Supplementary Information for

Spin wave-driven variable-phase mutual synchronization in spin Hall nano-oscillators

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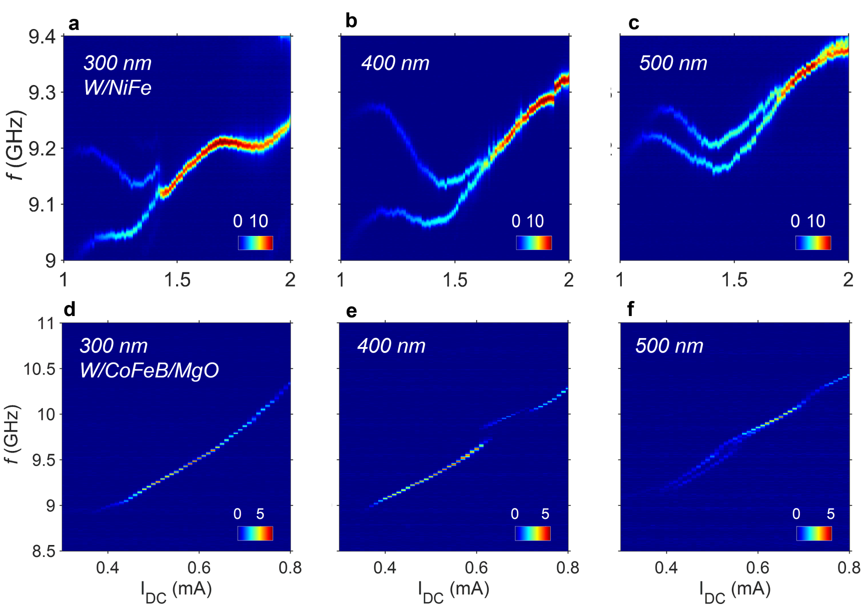
1 Electrical Measurements

Supplementary Note 1. In-plane *vs.* out-of-plane magnetized system

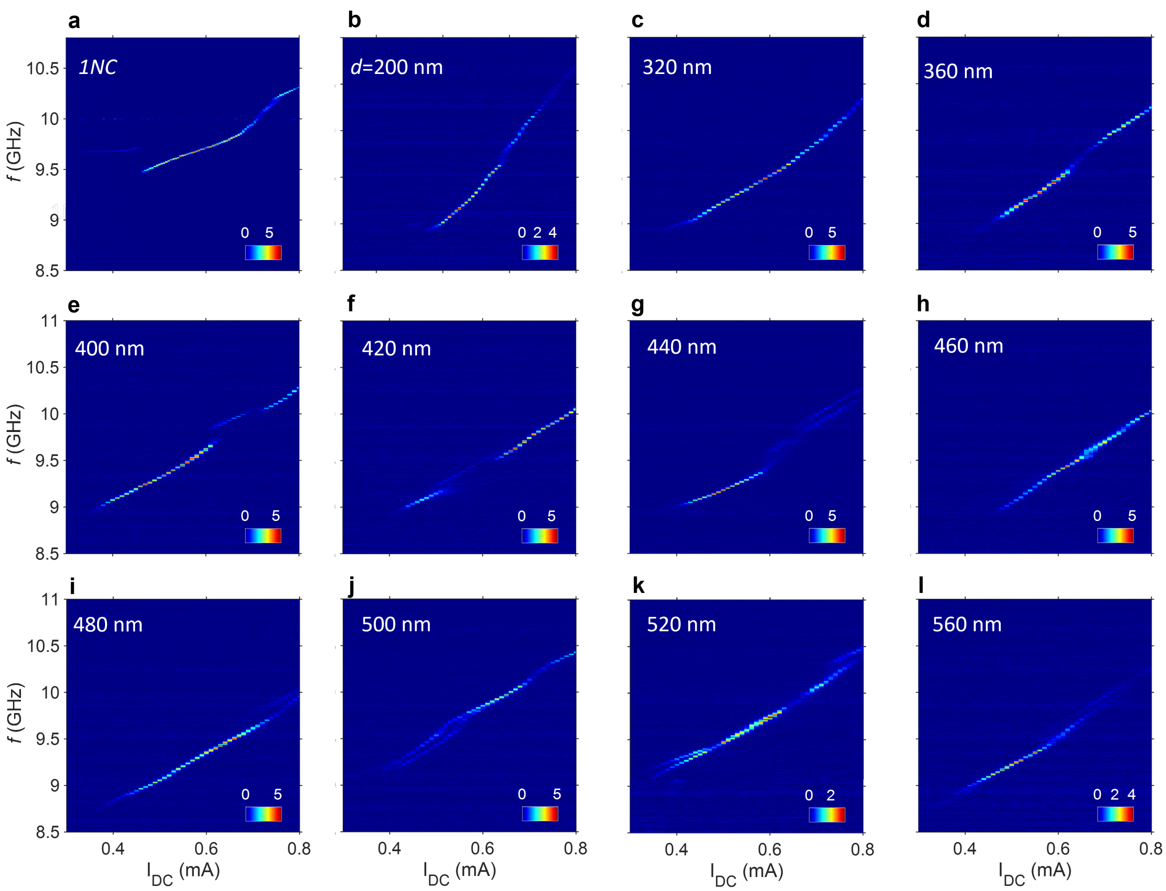
Figure S1 compares double nano-constriction Spin Hall Nano-Oscillators (SHNOs) with different widths (d = 300 nm, 400 nm, and 500 nm) fabricated using in-plane magnetized W/NiFe and perpendicular magnetic anisotropy (PMA)-based W/CoFeB/MgO thin films. It is evident that, although the W/NiFe system exhibits positive non-linearity, no out-of-phase synchronized modes (Region III, as discussed in the manuscript) are observed for W/NiFe due to the absence of propagating spin wave (PSW) modes. These results underscore the necessity of PSWs to observe such phenomena. The double nano-constriction SHNOs based on W/CoFeB/MgO clearly exhibit Region III (out-of-phase synchronization) for varying current values.



Supplementary Figure 1: (a-c): PSD *vs.* direct current for W/NiFe thin film nano-constriction SHNO with varying separation (d) between SHNOs (a. 300 nm, b. 400 nm and c. 500 nm, respectively), measured at H = 0.7 T, θ=82° and ϕ=20° angle. (d-f): PSD *vs.* direct current for W/CoFeB/MgO thin film nano-constriction SHNO with varying separation between SHNOs (d: 300 nm, e: 400 nm and f: 500 nm, respectively, measured at H = 0.4 T, θ=65° and ϕ=22° angle.

Supplementary Note 2. Separation dependence of variable synchronization

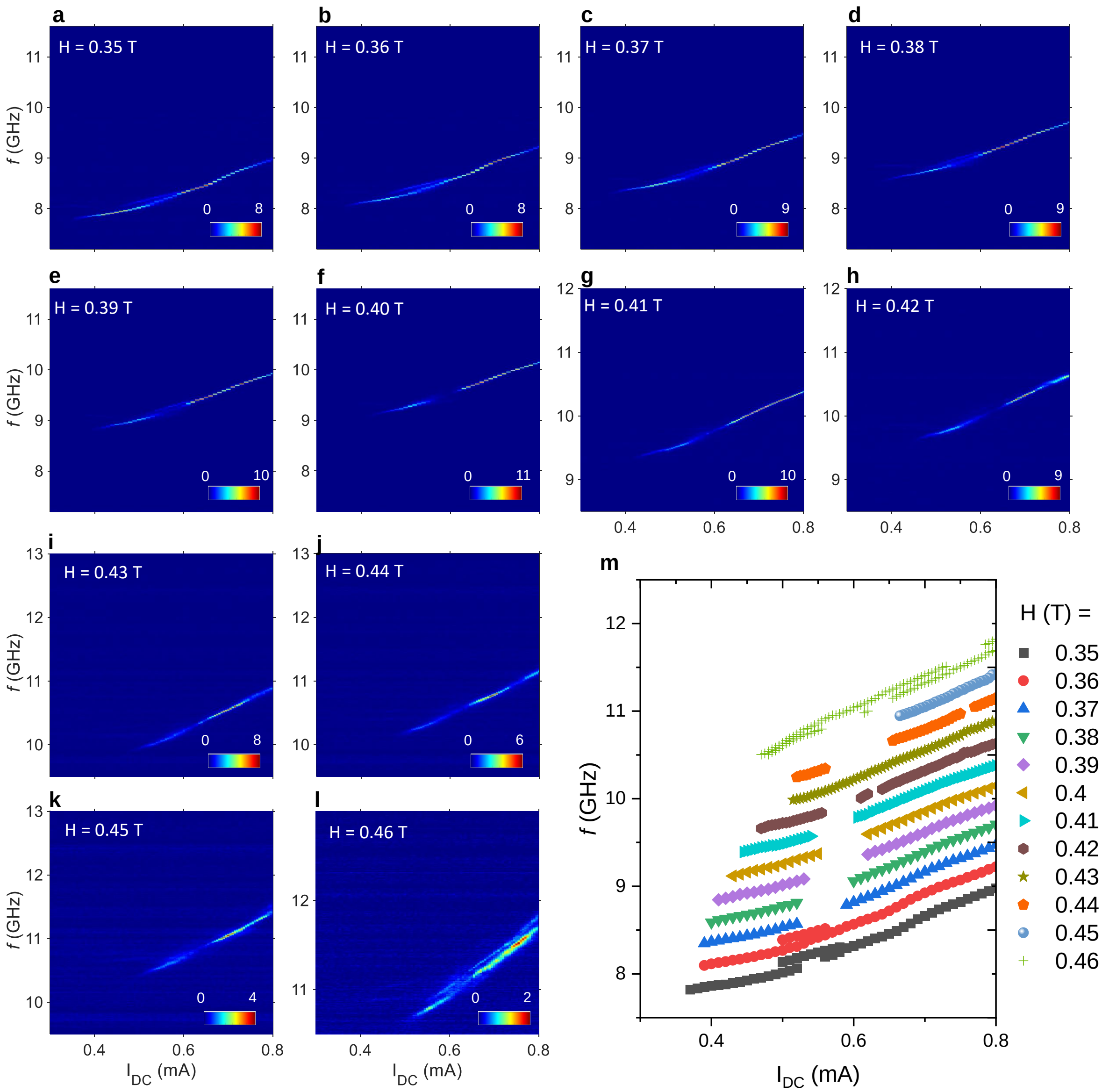
Figure S2 presents the power spectral density (PSD) *vs.* direct current with a fine variation in the separation between SHNOs fabricated using PMA induced W/CoFeB/MgO stacks. The PSD is shown for *d*=200-560 nm. A noticeable shift in out-of-phase synchronizing states is observed, ranging from 200 nm separation SHNOs to 560 nm separation SHNOs. This phenomenon can be attributed to the changing distance for propagating spin-waves. However, proposing a clear dependence of Region III with varying *d* (with respect to current values) is challenging, primarily due to the intricate dynamics in varying bridge (Rhombus) sizes with separation.



Supplementary Figure 2: (a-l): PSD *vs.* direct current for single and double nano-constriction SHNOs of W/CoFeB/MgO with varying *d* (= 200-560 nm).

Supplementary Note 3. Magnetic field dependence of variable synchronization

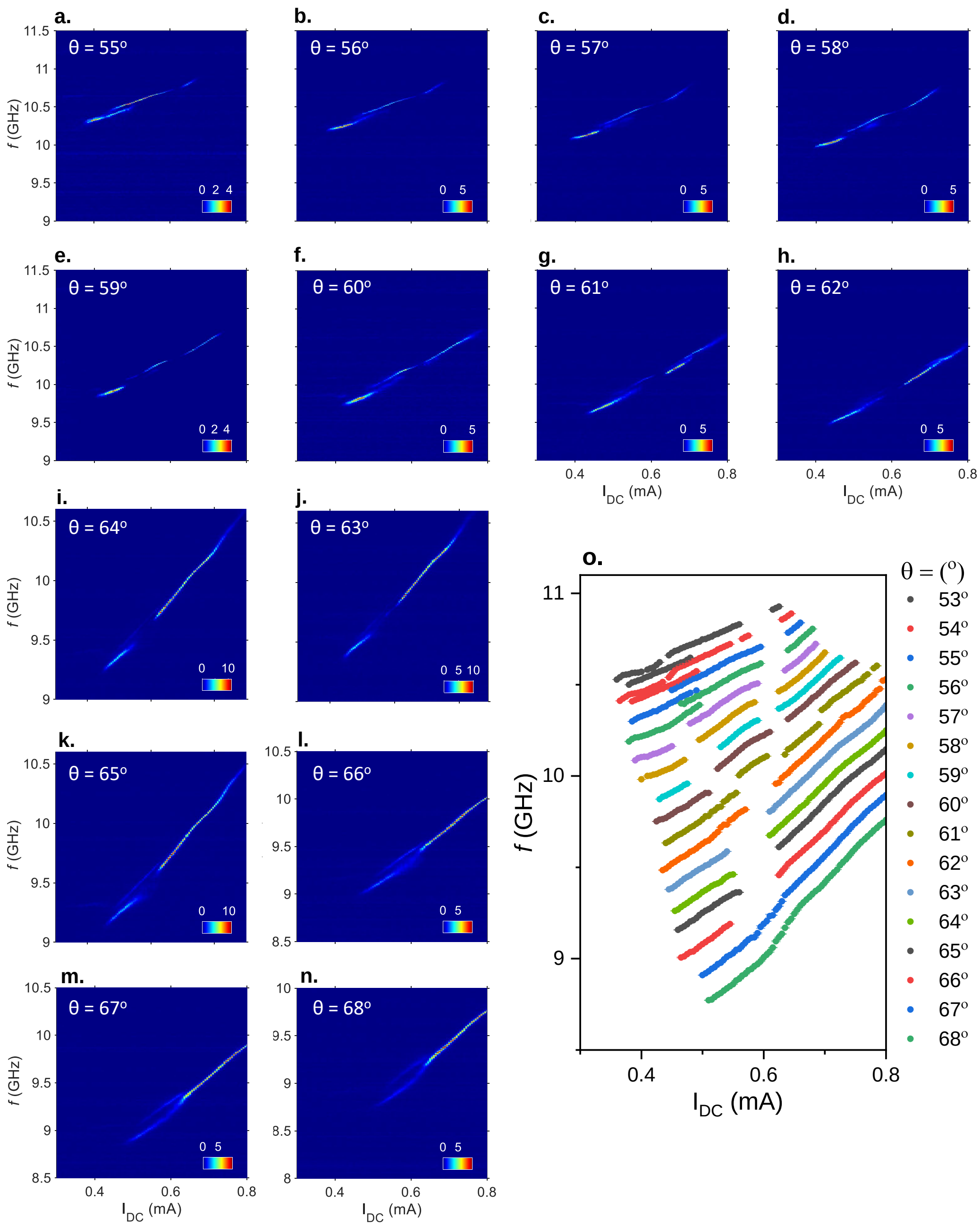
Figure S3a-l depict the PSD *vs.* direct current measured for varying field strengths (ranging from 0.35 T – 0.46 T with a step size of 0.01 T) for double nano-constriction SHNOs separated by 420 nm. One can clearly see a varying change in the strength of mutual synchronization in these measurements. A noticeable variation in the strength of mutual synchronization is evident in these measurements. Figure S3m illustrates the operating frequency extracted from the PSD plots, revealing that the out-of-phase synchronization regime varies among oscillators and is most pronounced around 0.4 T. This emphasizes the influence of the applied magnetic field on the control of relative phase. However, it is intriguing to observe that the position of out-of-phase locking in terms of direct current does not exhibit significant variation with the magnetic field. This indicates a consistent shift in the spin-wave (SW) gap compared to the ferromagnetic resonance (FMR) gap with the increasing magnetic field, as discussed in the main manuscript.



Supplementary Figure 3: (a-l): PSD *vs.* direct current of double nano-constriction SHNO (W/CoFeB/MgO) with d= 420 nm for varying magnetic field strength (ranging from 0.35 T – 0.46 T with a step size of 0.01 T). All measurements are performed for θ = 65° and ϕ = 22°. (m): Summarized frequency *vs.* direct current for varying magnetic field.

Supplementary Note 4. Magnetic field orientation of variable synchronization

Figure S4 shows the dependence of PSD *vs.* direct current on the out-of-plane angle (θ) for double nano-constriction SHNO with d = 420 nm. We observe a very complex dependence where for lower OOP angle (53°-60°), we observe two region III in the PSD plots which starts to merge around 61° and shows single region of out-of phase synchronization until 67°, where we observe two individual modes due to unsynchronization of SHNOs. The complex evolution of mutual synchronization with θ is not clearly understood and would require further detailed analysis.

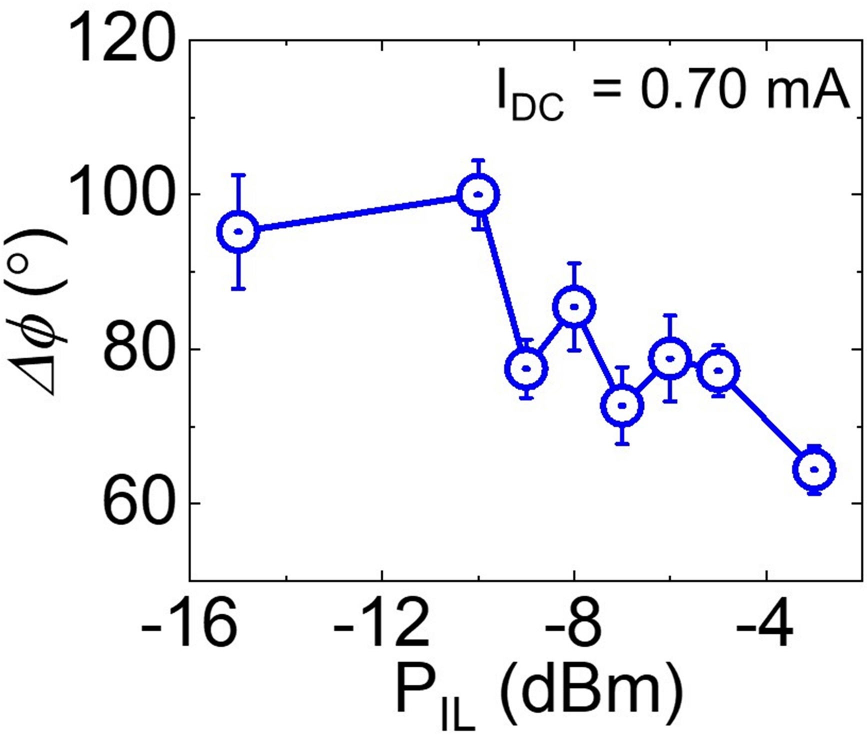


Supplementary Figure 4: (a-n): PSD *vs.* direct current for varying out-of-plane angle (θ = $55^{\circ} - 68^{\circ}$) in W/CoFeB/MgO thin film based double nano-constriction SHNO with d= 420 nm. (o.) Summarized extracted frequency *vs.* direct current for PSD plots shown in (a-n).

2 Micro-focused BLS microscopy

Supplementary Note 5. The relative phase difference between two SHNOs measured at $I_{DC} = 0.70$ mA for different P_{IL}

To justify the applied $P_{IL} =$ to be -10 dBm, we have measured the phase relation at various P_{IL} for $I_{DC} = 0.70$ mA (out-of-phase synchronization case) as shown in Fig. S7. It can be seen that even -3 dBm RF power is not enough to drive both oscillators in in-phase synchronization forcibly. Hence, the choice of P_{IL} to be -10 dBm seems convincing for our study.

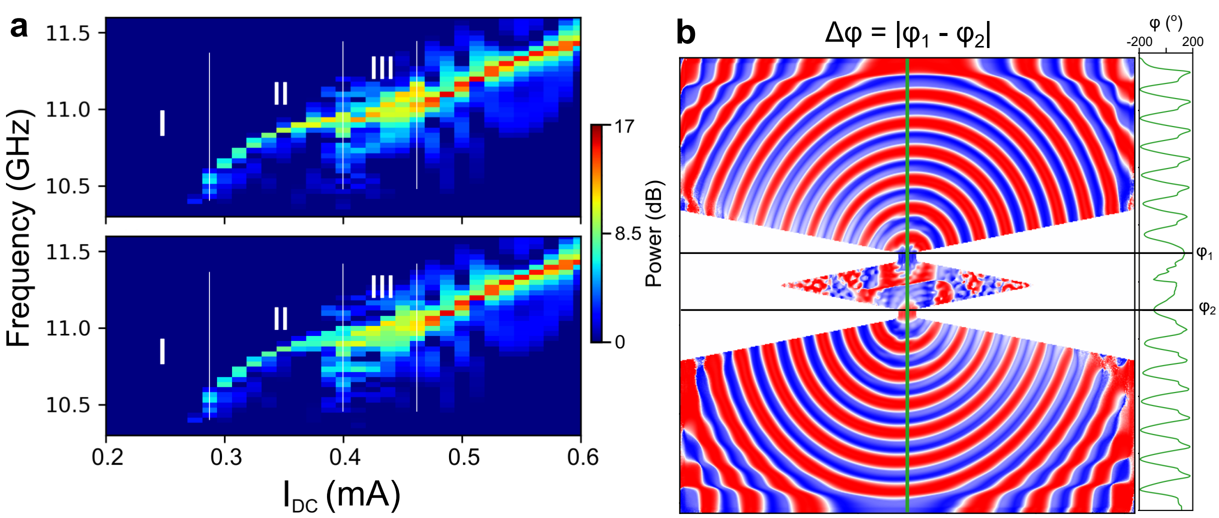


Supplementary Figure 5: The variation of relative phase difference as a function of P_{IL}. The solid line is just a guide to the eye.

3 Micromagnetic simulations

Supplementary Note 6. Individual oscillators

To completely characterize the dynamics of the system, we extracted the magnetization precession of individual constrictions as a function of time. The spectral modes of the devices were obtained via FFT and are shown in fig.6a. We observe that both constrictions auto-oscillate at all current values above the threshold, thus showing that there is no oscillator death, differing from previously reported results in spin torque nano-oscillators (Ref. 49). We can observe that the constrictions never cease to oscillate and the apparent amplitude death is due to a broadening of the linewidth of each individual oscillator as they pull on each other's frequency. This broadening lowers the overall device power below the noise floor of the measurement and thus the apparent amplitude death.



Supplementary Figure 6: **a** PSDs of individual oscillators. The anti-phase synchronization in regime III causes a broadening of the linewidth of the oscillators and a reduction of the overall device power. **b** Procedure for calculating the phase difference $\Delta\phi$ between the nano-constrictions. The phase along the green line is shown in the right.

Supplementary Note 7. Phase difference between oscillators

We obtained the phase differences between the oscillators as a function of the applied direct current shown in Fig. 5 (main manuscript), by averaging the phase calculated from the complex part of the point-resolved FFT maps. Specifically, we extracted the phase along the green line in Fig. S6b of the device and subtracted one from the other.