Supporting material to "An integration test based only on the maximum and minimum Z-scores for meta analysis"

Yuxin Zhang[1,2*] and Zhengbang Li[2,3†]

[1*]School of Mathematics and Statistics, Central China Normal University, No. 152 Luoyu Road, 430000, Wuhan, China.
[2]School of Mathematics and Statistics, Central China Normal University, No. 152 Luoyu Road, 430000, Wuhan, China.

*Corresponding author(s). E-mail(s): a1443196130@163.com;
Contributing authors: lizhengbang@mail.ccnu.edu.cn;
[†]These authors contributed equally to this work.

Abstract

In this supporting material, we provide some technical lemmas and detailed proofs for all lemmas and theorems.

Keywords: Z-score value, Maximum value, Minimum value, Asymptotic distribution

1 TECHNICAL LEMMAS AND THEIR PROOFS

Denotes $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$. Let $X_1, \cdots, X_p$ independently follow a common standard normal distribution. We can obtain that,
(i) For any real number x and y,

$$\lim_{p\to+\infty} P_r \left\{ \frac{\left[\max\limits_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x \right\} = \exp\{-\exp(-x)\}$$

and

$$\lim_{p\to+\infty} P_r\left\{\frac{\left[\min\limits_{s=1,\cdots,p}(X_s)\right]^2-c_p}{2-(\log p)^{-1}}<y\right\}=\exp\{-\exp(-y)\}$$

(ii) For any real number x and y, $\left[\max\limits_{s=1,\cdots,p}(X_s)\right]^2$ and $\left[\min\limits_{s=1,\cdots,p}(X_s)\right]^2$ are asymptotically independent,namely,

$$\lim_{p\to+\infty} P_r\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(X_s)\right]^2-c_p}{2-(\log p)^{-1}}<x,\frac{\left[\min\limits_{s=1,\cdots,p}(X_s)\right]^2-c_p}{2-(\log p)^{-1}}<y\right\}$$
$$=G(x)G(y)$$

where distribution function $G(x)=\exp\{-\exp(-x)\}$ and $G(y)=\exp\{-\exp(-y)\}$.

Proof. Please refer to literature [1]. □

If ξ_n is an i.i.d.(standard) normal sequence of r.v.'s, then the asymptotic distribution of $M_n=\max(\xi_1,\xi_2,\cdots,\xi_n)$ is of Type I. Specifically,

$$P\left\{a_n(M_n-b_n)\le x\right\}\to\exp\left(-e^{-x}\right),$$

where

$$a_n=(2\log n)^{1/2}$$

and

$$b_n=(2\log n)^{1/2}-\frac{1}{2}(2\log n)^{-1/2}(\log\log n+\log 4\pi)$$

Proof. Please refer to Theorem 1.5.3 in literature [2]. □

Writing $m_n=\min(\xi_1,\xi_2,\cdots,\xi_n)$, clearly $m_n=-\max(-\xi_1,-\xi_2,\cdots,-\xi_n)$, then

$$P\left\{a_n(m_n+b_n)\le x\right\}\to 1-\exp\left(-e^{x}\right). \tag{1}$$

Suppose $\xi_1,\xi_2,\cdots,\xi_n$ are standard normal variables with covariance matrix $\Lambda^1=(\Lambda^1_{ij})$, and $\eta_1,\eta_2,\cdots,\eta_n$ similarly with covariance matrix $\Lambda^0=(\Lambda^0_{ij})$,and let $\sigma_{ij}=\max(|\Lambda^1_{ij}|,|\Lambda^0_{ij}|)$. Then for any real u ,

$$\begin{aligned}&P\{\xi_j\le u\ for\ j=1,\cdots,n\}-P\{\eta_j\le u\ for\ j=1,\cdots,n\}\\&\le\frac{1}{2\pi}\sum_{1\le i<j\le n}(\Lambda^1_{ij}-\Lambda^0_{ij})^+(1-\sigma^2_{ij})^{-1/2}\exp\left\{-\frac{u^2}{1+\sigma_{ij}}\right\}\end{aligned}$$

where $(x)^+=\max(0,x)$.

Proof. Please refer to Theorem 4.2.1 in literature [2]. ☐

First we recall some notations,

$$d_p(z) = [2 - (\log p)^{-1}]z + c_p$$

where z be any real number, and p be any positive integer.
Condition C.1:

$$\lim_{p\to+\infty} \sum_{1\le i<j\le p} (1-\rho_{i,j}^2)^{-1/2} \exp\left\{-\frac{d_p(z)}{1+|\rho_{i,j}|}\right\} = 0.$$

Let sequence $Z = (Z_1, Z_2, \cdots, Z_p)^T$ follows a p-dimension multivariate normal distribution with mean vector $\mathbf{u} = (0, 0, \cdots, 0)^T$ and covariance matrix $\widetilde{R} = (\rho_{q,s})_{q,s=1,\cdots,p}$. Suppose the above condition C.1 is satisfied for any real x and y, we can obtain that,
(i) For any real number x and y,

$$\lim_{p\to+\infty} P_r\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x\right\} = \exp\{-\exp(-x)\}$$

and

$$\lim_{p\to+\infty} P_r\left\{\frac{\left[\min\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < y\right\} = \exp\{-\exp(-y)\}$$

(ii) For any real number x and y, $\left[\max\limits_{s=1,\cdots,p}(Z_s)\right]^2$ and $\left[\min\limits_{s=1,\cdots,p}(Z_s)\right]^2$ are asymptotically independent,namely,

$$\lim_{p\to+\infty} P_r\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x, \frac{\left[\min\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < y\right\}$$
$$= G(x)G(y)$$

where distribution function $G(x) = \exp\{-\exp(-x)\}$ and $G(y) = \exp\{-\exp(-y)\}$.

Proof. We first prove that (i).

Let $d_p(x) = [2-(\log p)^{-1}]x + c_p,\ d_p(y) = [2-(\log p)^{-1}]y + c_p$, then

$$
\begin{aligned}
&P\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x\right\}\\
=&P\left\{\left[\max_{s=1,\cdots,p}(X_s)\right]^2 < d_p(x)\right\}\\
=&P\left\{-\sqrt{d_p(x)} < \max_{s=1,\cdots,p}(X_s) < \sqrt{d_p(x)}\right\}\\
=&P\left\{\max_{s=1,\cdots,p}(X_s) < \sqrt{d_p(x)}\right\} - P\left\{\max_{s=1,\cdots,p}(X_s) \le -\sqrt{d_p(x)}\right\}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&P\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x\right\}\\
=&P\left\{\left[\max_{s=1,\cdots,p}(Z_s)\right]^2 < d_p(x)\right\}\\
=&P\left\{\max_{s=1,\cdots,p}(Z_s) < \sqrt{d_p(x)}\right\} - P\left\{\max_{s=1,\cdots,p}(Z_s) \le -\sqrt{d_p(x)}\right\}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&P\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x\right\} - P\left\{\frac{\left[\max\limits_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2-(\log p)^{-1}} < x\right\}\\
=&P\left\{\max_{s=1,\cdots,p}(Z_s) < \sqrt{d_p(x)}\right\} - P\left\{\max_{s=1,\cdots,p}(X_s) < \sqrt{d_p(x)}\right\}\\
+&\left\{P\left\{\max_{s=1,\cdots,p}(X_s) \le -\sqrt{d_p(x)}\right\} - P\left\{\max_{s=1,\cdots,p}(Z_s) \le -\sqrt{d_p(x)}\right\}\right\}\\
\le&\frac{1}{2\pi}\sum_{1\le i<j\le p}(\rho_{i,j}-a_{i,j})^+(1-\rho_{i,j}^2)^{-1/2}\exp\left\{-\frac{d_p(x)}{1+|\rho_{i,j}|}\right\}\\
+&\frac{1}{2\pi}\sum_{1\le i<j\le p}(a_{i,j}-\rho_{i,j})^+(1-\rho_{i,j}^2)^{-1/2}\exp\left\{-\frac{d_p(x)}{1+|\rho_{i,j}|}\right\}
\end{aligned}
$$

$$\leq \frac{1}{\pi} \sum_{1 \leq i < j \leq p} (1 - \rho_{i,j}^2)^{-1/2} \exp\left\{ -\frac{d_p(x)}{1 + |\rho_{i,j}|} \right\} \to 0 \; as \; p \to +\infty \tag{2}$$

where the first inequality is obtained by Lemma 1, and the second inequality is due to the condition C.1.

According to formula (2) and Lemma 1, we can obtain

$$\begin{aligned}
&\lim_{p\to+\infty} P_r \left\{ \frac{\left[\max_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2 - (\log p)^{-1}} < x \right\} \\
=&\lim_{p\to+\infty} P_r \left\{ \frac{\left[\max_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2 - (\log p)^{-1}} < x \right\} \\
=&\exp\{-\exp(-x)\}.
\end{aligned}$$

Similarly, when condition C.1 is satisfied, we can obtain

$$\begin{aligned}
&P \left\{ \frac{\left[\min_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2 - (\log p)^{-1}} < x \right\} - P \left\{ \frac{\left[\min_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2 - (\log p)^{-1}} < x \right\} \\
=&P\left\{ \min_{s=1,\cdots,p}(Z_s) < \sqrt{d_p(x)} \right\} - P\left\{ \min_{s=1,\cdots,p}(X_s) < \sqrt{d_p(x)} \right\} \\
+&\left\{ P\left\{ \min_{s=1,\cdots,p}(X_s) \leq -\sqrt{d_p(x)} \right\} - P\left\{ \min_{s=1,\cdots,p}(Z_s) \leq -\sqrt{d_p(x)} \right\} \right\} \\
\leq&\frac{1}{2\pi} \sum_{1 \leq i < j \leq p} (\rho_{i,j} - a_{i,j})^+ (1 - \rho_{i,j}^2)^{-1/2} \exp\left\{ -\frac{d_p(x)}{1 + |\rho_{i,j}|} \right\} \\
+&\frac{1}{2\pi} \sum_{1 \leq i < j \leq p} (a_{i,j} - \rho_{i,j})^+ (1 - \rho_{i,j}^2)^{-1/2} \exp\left\{ -\frac{d_p(x)}{1 + |\rho_{i,j}|} \right\} \\
\leq&\frac{1}{\pi} \sum_{1 \leq i < j \leq p} (1 - \rho_{i,j}^2)^{-1/2} \exp\left\{ -\frac{d_p(x)}{1 + |\rho_{i,j}|} \right\} \to 0 \; as \; p \to +\infty
\end{aligned} \tag{3}$$

then, according to formula (3) and Lemma 1, we can obtain

$$\lim_{p\to+\infty} P_r\left\{\frac{\left[\min\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}} < y\right\}$$

$$= \lim_{p\to+\infty} P_r\left\{\frac{\left[\min\limits_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2-(\log p)^{-1}} < y\right\}$$

$$= \exp\{-\exp(-y)\}$$

Next, we will prove (ii). According to Theorem 11.1.5 in literature [2], we can know that $\max\limits_{s=1,\cdots,p}(Z_s)$ and $\min\limits_{s=1,\cdots,p}(Z_s)$ are asymptotically independent. Obviously, $\left[\max\limits_{s=1,\cdots,p}(Z_s)\right]^2$ and $\left[\min\limits_{s=1,\cdots,p}(Z_s)\right]^2$ are also asymptotically independent.

□

Denote $X = (X_1, X_2, \cdots, X_p)^\top$ follows a p-dimension multivariate standard normal distribution with mean $\mathbf{u} = (0, 0, \cdots, 0)^\top$ and covariance matrix $I_{p\times p} = (a_{q,s})_{q,s=1,\cdots,p}$ where $a_{q,q} = 1$ and $a_{q,s} = 0, q \neq s$. Denotes $\widetilde{\lambda_{1,p}} = \exp\left\{-\frac{\left[\max\limits_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2-(\log p)^{-1}}\right\}$, $\widetilde{\lambda_{2,p}} = \exp\left\{-\frac{\left[\min\limits_{s=1,\cdots,p}(X_s)\right]^2 - c_p}{2-(\log p)^{-1}}\right\}$ and $\widetilde{T_5} = -2\log(1 - e^{-\widetilde{\lambda_{1,p}}}) - 2\log(1 - e^{-\widetilde{\lambda_{2,p}}})$.

Let the non-zero mean $|\mu_s| \geq \log p$, $s \in \Omega$. Suppose $0 < v < 1 - \frac{1}{p^{1/2}\log p}$. Under significance level $\alpha > 0$ and alternative hypothesis $H_1^{'}$, we can obtain

$$\lim_{p\to+\infty} P\{R_\alpha(\widetilde{T_5}) = 1\} = 1.$$

Proof.

$$\begin{aligned}
&P\{R_\alpha(\widetilde{T_5}) = 1\} \\
=&P\{\widetilde{T_5} > C_\alpha\} \\
=&P\left\{-2\log(1 - e^{-\widetilde{\lambda_{1,p}}}) - 2\log(1 - e^{-\widetilde{\lambda_{2,p}}}) > C_\alpha\right\} \\
\geq&P\left\{-2\log(1 - e^{-\widetilde{\lambda_{1,p}}}) > C_\alpha\right\} \\
=&P\left\{\log(1 - e^{-\widetilde{\lambda_{1,p}}}) < -\frac{C_\alpha}{2}\right\} \\
=&P\left\{1 - e^{-\widetilde{\lambda_{1,p}}} < e^{-\frac{C_\alpha}{2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= P\left\{ e^{-\widetilde{\lambda_{1,p}}} > 1 - e^{-\frac{C_\alpha}{2}} \right\} \\
&= P\left\{ -\widetilde{\lambda_{1,p}} > \log\left(1 - e^{-\frac{C_\alpha}{2}}\right) \right\} \\
&= P\left\{ \widetilde{\lambda_{1,p}} < -\log\left(1 - e^{-\frac{C_\alpha}{2}}\right) \right\} \\
&= P\left\{ \exp\left(-\frac{\left(\max\limits_{s=1,\cdots,p}(X_s)\right)^2 - c_p}{2 - (\log p)^{-1}} \right) < -\log\left(1 - e^{-\frac{C_\alpha}{2}}\right) \right\} \\
&= P\left\{ \frac{\left(\max\limits_{s=1,\cdots,p}(X_s)\right)^2 - c_p}{2 - (\log p)^{-1}} > -\log\left(-\log\left(1 - e^{-\frac{C_\alpha}{2}}\right)\right) \right\}, \qquad (4)
\end{aligned}
$$

where we let $h_\alpha = -\log\left(-\log\left(1 - e^{-\frac{C_\alpha}{2}}\right)\right)$, h_α is a constant about α, then we have

$$
\begin{aligned}
& P\left\{ \frac{\left\{\max\limits_{s=1,\cdots,p}(X_s)\right\}^2 - c_p}{2 - (\log p)^{-1}} > -\log\left(-\log\left(1 - e^{-\frac{C_\alpha}{2}}\right)\right) \right\} \\
&= P\left\{ \left(\max_{s=1,\cdots,p}(X_s)\right)^2 > h_\alpha \left[2 - (\log p)^{-1}\right] + c_p \right\} \\
&= 1 - P\left\{ \left(\max_{s=1,\cdots,p}(X_s)\right)^2 \leq h_\alpha \left[2 - (\log p)^{-1}\right] + c_p \right\} \\
&= 1 - P\left\{ \left(\max_{s=1,\cdots,p}(X_s)\right)^2 \leq d_p(h_\alpha) \right\} \\
&= 1 - P\left\{ -\left[d_p(h_\alpha)\right]^{\frac{1}{2}} \leq \max_{s=1,\cdots,p}(X_s) \leq \left[d_p(h_\alpha)\right]^{\frac{1}{2}} \right\} \\
&\geq 1 - P\left\{ \max_{s=1,\cdots,p}(X_s) \leq \left[d_p(h_\alpha)\right]^{\frac{1}{2}} \right\} \\
&= 1 - P\left\{ \max\left[\max_{s\in\Omega}(X_s), \max_{s\in\Omega^c}(X_s)\right] \leq \left[d_p(h_\alpha)\right]^{\frac{1}{2}} \right\} \\
&= 1 - P\left\{ \max_{s\in\Omega}(X_s) \leq \left[d_p(h_\alpha)\right]^{\frac{1}{2}} \right\} \times P\left\{ \max_{s\in\Omega^c}(X_s) \leq \left[d_p(h_\alpha)\right]^{\frac{1}{2}} \right\} \\
&\geq 1 - P\left\{ \max_{s\in\Omega}(X_s) \leq \left[d_p(h_\alpha)\right]^{\frac{1}{2}} \right\}. \qquad (5)
\end{aligned}
$$

The reciprocal third equation is derived from the alternative hypothesis $H_1^{'}$.

We first prove the limit of $P\left\{\max_{s\in\Omega}(X_s) \le [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \to 0$ when $p \to +\infty$, then, by Lemma1, we have

$$
\begin{aligned}
&P\left\{\max_{s\in\Omega}(X_s) \le [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\max_{s\in\Omega}(X_s-\mu_s) \le [d_p(h_\alpha)]^{\frac{1}{2}}-\mu_s\right\}\\
\le&P\left\{\max_{s\in\Omega}(X_s-\mu_s) \le [d_p(h_\alpha)]^{\frac{1}{2}}-\log p\right\}\\
=&P\left\{a_{p^v}\left[\max_{s\in\Omega}(X_s-\mu_s)-b_{p^v}\right] \le a_{p^v}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^v}\right]\right\}\\
\to&\exp\left\{-\exp\left\{-a_{p^v}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^v}\right]\right\}\right\},
\end{aligned}
$$

where $a_{p^v}=(2\log p^v)^{1/2}$, $b_{p^v}=(2\log p^v)^{1/2}-\frac{1}{2}(2\log p^v)^{-1/2}(\log\log p^v+\log 4\pi)$, $c_p=2\log(p)-[\log(\log p)+\log(4\pi)]+\frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.

Since $0<v<1-\frac{1}{p^{1/2}\log p}$, when $p\to+\infty$, it is obvious that

$$
a_{p^v}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^v}\right]=O\left(-\sqrt{2v}(\log p)^{\frac{3}{2}}\right)\to-\infty,
$$

which means that

$$
\exp\left\{-\exp\left\{-a_{p^v}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^v}\right]\right\}\right\}\to 0.
$$

So, we have

$$
P\left\{\max_{s\in\Omega}(X_s) \le [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \to 0, as\ p\to+\infty. \tag{6}
$$

Next, according to formula (5) and (6), applying the squeeze theorem, we have

$$
\lim_{p\to+\infty}\left\{1-P\left\{\max_{s=1,\cdots,p}(X_s)\le[d_p(h_\alpha)]^{\frac{1}{2}}\right\}\right\}=1. \tag{7}
$$

According to formulas (4) and (5), we have the following inequalities

$$
1\ge P\{R_\alpha(\widetilde{T_5})=1\}\ge 1-P\left\{\max_{s=1,\cdots,p}(X_s)\le[d_p(h_\alpha)]^{\frac{1}{2}}\right\},
$$

applying the squeeze theorem, by formula (7), we have

$$
\lim_{p\to+\infty}P\{R_\alpha(\widetilde{T_5})=1\}=1.
$$

□

Let the non-zero mean $|\mu_s| \geq \log p$, $s \in \Omega$. Suppose $0 < v < 1 - \frac{1}{p^{1/2}\log p}$. Under significance level $\alpha > 0$ and alternative hypothesis H_1, we can obtain

$$\lim_{p\to+\infty} P\{R_\alpha(\widetilde{T_5}) = 1\} = 1.$$

Proof. According to the proof of Lemma 1, by applying formula (4) and (5) , we have

$$\begin{aligned}
& P\{R_\alpha(\widetilde{T_5}) = 1\} \\
=& P\{\widetilde{T_5} > C_\alpha\} \\
=& P\left\{-2\log(1 - e^{-\widetilde{\lambda_{1,p}}}) - 2\log(1 - e^{-\widetilde{\lambda_{2,p}}}) > C_\alpha\right\} \\
\geq& P\left\{-2\log(1 - e^{-\widetilde{\lambda_{1,p}}}) > C_\alpha\right\} \\
=& P\left\{\frac{\left(\max\limits_{s=1,\cdots,p}(X_s)\right)^2 - c_p}{2 - (\log p)^{-1}} > -\log\left(-\log\left(1 - e^{-\frac{C_\alpha}{2}}\right)\right)\right\} \\
=& 1 - P\left\{-[d_p(h_\alpha)]^{\frac{1}{2}} \leq \max_{s=1,\cdots,p}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \\
\geq& 1 - P\left\{\max_{s=1,\cdots,p}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \\
=& 1 - P\left\{\max\left[\max_{s\in\Omega_1}(X_s), \max_{s\in\Omega_2}(X_s), \max_{s\in\Omega^c}(X_s)\right] \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \\
=& 1 - P\left\{\max_{s\in\Omega_1}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \times P\left\{\max_{s\in\Omega_2}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \times \\
& P\left\{\max_{s\in\Omega^c}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \\
\geq& 1 - P\left\{\max_{s\in\Omega_1}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\}.
\end{aligned} \tag{8}$$

The reciprocal third equation is derived from the alternative hypothesis H_1.

We first prove the limit of $P\left\{\max\limits_{s\in\Omega_1}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \to 0$ when $p \to +\infty$, then, by Lemma1, we have

$$\begin{aligned}
& P\left\{\max_{s\in\Omega_1}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \\
=& P\left\{\max_{s\in\Omega_1}(X_s - \mu_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}} - \mu_s\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le P\left\{\max_{s\in\Omega_1}(X_s-\mu_s)\le [d_p(h_\alpha)]^{\frac{1}{2}}-\log p\right\}\\
&=P\left\{a_{p^{v_1}}\left[\max_{s\in\Omega_1}(X_s-\mu_s)-b_{p^{v_1}}\right]\le a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^{v_1}}\right]\right\}\\
&\to\exp\left\{-\exp\left\{-a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^{v_1}}\right]\right\}\right\},
\end{aligned}
$$

where $a_{p^{v_1}} = (2\log p^{v_1})^{1/2}$, $b_{p^{v_1}} = (2\log p^{v_1})^{1/2} - \frac{1}{2}(2\log p^{v_1})^{-1/2}(\log\log p^{v_1} + \log 4\pi)$, $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.

Since $0 < v_1 < v < 1 - \frac{1}{p^{1/2}\log p}$, when $p \to +\infty$, it is obvious that

$$
a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p-b_{p^{v_1}}\right]=O\left(-\sqrt{2v_1}(\log p)^{\frac{3}{2}}\right)\to-\infty,
$$

which means that

$$
\exp\left\{-\exp\left\{-a_{p^{v_1}}\left[\left(h_\alpha\left(2-(\log p)^{-1}\right)+c_p\right)^{\frac{1}{2}}-\log p-b_{p^{v_1}}\right]\right\}\right\}\to 0.
$$

So, we have

$$
P\left\{\max_{s\in\Omega_1}(X_s)\le [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\to 0, as\ p\to+\infty. \tag{9}
$$

According to formula (8), we have the following inequalities

$$
1\ge P\{R_\alpha(\widetilde{T_5})=1\}\ge 1-P\left\{\max_{s\in\Omega_1}(X_s)\le [d_p(h_\alpha)]^{\frac{1}{2}}\right\},
$$

applying the squeeze theorem, by formula (9) we have

$$
\lim_{p\to+\infty}P\{R_\alpha(\widetilde{T_5})=1\}=1.
$$

□

2 PROOFS OF LEMMAS AND THEOREMS

Proof of Theorem 1. For convenience, let $W_1 = \max_{s=1,\cdots,p}\{Z_s\}$, $W_2 = \min_{s=1,\cdots,p}\{Z_s\}$ According to Lemma 1, for any real numbers x and y,

$$
\begin{aligned}
c_p&=2\log(p)-[\log(\log p)+\log(4\pi)]+\frac{\log(\log p)+\log(4\pi)}{2\log(p)}\\
\lim_{p\to+\infty}P_r&\left(\frac{{W_1}^2-c_p}{2-(\log p)^{-1}}<x\right)=\exp\{-\exp(-x)\}=G(x)\\
\lim_{p\to+\infty}P_r&\left(\frac{{W_2}^2-c_p}{2-(\log p)^{-1}}<y\right)=\exp\{-\exp(-y)\}=G(y)
\end{aligned}
$$

and ${W_1}^2$ and ${W_2}^2$ are asymptotically independent we can obtain that,

$$P_{{W_1}^2} = 1 - G\left(\frac{{W_1}^2 - c_p}{2 - (\log p)^{-1}}\right) = 1 - \exp\left\{-\exp\left(-\frac{{W_1}^2 - c_p}{2 - (\log p)^{-1}}\right)\right\},$$
$$P_{{W_2}^2} = 1 - G\left(\frac{{W_2}^2 - c_p}{2 - (\log p)^{-1}}\right) = 1 - \exp\left\{-\exp\left(-\frac{{W_2}^2 - c_p}{2 - (\log p)^{-1}}\right)\right\}.$$

Denote $\lambda_{1,p} = \exp\left(-\frac{{W_1}^2 - c_p}{2-(\log p)^{-1}}\right)$ and $\lambda_{2,p} = \exp\left(-\frac{{W_2}^2 - c_p}{2-(\log p)^{-1}}\right)$, we can obtain that,

$$P_{{W_1}^2} = 1 - e^{-\lambda_{1,p}} \sim U(0,1)$$
$$P_{{W_2}^2} = 1 - e^{-\lambda_{2,p}} \sim U(0,1).$$

Then, according to the literature [3], we have

$$-2\log(P_{{W_1}^2}) = -2\log(1 - e^{-\lambda_{1,p}}) \sim \chi^2(2)$$
$$-2\log(P_{{W_2}^2}) = -2\log(1 - e^{-\lambda_{2,p}}) \sim \chi^2(2),$$

where $\chi^2(2)$ denotes the chi-square distribution with degrees of freedom 2, and the distribution function is $F(x,n) = \int_0^x \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} t^{\frac{n}{2}-1} e^{-\frac{t}{2}}\,\mathrm{d}t$.

Because ${W_1}^2$ and ${W_2}^2$ are asymptotically independent when $p \to +\infty$, and $-2\log(1 - e^{-\lambda_{1,p}})$ and $-2\log(1 - e^{-\lambda_{2,p}})$ are functions of ${W_1}^2$ and ${W_2}^2$, respectively, then $-2\log(1 - e^{-\lambda_{1,p}})$ and $-2\log(1 - e^{-\lambda_{2,p}})$ are also asymptotically independent when $p \to +\infty$.

Through the regeneration of chi-square distribution,when $p \to +\infty$, we can obtain that,

$$T_5 = -2\log(1 - e^{-\lambda_{1,p}}) - 2\log(1 - e^{-\lambda_{2,p}}) \sim \chi^2(4).$$

□

Next is the proof of the asymptotic power $R_\alpha(T_5)$ of T_5. Before the proof, we first recall some symbols about T_5.

$$c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p) + \log(4\pi)}{2\log(p)},$$
$$\lambda_{1,p} = \exp\left(-\frac{\left[\max_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2 - (\log p)^{-1}}\right),$$

$$\lambda_{2,p} = \exp\left(-\frac{\left[\min\limits_{s=1,\cdots,p}(Z_s)\right]^2 - c_p}{2-(\log p)^{-1}}\right),$$

$$T_5 = -2\log(1-e^{-\lambda_{1,p}}) - 2\log(1-e^{-\lambda_{2,p}}).$$

Proof of Theorem 2.

$$\begin{aligned}
&P\{R_\alpha(T_5)=1\}\\
=&P\{T_5 > C_\alpha\}\\
=&P\left\{-2\log(1-e^{-\lambda_{1,p}}) - 2\log(1-e^{-\lambda_{2,p}}) > C_\alpha\right\}\\
\geq&P\left\{-2\log(1-e^{-\lambda_{1,p}}) > C_\alpha\right\}\\
=&P\left\{\log(1-e^{-\lambda_{1,p}}) < -\frac{C_\alpha}{2}\right\}\\
=&P\left\{-\lambda_{1,p} > \log\left(1-e^{-\frac{C_\alpha}{2}}\right)\right\}\\
=&P\left\{\lambda_{1,p} < -\log\left(1-e^{-\frac{C_\alpha}{2}}\right)\right\}\\
=&P\left\{\exp\left(-\frac{\left(\max\limits_{s=1,\cdots,p}(Z_s)\right)^2 - c_p}{2-(\log p)^{-1}}\right) < -\log\left(1-e^{-\frac{C_\alpha}{2}}\right)\right\}\\
=&P\left\{\frac{\left(\max\limits_{s=1,\cdots,p}(Z_s)\right)^2 - c_p}{2-(\log p)^{-1}} > -\log\left(-\log\left(1-e^{-\frac{C_\alpha}{2}}\right)\right)\right\}\\
=&P\left\{\left(\max\limits_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right] + c_p\right\}\\
=&1-P\left\{\left(\max\limits_{s=1,\cdots,p}(Z_s)\right)^2 \leq h_\alpha\left[2-(\log p)^{-1}\right] + c_p\right\}
\end{aligned} \tag{10}$$

Since condition C.1 holds, according to (2),we can deduce that

$$\begin{aligned}
&\lim_{p\to+\infty} P\left\{\left(\max\limits_{s=1,\cdots,p}(Z_s)\right)^2 \leq h_\alpha\left[2-(\log p)^{-1}\right] + c_p\right\}\\
=&\lim_{p\to+\infty} P\left\{\left(\max\limits_{s=1,\cdots,p}(X_s)\right)^2 \leq h_\alpha\left[2-(\log p)^{-1}\right] + c_p\right\}.
\end{aligned} \tag{11}$$

According to the formulas (5) and (7), applying the Squeeze Theorem, we have

$$\lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(X_s)\right)^2 \le h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}=0$$

According to (11),we have

$$\begin{aligned}
&\lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&1-\lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 \le h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&1-\lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(X_s)\right)^2 \le h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&1.
\end{aligned}$$

According to formula (10), we can obtain the following inequalities

$$1 \ge P\{R_\alpha(T_5)=1\} \ge P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}, \tag{12}$$

applying the squeeze theorem, we have

$$\lim_{p\to+\infty} P\{R_\alpha(T_5)=1\}=1$$

□

Proof of Theorem 3. The proof of Theorem 3.2 and Lemma 1 is similar. According to (10), we have

$$\begin{aligned}
&P\{R_\alpha(T_5)=1\}\\
=&P\{T_5 > C_\alpha\}\\
=&P\left\{-2\log(1-e^{-\lambda_{1,p}})-2\log(1-e^{-\lambda_{2,p}}) > C_\alpha\right\}\\
\ge&P\left\{-2\log(1-e^{-\lambda_{1,p}}) > C_\alpha\right\}\\
=&P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&1-P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 \le h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}
\end{aligned}$$

$$=1 - P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 \leq d_p(h_\alpha)\right\}$$

Since condition C.1 holds, according to (2), (8) and (9), under the alternative hypothesis H_1, we have

$$\begin{aligned}
&\lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(h_\alpha)\right\}\\
=&1 - \lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 \leq d_p(h_\alpha)\right\}\\
=&1 - \lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(X_s)\right)^2 \leq d_p(h_\alpha)\right\}\\
=&1 - \lim_{p\to+\infty} P\left\{-[d_p(h_\alpha)]^{\frac{1}{2}} \leq \max_{s=1,\cdots,p}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
\geq&1 - \lim_{p\to+\infty} P\left\{\max_{s=1,\cdots,p}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&1 - \lim_{p\to+\infty} P\left\{\max\left[\max_{s\in\Omega_1}(X_s), \max_{s\in\Omega_2}(X_s), \max_{s\in\Omega^c}(X_s)\right] \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
\geq&1 - \lim_{p\to+\infty} P\left\{\max_{s\in\Omega_1}(X_s) \leq [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&1 - 0\\
=&1.
\end{aligned}$$

Which means that

$$\lim_{p\to+\infty} P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(h_\alpha)\right\} = 1.$$

According to the inequality

$$1 \geq P\{R_\alpha(T_5) = 1\} \geq P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(h_\alpha)\right\},$$

applying the squeeze theorem, we have

$$\lim_{p\to+\infty} P\{R_\alpha(T_5) = 1\} = 1.$$

□

Proof of Theorem 4. Since $P(T_5 > C_\alpha) = P\{R_\alpha(T_5) = 1\}$,according to the above proof of Theorem 3, we can get the following inequality

$$\beta_{T_5} \geq P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}, \tag{13}$$

there the inequality follow from (12), C_α is a constant about α, and it may represent different constants at different positions.

$$\begin{aligned}
\beta_{T_4} &= P\left\{\frac{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2+\left(\min_{s=1,\cdots,p}(Z_s)\right)^2-2c_p}{2-(\log p)^{-1}} > C_\alpha\right\} \\
=&P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2+\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha[2-(\log p)^{-1}]+2c_p\right\} \\
\leq&P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha[2-(\log p)^{-1}]+2c_p\right\}+ \\
&P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha[2-(\log p)^{-1}]+2c_p\right\}
\end{aligned} \tag{14}$$

Because when α is fixed, $p \to +\infty$, we have

$$C_\alpha[2-(\log p)^{-1}]+2c_p \geq h_\alpha\left[2-(\log p)^{-1}\right]+c_p.$$

Then when $p \to +\infty$, we have

$$\begin{aligned}
&P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha[2-(\log p)^{-1}]+2c_p\right\} \\
\leq&P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\} \\
\leq&P(T_5 > C_\alpha) \\
=&\beta_{T_5}
\end{aligned} \tag{15}$$

there the second inequality follow from (13).
Similarly, we can also conclude that

$$P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha[2-(\log p)^{-1}]+2c_p\right\}$$

$$\leq P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\} \tag{16}$$

Since condition C.1 holds, according to (3),we can deduce that

$$\begin{aligned}
&\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&1-\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 \leq h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&1-\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 \leq h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}. \qquad (17)
\end{aligned}$$

First we recall some notations.Under alternative hypothesis H_1, denote the index set of nonzero signals by $\Omega = \{i : 1 \leq i \leq p, \mu_i \neq 0\}$ and let the total number of nonzero signals be p^v, where the parameter $v(0 < v < 1)$ measures the sparsity of the nonzero mean. Similarly, denote the index set of zero signals by $\Omega^c = \{i : 1 \leq i \leq p, \mu_i = 0\}$ and let the total number of zero signals be p^k, where $0 < k < 1$ and $p^v + p^k = p$. Since non-zero signals include positive and negative, we divide Ω into two parts, where $\Omega_1 = \{i : 1 \leq i \leq p, \mu_i > 0\}$ represents the index set of positive non-zero signals, and let the total number of non-zero signals be p^{v_1} $(0 < v_1 < v)$, $\Omega_2 = \{i : 1 \leq i \leq p, \mu_i < 0\}$ represents the index set of negative non-zero signals, and let the total number of non-zero signals be p^{v_2} $(0 < v_2 < v)$, $p^{v_1} + p^{v_2} = p^v$.

Now, we have

$$\begin{aligned}
&P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 > h_\alpha\left[2-(\log p)^{-1}\right]+c_p\right\}\\
=&P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 > d_p(h_\alpha)\right\}\\
=&P\left\{\min_{s=1,\cdots,p}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}+P\left\{\min_{s=1,\cdots,p}(X_s) < -[d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
\leq&P\left\{\min_{s=1,\cdots,p}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min\left[\min_{s\in\Omega_1}(X_s),\min_{s\in\Omega_2}(X_s),\min_{s\in\Omega^c}(X_s)\right] > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\times P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\times
\end{aligned}$$

$$P\left\{\min_{s\in\Omega^c}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}. \tag{18}$$

The penultimate equation is from is obtained under the alternative hypothesis H_1.

We first prove the limit of $P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \to 1$ when $p \to +\infty$, then, by formula (1), we have

$$\begin{aligned}
&P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min_{s\in\Omega_1}(X_s-\mu_s) > [d_p(h_\alpha)]^{\frac{1}{2}}-\mu_s\right\}\\
\geq&P\left\{\min_{s\in\Omega_1}(X_s-\mu_s) > [d_p(h_\alpha)]^{\frac{1}{2}}-\log p\right\}\\
=&P\left\{a_{p^{v_1}}\left[\min_{s\in\Omega_1}(X_s-\mu_s)+b_{p^{v_1}}\right] > a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p+b_{p^{v_1}}\right]\right\}\\
\to&\exp\left\{-\exp\left\{a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p+b_{p^{v_1}}\right]\right\}\right\},
\end{aligned}$$

where $a_{p^{v_1}} = (2\log p^{v_1})^{1/2}$, $b_{p^{v_1}} = (2\log p^{v_1})^{1/2} - \frac{1}{2}(2\log p^{v_1})^{-1/2}(\log\log p^{v_1} + \log 4\pi)$, $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.

Since $0 < v_1 < v < 1 - \frac{1}{p^{1/2}\log p}$, when $p \to +\infty$, it is obvious that

$$a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p+b_{p^{v_1}}\right] = O\left(-\sqrt{2v_1}(\log p)^{\frac{3}{2}}\right) \to -\infty,$$

which means that

$$\exp\left\{-\exp\left\{a_{p^{v_1}}\left[(d_p(h_\alpha))^{\frac{1}{2}}-\log p+b_{p^{v_1}}\right]\right\}\right\} \to 1,$$

So, we have

$$P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \to 1, as\ p \to +\infty. \tag{19}$$

Secondly, we prove that $P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \to 0$ as $p \to +\infty$, then, by formula (1), we have

$$\begin{aligned}
&P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min_{s\in\Omega_2}(X_s-\mu_s) > [d_p(h_\alpha)]^{\frac{1}{2}}-\mu_s\right\}\\
\leq&P\left\{\min_{s\in\Omega_2}(X_s-\mu_s) > [d_p(h_\alpha)]^{\frac{1}{2}}+\log p\right\}
\end{aligned}$$

$$
\begin{aligned}
&=P\left\{a_{p^{v_2}}\left[\min_{s\in\Omega_2}(X_s-\mu_s)+b_{p^{v_2}}\right]>a_{p^{v_2}}\left[(d_p(h_\alpha))^{\frac{1}{2}}+\log p+b_{p^{v_2}}\right]\right\}\\
&\rightarrow\exp\left\{-\exp\left\{a_{p^{v_2}}\left[(d_p(h_\alpha))^{\frac{1}{2}}+\log p+b_{p^{v_2}}\right]\right\}\right\},
\end{aligned}
$$

where $a_{p^{v_2}} = (2\log p^{v_2})^{1/2}$, $b_{p^{v_2}} = (2\log p^{v_2})^{1/2} - \frac{1}{2}(2\log p^{v_2})^{-1/2}(\log\log p^{v_2} + \log 4\pi)$, $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.

Since $0 < v_2 < v < 1 - \frac{1}{p^{1/2}\log p}$, when $p \to +\infty$, it is obvious that

$$
a_{p^{v_2}}\left[(d_p(h_\alpha))^{\frac{1}{2}}+\log p+b_{p^{v_2}}\right]=O\left(\sqrt{2v_2}(\log p)^{\frac{3}{2}}\right)\rightarrow+\infty,
$$

which means that

$$
\exp\left\{-\exp\left\{a_{p^{v_2}}\left[(d_p(h_\alpha))^{\frac{1}{2}}+\log p+b_{p^{v_2}}\right]\right\}\right\}\rightarrow 0, as\ p\rightarrow+\infty.
$$

So, we have

$$
P\left\{\min_{s\in\Omega_2}(X_s)>[d_p(h_\alpha)]^{\frac{1}{2}}\right\}\rightarrow 0, as\ p\rightarrow+\infty. \tag{20}
$$

Finally,we prove the limit of $P\left\{\min_{s\in\Omega^c}(X_s)>[d_p(h_\alpha)]^{\frac{1}{2}}\right\}\rightarrow 0$ when $p\rightarrow+\infty$. By formula (1), we have

$$
\begin{aligned}
&P\left\{\min_{s\in\Omega^c}(X_s)>[d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{a_{p^k}\left[\min_{s\in\Omega^c}(X_s)+b_{p^k}\right]>a_{p^k}\left[(d_p(h_\alpha))^{\frac{1}{2}}+b_{p^k}\right]\right\}\\
\rightarrow&\exp\left\{-\exp\left\{a_{p^k}\left[(d_p(h_\alpha))^{\frac{1}{2}}+b_{p^k}\right]\right\}\right\},
\end{aligned}
$$

where $a_{p^k} = (2\log p^k)^{1/2}$,$c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$, $b_{p^k} = (2\log p^k)^{1/2} - \frac{1}{2}(2\log p^k)^{-1/2}(\log\log p^k + \log 4\pi)$.

Since $0 < k < 1$, it is obvious that

$$
a_{p^k}\left[(d_p(h_\alpha))^{\frac{1}{2}}+b_{p^k}\right]=O\left(2\sqrt{k}\log p\right)\rightarrow+\infty, as\ p\rightarrow+\infty.
$$

which means that

$$
\exp\left\{-\exp\left\{a_{p^k}\left[(d_p(h_\alpha))^{\frac{1}{2}}+b_{p^k}\right]\right\}\right\}\rightarrow 0, as\ p\rightarrow+\infty.
$$

So, we have

$$
P\left\{\max_{s\in\Omega^c}(X_s)>[d_p(h_\alpha)]^{\frac{1}{2}}\right\}\rightarrow 0, as\ p\rightarrow+\infty. \tag{21}
$$

Next, according to formula (18),(19),(20) and (21) we have

$$
\begin{aligned}
&\lim_{p\to+\infty} P\left\{\min_{s=1,\cdots,p}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&\lim_{p\to+\infty} P\left\{\min\left[\min_{s\in\Omega_1}(X_s), \min_{s\in\Omega_2}(X_s), \min_{s\in\Omega^c}(X_s)\right] > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&\lim_{p\to+\infty} P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \times \lim_{p\to+\infty} P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\} \times\\
&\lim_{p\to+\infty} P\left\{\min_{s\in\Omega^c}(X_s) > [d_p(h_\alpha)]^{\frac{1}{2}}\right\}\\
=&1\times 0\times 0\\
=&0. \qquad (22)
\end{aligned}
$$

Next, according to formula (16), (17) and (18),applying the squeeze theorem, we have

$$
\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 \geq C_\alpha[2-(\log p)^{-1}] + 2c_p\right\} = 0. \qquad (23)
$$

Now, according to (14),(15) and (23),we have

$$
\beta_{T_4} \leq \beta_{T_5} + o(1).
$$

□

Proof of Theorem 5.

$$
\begin{aligned}
\beta_{T_5} =&P\{T_5 > C_\alpha\}\\
=&P\left\{-2\log(1-e^{-\lambda_{1,p}}) - 2\log(1-e^{-\lambda_{2,p}}) > C_\alpha\right\}\\
\geq&P\left\{-4\log(1-e^{-\lambda_{1,p}}) > C_\alpha\right\}\\
=&P\left\{\log(1-e^{-\lambda_{1,p}}) < -\frac{C_\alpha}{4}\right\}\\
=&P\left\{1-e^{-\lambda_{1,p}} < e^{-\frac{C_\alpha}{4}}\right\}\\
=&P\left\{e^{-\lambda_{1,p}} > 1-e^{-\frac{C_\alpha}{4}}\right\}\\
=&P\left\{-\lambda_{1,p} > \log\left(1-e^{-\frac{C_\alpha}{4}}\right)\right\}\\
=&P\left\{\lambda_{1,p} < -\log\left(1-e^{-\frac{C_\alpha}{4}}\right)\right\}\\
=&P\left\{\exp\left(-\frac{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 - c_p}{2-(\log p)^{-1}}\right) < -\log\left(1-e^{-\frac{C_\alpha}{4}}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= P\left\{\frac{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 - c_p}{2-(\log p)^{-1}} > -\log\left(-\log\left(1-e^{-\frac{C_\alpha}{4}}\right)\right)\right\} \\
&= P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(g_\alpha)\right\}, \qquad (24)
\end{aligned}
$$

where $g_\alpha = -\log\left(-\log\left(1-e^{-\frac{C_\alpha}{4}}\right)\right) = -\log\left(-\log\left(1-e^{-\frac{\chi^2_\alpha(4)}{4}}\right)\right)$.

$$
\begin{aligned}
&\beta_{T_2} = P\left\{\max_{s=1,\cdots,p}(Z_s^2) - 2\log p + \log(\log p) > C_\alpha\right\} \\
&= P\left\{\max_{s=1,\cdots,p}(Z_s^2) > C_\alpha + 2\log p - \log(\log p)\right\} \\
&= P\left\{\max\left[\left(\max_{s=1,\cdots,p}(Z_s)\right)^2, \left(\min_{s=1,\cdots,p}(Z_s)\right)^2\right] > C_\alpha + \log\frac{p^2}{\log p}\right\} \\
&\leq P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha + \log\frac{p^2}{\log p}\right\} + \\
&\quad P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha + \log\frac{p^2}{\log p}\right\}, \qquad (25)
\end{aligned}
$$

where $C_\alpha = -2\log[-\sqrt{\pi}\log(1-\alpha)]$.

For a fixed α, we have

$$
\begin{aligned}
&\lim_{p\to+\infty} C_\alpha + \log\frac{p^2}{\log p} - g_\alpha\left[2-(\log p)^{-1}\right] - c_p \\
&= \lim_{p\to+\infty} C_\alpha - g_\alpha\left[2-(\log p)^{-1}\right] + \log 4\pi - \frac{\log(\log p) + \log 4\pi}{2\log p} \\
&= C_\alpha - 2g_\alpha + \log 4\pi \\
&= -2\log[-\sqrt{\pi}\log(1-\alpha)] + 2\log\left(-\log\left(1-e^{-\frac{\chi^2_\alpha(4)}{4}}\right)\right) + \log 4\pi \\
&> 0
\end{aligned}
$$

This means that when $p \to +\infty$, $C_\alpha + \log\frac{p^2}{\log p} > d_p(g_\alpha)$.

Then when $p \to +\infty$, we have

$$
P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha + \log\frac{p^2}{\log p}\right\}
$$

$$
\begin{aligned}
&\leq P\left\{\left(\max_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(g_\alpha)\right\}\\
&\leq P\left(T_5 > C_\alpha\right)\\
&=\beta_{T_5}
\end{aligned}
\tag{26}
$$

there the second inequality follow from (24).

Similarly, we can also conclude that

$$
\begin{aligned}
&P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > C_\alpha + \log\frac{p^2}{\log p}\right\}\\
\leq &P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(g_\alpha)\right\}
\end{aligned}
\tag{27}
$$

Since condition C.1 holds, according to (3),we can deduce that

$$
\begin{aligned}
&\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 > d_p(g_\alpha)\right\}\\
=&1-\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 \leq d_p(g_\alpha)\right\}\\
=&1-\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 \leq d_p(g_\alpha)\right\}\\
=&\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 > d_p(g_\alpha)\right\}.
\end{aligned}
\tag{28}
$$

First we recall some notations.Under alternative hypothesis H_1, denote the index set of nonzero signals by $\Omega = \{i : 1 \leq i \leq p, \mu_i \neq 0\}$ and let the total number of nonzero signals be p^v, where the parameter $v(0 < v < 1)$ measures the sparsity of the nonzero mean. Similarly, denote the index set of zero signals by $\Omega^c = \{i : 1 \leq i \leq p, \mu_i = 0\}$ and let the total number of zero signals be p^k, where $0 < k < 1$ and $p^v + p^k = p$. Since non-zero signals include positive and negative, we divide Ω into two parts, where $\Omega_1 = \{i : 1 \leq i \leq p, \mu_i > 0\}$ represents the index set of positive non-zero signals, and let the total number of non-zero signals be p^{v_1} $(0 < v_1 < v)$, $\Omega_2 = \{i : 1 \leq i \leq p, \mu_i < 0\}$ represents the index set of negative non-zero signals, and let the total number of non-zero signals be p^{v_2} $(0 < v_2 < v)$, $p^{v_1} + p^{v_2} = p^v$.

Now, we have

$$
P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 > d_p(g_\alpha)\right\}
$$

$$
\begin{aligned}
=&P\left\{\left(\min_{s=1,\cdots,p}(X_s)\right)^2 > d_p(g_\alpha)\right\}\\
=&P\left\{\min_{s=1,\cdots,p}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} + P\left\{\min_{s=1,\cdots,p}(X_s) < -[d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
\leq&P\left\{\min_{s=1,\cdots,p}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min\left[\min_{s\in\Omega_1}(X_s), \min_{s\in\Omega_2}(X_s), \min_{s\in\Omega^c}(X_s)\right] > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \times P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \times\\
&P\left\{\min_{s\in\Omega^c}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}. \qquad (29)
\end{aligned}
$$

The penultimate equation is from is obtained under the alternative hypothesis H_1.

We first prove the limit of $P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \to 1$ when $p \to +\infty$, then, by formula (1), we have

$$
\begin{aligned}
&P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
=&P\left\{\min_{s\in\Omega_1}(X_s-\mu_s) > [d_p(g_\alpha)]^{\frac{1}{2}} - \mu_s\right\}\\
\geq&P\left\{\min_{s\in\Omega_1}(X_s-\mu_s) > [d_p(g_\alpha)]^{\frac{1}{2}} - \log p\right\}\\
=&P\left\{a_{p^{v_1}}\left[\min_{s\in\Omega_1}(X_s-\mu_s) + b_{p^{v_1}}\right] > a_{p^{v_1}}\left[(d_p(g_\alpha))^{\frac{1}{2}} - \log p + b_{p^{v_1}}\right]\right\}\\
\to&\exp\left\{-\exp\left\{a_{p^{v_1}}\left[(d_p(g_\alpha))^{\frac{1}{2}} - \log p + b_{p^{v_1}}\right]\right\}\right\},
\end{aligned}
$$

where $a_{p^{v_1}} = (2\log p^{v_1})^{1/2}$, $b_{p^{v_1}} = (2\log p^{v_1})^{1/2} - \frac{1}{2}(2\log p^{v_1})^{-1/2}(\log\log p^{v_1} + \log 4\pi)$, $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.

Since $0 < v_1 < v < 1 - \frac{1}{p^{1/2}\log p}$, when $p \to +\infty$, it is obvious that

$$
a_{p^{v_1}}\left[(d_p(g_\alpha))^{\frac{1}{2}} - \log p + b_{p^{v_1}}\right] = O\left(-\sqrt{2v_1}(\log p)^{\frac{3}{2}}\right) \to -\infty,
$$

which means that

$$
\exp\left\{-\exp\left\{a_{p^{v_1}}\left[(d_p(g_\alpha))^{\frac{1}{2}} - \log p + b_{p^{v_1}}\right]\right\}\right\} \to 1, as\ p \to +\infty.
$$

So, we have

$$P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \to 1, as\ p \to +\infty. \tag{30}$$

Secondly, we prove that $P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \to 0$ as $p \to +\infty$, then, by formula (1), we have

$$\begin{aligned}
0 \leq & P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \\
= & P\left\{\min_{s\in\Omega_2}(X_s-\mu_s) > [d_p(g_\alpha)]^{\frac{1}{2}} - \mu_s\right\} \\
\leq & P\left\{\min_{s\in\Omega_2}(X_s-\mu_s) > [d_p(g_\alpha)]^{\frac{1}{2}} + \log p\right\} \\
= & P\left\{a_{p^{v_2}}\left[\min_{s\in\Omega_2}(X_s-\mu_s) + b_{p^{v_2}}\right] > a_{p^{v_2}}\left[(d_p(g_\alpha))^{\frac{1}{2}} + \log p + b_{p^{v_2}}\right]\right\} \\
\to & \exp\left\{-\exp\left\{a_{p^{v_2}}\left[(d_p(g_\alpha))^{\frac{1}{2}} + \log p + b_{p^{v_2}}\right]\right\}\right\},
\end{aligned}$$

where $a_{p^{v_2}} = (2\log p^{v_2})^{1/2}$, $b_{p^{v_2}} = (2\log p^{v_2})^{1/2} - \frac{1}{2}(2\log p^{v_2})^{-1/2}(\log\log p^{v_2} + \log 4\pi)$, $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.
Since $0 < v_2 < v < 1 - \frac{1}{p^{1/2}\log p}$, when $p \to +\infty$, it is obvious that

$$a_{p^{v_2}}\left[(d_p(g_\alpha))^{\frac{1}{2}} + \log p + b_{p^{v_2}}\right] = O\left(\sqrt{2v_2}(\log p)^{\frac{3}{2}}\right) \to +\infty,$$

which means that

$$\exp\left\{-\exp\left\{a_{p^{v_2}}\left[(d_p(g_\alpha))^{\frac{1}{2}} + \log p + b_{p^{v_2}}\right]\right\}\right\} \to 0, as\ p \to +\infty.$$

So, we have

$$P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \to 0, as\ p \to +\infty. \tag{31}$$

Finally,we prove the limit of $P\left\{\min_{s\in\Omega^c}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \to 0$ when $p \to +\infty$. By formula (1), we have

$$\begin{aligned}
& P\left\{\min_{s\in\Omega^c}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \\
= & P\left\{a_{p^k}\left[\min_{s\in\Omega^c}(X_s) + b_{p^k}\right] > a_{p^k}\left[(d_p(g_\alpha))^{\frac{1}{2}} + b_{p^k}\right]\right\} \\
\to & \exp\left\{-\exp\left\{a_{p^k}\left[(d_p(g_\alpha))^{\frac{1}{2}} + b_{p^k}\right]\right\}\right\},
\end{aligned}$$

where $a_{p^k} = (2\log p^k)^{1/2}$, $b_{p^k} = (2\log p^k)^{1/2} - \frac{1}{2}(2\log p^k)^{-1/2}(\log\log p^k + \log 4\pi)$, $c_p = 2\log(p) - [\log(\log p) + \log(4\pi)] + \frac{\log(\log p)+\log(4\pi)}{2\log(p)}$.

Since $0 < k < 1$, it is obvious that

$$a_{p^k}\left[(d_p(g_\alpha))^{\frac{1}{2}} + b_{p^k}\right] = O\left(2\sqrt{k}\log p\right) \to +\infty, as\ p \to +\infty.$$

which means that

$$\exp\left\{-\exp\left\{a_{p^k}\left[(d_p(g_\alpha))^{\frac{1}{2}} + b_{p^k}\right]\right\}\right\} \to 0, as\ p \to +\infty.$$

So, we have

$$P\left\{\max_{s\in\Omega^c}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \to 0, as\ p \to +\infty. \tag{32}$$

Next, according to formula (29),(30),(31) and (32) we have

$$\begin{aligned}
&\lim_{p\to+\infty} P\left\{\min_{s=1,\cdots,p}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
=&\lim_{p\to+\infty} P\left\{\min\left[\min_{s\in\Omega_1}(X_s), \min_{s\in\Omega_2}(X_s), \min_{s\in\Omega^c}(X_s)\right] > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
=&\lim_{p\to+\infty} P\left\{\min_{s\in\Omega_1}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \times \lim_{p\to+\infty} P\left\{\min_{s\in\Omega_2}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\} \times\\
&\lim_{p\to+\infty} P\left\{\min_{s\in\Omega^c}(X_s) > [d_p(g_\alpha)]^{\frac{1}{2}}\right\}\\
=&1 \times 0 \times 0\\
=&0.
\end{aligned} \tag{33}$$

Next, according to formula (27), (28),(29)and (33),applying the squeeze theorem, we have

$$\lim_{p\to+\infty} P\left\{\left(\min_{s=1,\cdots,p}(Z_s)\right)^2 \geq C_\alpha + \log\frac{p^2}{\log p}\right\} = 0. \tag{34}$$

Now, according to (25),(26) and (34),we have

$$\beta_{T_2} \leq \beta_{T_5} + o(1).$$

□

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