma\nu^{N-1},\ldots,\gamma\nu,\gamma\},\{\nu^{N-1},\ldots,\nu,1\},\{\beta\nu^{N-1},\ldots,\beta\nu,\beta\}\rangle \end{cases}\end{aligned}$$

where $\nu = \cos(k/2)$, $\beta = (\sqrt{3}/2)\epsilon\cot(k/2)$, $\gamma = i(\sqrt{3}/2)\epsilon\csc(k/2)$ and the normalization constant $\xi = ((1+|\beta|^2+|\gamma|^2)(1-\nu^{2N})/(1-\nu^2))^{-1/2}$. At the maximum localized point ($k=\pm\pi$), the energy shift is given by $3\chi\sigma_z/4$. Our theoretical analysis clearly demonstrates the tunability of the edge-localized states by coupling to superconducting qubits.

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