Supplemental Material for "Quantification of Entanglement and Coherence with Purity Detection"

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I. DERIVATION OF THE BOUNDS

Given a quantum state ρ in a d-dimensional Hilbert space, our task is to bound the Von Neumann entropy of ρ with a function of the state purity $\mathcal{P}(\rho) := \mathrm{Tr}(\rho^2)$. The spectral decomposition of the quantum state is $\rho = \sum_{i=1}^d \lambda_i \left|\psi_i\right\rangle\left\langle\psi_i\right|$, where $\{\left|\psi_i\right\rangle\}$ forms an orthonormal basis of the d-dimensional Hilbert space. The variational problem is then formulated as

$$\begin{aligned}\max/\min S(\rho) &= -\sum_{i=1}^{d}\lambda_i\log(\lambda_i)\\ s.t.\ \ &\sum_{i=1}^{d}\lambda_i^2=\gamma\\ &\sum_{i=1}^{d}\lambda_i=1\\ &0\le\lambda_i\le 1,\forall i,\end{aligned}\tag{1}$$

where $\mathcal{P}(\rho)=\mathrm{Tr}\rho^2$ is the purity of ρ.

Intuitively, the vector λ that maximizes S is the one that spread as uniformly as possible; while the vector λ that minimizes S is the one that has the minimal number of nonzero large values. In the following, we will analytically solve this problem and confirm this intuition.

A. Maximization

First, we focus on the maximization problem with $d=3$. Note that when $d=2$, the solution to the constraints of Eq. (1) is unique and the optimization problem will be trivial. Without loss of generality, we assume $\lambda_1\ge\lambda_2\ge\lambda_3$. Then the problem can be stated as

$$\begin{aligned}\max S(\rho) = -\lambda_1\log(\lambda_1)-\lambda_2\log(\lambda_2)-\lambda_3\log(\lambda_3)\\ s.t.\ \ \lambda_1^2+\lambda_2^2+\lambda_3^2=\gamma\\ \lambda_1+\lambda_2+\lambda_3=1\\ 1\ge\lambda_1\ge\lambda_2\ge\lambda_3\ge 0.\end{aligned}\tag{2}$$

We prove that the maximum is reached with the following Lemma.

Lemma 1. *The solution to the maximization problem in Eq.* (2) *is given by*

$$\begin{aligned}\lambda_1 &= \frac{1}{3}+\sqrt{\frac{2}{3}\left(\mathcal{P}(\rho)-\frac{1}{3}\right)},\\ \lambda_2 &= \lambda_3=\frac{1-\lambda_1}{2}.\end{aligned}\tag{3}$$

Proof. The differential of the entropy function $S(\rho)$ and the constraints are given by

$$\begin{aligned}dS = &-\left(\frac{1}{\ln(2)}+\log\lambda_1\right)d\lambda_1-\left(\frac{1}{\ln(2)}+\log\lambda_2\right)d\lambda_2\\ &-\left(\frac{1}{\ln(2)}+\log\lambda_3\right)d\lambda_3\end{aligned}\tag{4}$$

and

$$\begin{aligned}\lambda_1 d\lambda_1+\lambda_2 d\lambda_2+\lambda_3 d\lambda_3=0\\ d\lambda_1+d\lambda_2+d\lambda_3=0,\end{aligned}\tag{5}$$

respectively. We rewrite Eq. (5) to

$$\begin{aligned}d\lambda_1 &= -\frac{(\lambda_3-\lambda_2)}{\lambda_1-\lambda_2}d\lambda_3,\\ d\lambda_2 &= -\frac{(\lambda_1-\lambda_3)}{\lambda_1-\lambda_2}d\,\lambda_3.\end{aligned}$$

Thus, the differential of the entropy function becomes

$$\begin{aligned}dS(\rho) = &\frac{d\lambda_3}{\lambda_1-\lambda_2}[(\lambda_3-\lambda_2)\log\lambda_1+(\lambda_1-\lambda_3)\log\lambda_2\\ &+(\lambda_2-\lambda_1)\log\lambda_3]\\ = &(\lambda_2-\lambda_3)\left[-\frac{\log\lambda_1-\log\lambda_2}{\lambda_1-\lambda_2}+\frac{\log\lambda_3-\log\lambda_2}{\lambda_3-\lambda_2}\right]d\lambda_3\end{aligned}\tag{6}$$

Since the function $\log\lambda$ is concave for $\lambda \in [0,1]$, for $\lambda_1 \geq \lambda_2 \geq \lambda_3$,

$$\frac{\log\lambda_1 - \log\lambda_2}{\lambda_1 - \lambda_2} \leq \frac{\log\lambda_3 - \log\lambda_2}{\lambda_3 - \lambda_2}.$$

Thus, $dS(\rho)/d\lambda_3 \geq 0$. To reach the maximum of $S(\rho)$, we thus only need to set λ_3 to be its maximum, which happens when $\lambda_2 = \lambda_3$. Together with the constraints, then we can solve the equations and show that the solution to the maximization problem is given in Eq. (3). □

Now, we can solve the maximization problem of Eq. (1) for a general case of d.

Theorem 1. *Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_d$. The solution to the maximization problem in Eq. (1) is*

$$\begin{aligned} \lambda_1 &= \frac{1}{d} + \sqrt{\frac{d-1}{d}\left(\mathcal{P}(\rho) - \frac{1}{d}\right)}, \\ \lambda_2 &= \lambda_3 = \cdots = \lambda_d = \frac{1-\lambda_1}{d-1}. \end{aligned} \tag{7}$$

Proof. The solution in Eq. (7) is exactly determined when setting $\lambda_2 = \lambda_3 = \cdots = \lambda_d$. Suppose the maximization problem solution is not this one, then we must have that $\lambda_2 > \lambda_d$. In the following, we prove the contradiction by showing that changing the values of $\lambda_1, \lambda_2, \lambda_d$ would make the entropy $S(\rho)$ larger while fixing all other values $(\lambda_3, \lambda_4, \ldots, \lambda_{d-1})$ and the constraints. Now the constraints for λ_1, λ_2, and λ_d becomes

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 + \lambda_d^2 &= a \\ \lambda_1 + \lambda_2 + \lambda_d &= b. \end{aligned} \tag{8}$$

By defining $\lambda_1' = \lambda_1/b$, $\lambda_2' = \lambda_2/b$, $\lambda_d' = \lambda_d/b$, the relations become

$$\begin{aligned} \lambda_1'^2 + \lambda_2'^2 + \lambda_d'^2 &= a/b^2 \\ \lambda_1' + \lambda_2' + \lambda_d' &= 1. \end{aligned} \tag{9}$$

The entropy function is

$$\begin{aligned} S(\rho) &= -\sum_{i=1}^{d} \lambda_i \log(\lambda_i), \\ &= S_{1,2,d}(\rho) + S_r(\rho), \end{aligned} \tag{10}$$

where $S_{1,2,d}(\rho) = -\lambda_1\log(\lambda_1) - \lambda_2\log(\lambda_2) - \lambda_d\log(\lambda_d)$ and $S_r(\rho) = -\sum_{i=3}^{d-1}\lambda_i\log(\lambda_i)$. Since $S_r(\rho)$ is fixed, we need to maximize $S_{1,2,d}(\rho)$, which can also be represented as

$$\begin{aligned} &S_{1,2,d}(\rho) = -b\lambda_1'\log(b\lambda_1') - b\lambda_2'\log(b\lambda_2') - b\lambda_d'\log(b\lambda_d') \\ &= b[-\lambda_1'\log(\lambda_1') - \lambda_2'\log(\lambda_2') - \lambda_d'\log(\lambda_d')] - b\log b \end{aligned} \tag{11}$$

Denoting $S_{1,2,d}'(\rho) = -\lambda_1'\log(\lambda_1') - \lambda_2'\log(\lambda_2') - \lambda_d'\log(\lambda_d')$, this optimization problem has the same form of Eq. (2). Then Lemma 1 indicates that the maximum of $S_{1,2,d}'(\rho)$ given the constraints in Eq. (9) is reached when $\lambda_2' = \lambda_d'$. In other words, the maximum of $S_{1,2,d}(\rho)$ given the constrains of Eq. (8) is saturated with $\lambda_2 = \lambda_d$, which contradicts $\lambda_2 > \lambda_d$. Therefore, the solution to the maximization problem is given by Eq. (7). □

B. Minimization

Now, we consider the solution to the minimization of Eq. (1). Similarly, we first consider the minimization with $d = 3$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3$,

$$\begin{aligned} \min S(\rho) &= -\lambda_1\log(\lambda_1) - \lambda_2\log(\lambda_2) - \lambda_3\log(\lambda_3) \\ s.t.\ & \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathcal{P}(\rho) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & 1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0. \end{aligned} \tag{12}$$

Lemma 2. *The solution to the minimization problem in Eq. (12) is reached either when $\lambda_1 = \lambda_2$ or $\lambda_3 = 0$.*

Proof. From the proof of Lemma 1, we already showed that $dS(\rho)/d\lambda_3 \geq 0$. Therefore, the lower bound of $S(\rho)$ is reached when λ_3 takes its minimum. As $2(\lambda_1^2 + \lambda_2^2) \geq (\lambda_1 + \lambda_2)^2$, according to Eq. (12), we have

$$2(\mathcal{P}(\rho) - \lambda_3^2) \geq (1-\lambda_3)^2. \tag{13}$$

The lower bound for λ_3 is

$$\lambda_3 \geq \max\left\{0, \frac{1-\sqrt{6\mathcal{P}(\rho)-2}}{3}\right\}$$

Thus, when $\mathcal{P}(\rho) \geq 1/2$, the minimal possible value for λ_3 is 0. When $1/3 \leq \mathcal{P}(\rho) < 1/2$, the minimal possible value for λ_3 is $\frac{1-\sqrt{6\mathcal{P}(\rho)-2}}{3}$ and $\lambda_1 = \lambda_2 = (1-\lambda_3)/2$. Note that $\mathcal{P}(\rho) \geq 1/3$ for $d = 3$. □

Now, we can show the general solution to the minimization of Eq. (1).

Theorem 2. *Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k$, the solution to the minimization problem in Eq. (1) is*

$$\begin{aligned} \lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} &= \frac{1-\alpha}{k-1}, \\ \lambda_k &= \alpha, \\ \lambda_{k+1} = \cdots = \lambda_d &= 0. \end{aligned} \tag{14}$$

Here,

$$\alpha = \frac{1}{k} - \sqrt{(1-1/k)(\mathcal{P}(\rho) - 1/k)} \tag{15}$$

and k is the integer such that $\frac{1}{k} \leq \mathcal{P}(\rho) \leq \frac{1}{k-1}$.

Proof. Suppose we always have the solution in the form as

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1}, \lambda_k, \lambda_{k+1} = \ldots \lambda_d = 0. \tag{16}$$

Otherwise, there must exist three $\lambda_i, \lambda_j, \lambda_k$ such that $\lambda_i > \lambda_j \geq \lambda_k$ and $\lambda_k \neq 0$. Following a similar argument in the proof of Theorem 1, we can show that this contradicts Lemma 2.

According to Eq. (16), we have

$$\begin{aligned} (k-1)\lambda_1^2 + \lambda_k^2 &= \mathcal{P}(\rho), \\ (k-1)\lambda_1 + \lambda_k &= 1, \\ k &\leq d \end{aligned}$$

We can show that the possible integer value for k is unique. That is,

$$k[(k-1)\lambda_1^2+\lambda_k^2] \geq [(k-1)\lambda_1+\lambda_k]^2 \\ \geq (k-1)[(k-1)\lambda_1^2+\lambda_k^2]$$

Equivalently, we have

$$k\mathcal{P}(\rho) \geq 1 \geq (k-1)\mathcal{P}(\rho),$$

hence

$$\frac{1}{\mathcal{P}(\rho)} \leq k \ \leq \frac{1}{\mathcal{P}(\rho)}+1, \\ \frac{1}{k} \leq \mathcal{P}(\rho) \ \leq \frac{1}{k-1}.$$

□

C. Upper and lower bounds to coherence and entanglement

We now call $\{\lambda^{\mathrm{M}}_{i,\rho}\}, \{\lambda^{\mathrm{m}}_{i,\rho}\}$ the vectors solving the maximization and the minimization, respectively. Given a bipartite state ρ_{AB}, by minimizing (maximizing) the marginal purity on B subsystem and maximizing (minimizing) the global purity, one has

Result 1 (Eq. (2) of the main text)— Given a quantum state $\rho_{AB} \in \mathcal{H}_{d_A} \otimes \mathcal{H}_{d_B}$, and defining $\rho_B = \mathrm{Tr}_A\rho_{AB}$, its coherent information $I(A\rangle B)$ is bounded as follows:

$$l_e(\rho_{AB}) \leq I(A\rangle B) \leq u_e(\rho_{AB}), \tag{17}$$

$$\begin{aligned} &l_e(\rho_{AB}) = \\ &-(1-\lambda^{\mathrm{m}}_{k_{\rho_B},\rho_B})\log\lambda^{\mathrm{m}}_{1,\rho_B} - \lambda^{\mathrm{m}}_{k_{\rho_B},\rho_B}\log\lambda^{\mathrm{m}}_{k_{\rho_B},\rho_B} \\ &+(1-\lambda^{\mathrm{M}}_{1,\rho_{AB}})\log\frac{(1-\lambda^{\mathrm{M}}_{1,\rho_{AB}})}{(d-1)} + \lambda^{\mathrm{M}}_{1,\rho_{AB}}\log\lambda^{\mathrm{M}}_{1,\rho_{AB}}, \\ &u_c(\rho_{AB}) = \\ &(1-\lambda^{\mathrm{m}}_{k_{\rho_{AB}},\rho_{AB}})\log\lambda^{\mathrm{m}}_{1,\rho_{AB}} + \lambda^{\mathrm{m}}_{k_{\rho_{AB}},\rho_{AB}}\log\lambda^{\mathrm{m}}_{k_{\rho_{AB}},\rho_{AB}} \\ &-(1-\lambda^{\mathrm{M}}_{1,\rho_B})\log\frac{(1-\lambda^{\mathrm{M}}_{1,\rho_B})}{(d_B-1)} - \lambda^{\mathrm{M}}_{1,\rho_B}\log\lambda^{\mathrm{M}}_{1,\rho_B}. \end{aligned}$$

By minimizing (maximizing) the coherence of the dephased state $\rho_d = \sum_i |i\rangle\langle i|\,\rho\,|i\rangle\langle i|$, and maximizing (minimizing) the coherence of the state under study, we obtain lower (upper) bounds to the relative entropy of coherence:

Result 2 (Eq. (5) of the main text) — The relative entropy of coherence $C_{\mathrm{RE}}(\rho)$ is bounded as follows:

$$l_c(\rho) \leq C_{\mathrm{RE}}(\rho) \leq u_c(\rho), \tag{18}$$

$$\begin{aligned} &l_c(\rho) = -(1-\lambda^{\mathrm{m}}_{k_{\rho_d},\rho_d})\log\lambda^{\mathrm{m}}_{1,\rho_d} - \lambda^{\mathrm{m}}_{k_{\rho_d},\rho_d}\log\lambda^{\mathrm{m}}_{k_{\rho_d},\rho_d} \\ &+(1-\lambda^{\mathrm{M}}_{1,\rho})\log\frac{(1-\lambda^{\mathrm{M}}_{1,\rho})}{(d-1)} + \lambda^{\mathrm{M}}_{1,\rho}\log\lambda^{\mathrm{M}}_{1,\rho}, \\ &u_c(\rho) = (1-\lambda^{\mathrm{m}}_{k_\rho,\rho})\log\lambda^{\mathrm{m}}_{1,\rho} + \lambda^{\mathrm{m}}_{k_\rho,\rho}\log\lambda^{\mathrm{m}}_{k_\rho,\rho} \\ &-(1-\lambda^{\mathrm{M}}_{1,\rho_d})\log\frac{(1-\lambda^{\mathrm{M}}_{1,\rho_d})}{(d-1)} - \lambda^{\mathrm{M}}_{1,\rho_d}\log\lambda^{\mathrm{M}}_{1,\rho_d}. \end{aligned}$$

II. DETAILS OF EXPERIMENTAL REALIZATIONS AND RESULTS

Before discussing the experimental details, we provide an overview of the crucial optical components and their respective functions employed in our experiment:

- Waveplates: We utilize both half-wave plates (HWPs) and quarter-wave plates (QWPs) to perform unitary operations (U) on photons. The angle between the fast axis of the waveplate and the vertical polarization direction is denoted as ω for HWPs and ϑ for QWPs. These waveplates induce specific unitary transformations on the quantum state of photons, enabling the desired experimental operations. The unitary transformations of HWPs and QWPs can be expressed as:

$$U_{\mathrm{HWP}}(\omega) = -\begin{pmatrix}\cos 2\omega & \sin 2\omega \\ \sin 2\omega & -\cos 2\omega\end{pmatrix}, \tag{19}$$

$$U_{\mathrm{QWP}}(\vartheta) = \frac{1}{\sqrt{2}}\begin{pmatrix}1+i\cos 2\vartheta & i\sin 2\vartheta \\ i\sin 2\vartheta & 1-i\cos 2\vartheta\end{pmatrix} \tag{20}$$

- Polarization beam splitter (PBS): This component separates photons with perpendicular polarization directions. It transmits horizontally polarized photons while reflecting vertically polarized photons.

- Beam displacer (BD): The BD allows vertically polarized photons to pass through in the original direction, while horizontally polarized photons are displaced. In our experiment, we use a BD with a length of 28.3 mm, resulting in a displacement offset of 3 mm.

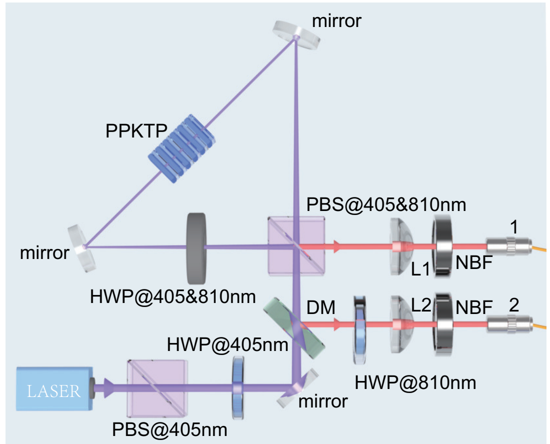

FIG. 1. Schematic illustration of biased polarization-entangled photon source.

A. Biased polarization-entangled photon source

We use a continuous-wave laser operating at a central wavelength of 405 nm with a full width at half maximum (FWHM) of 0.012 nm as our pump light source Fig. 1. The pump light passes through a PBS followed by an HWP set at $\theta/2$, which transforms the polarization of the pump light into $\cos\theta\,|H\rangle_p+\sin\theta\,|V\rangle_p$. The pump light passes PBS that transmits the component of $|H\rangle$ and reflects component of $|V\rangle$. Then, the periodically poled potassium titanyl phosphate (PPKTP) is coherently pumped from anticlockwise and clockwise directions respectively, and the generated photons are superposed on the PBS leading to the outcome state of $\cos\theta\,|HV\rangle_{12}+\sin\theta\,|VH\rangle_{12}$. An HWP set at $45°$ is applied on photon 2, which leads to a biased polarization-entangled state in form of $\cos\theta\,|HH\rangle_{12}+\sin\theta\,|VV\rangle_{12}$

To enhance collective efficiency, we employ lens L1 with a focal length of 200 mm and lens L2 with a focal length of 250 mm. The two photons pass through narrowband filters (NBFs) with an FWHM of 3 nm and then are coupled into single-mode fibres.

B. Preparation of $|\mathrm{GHZ}_\theta\rangle$ and $|\psi_{2,\theta}\rangle$

To generate $|\mathrm{GHZ}_\theta\rangle$, photon 1 and 2 pass through two BDs after calibration of their polarization as shown in Fig. 2(a). Calling $h(v)$ the deviated (transmitted) spatial modes, the step-by-step description of optical elements to generate $|\mathrm{GHZ}_\theta\rangle$ is

$$\begin{aligned}&\cos\theta\,|HH\rangle_{12}+\sin\theta\,|VV\rangle_{12}\\ \xrightarrow{\text{BD1}}\,&\cos\theta\,|HhH\rangle_{11'2}+\sin\theta\,|VvV\rangle_{11'2}\\ \xrightarrow{\text{BD2}}\,&\cos\theta\,|HhHh\rangle_{11'22'}+\sin\theta\,|VvVv\rangle_{11'22'}\,.\end{aligned}\tag{21}$$

In the experiment, by changing the value of θ, we totally generate eleven $\rho_{\mathrm{GHZ}_\theta}$. For each $\rho_{\mathrm{GHZ}_\theta}$, we reconstruct the corresponding density matrix $\rho_{\mathrm{GHZ}_\theta}$ using standard quantum state tomography (SQT). We calculate the fidelity $F=\mathrm{Tr}(\rho_{\mathrm{GHZ}_\theta}\,|\mathrm{GHZ}_\theta\rangle\langle\mathrm{GHZ}_\theta|)$. The results are summarized in Table I.

The setup to prepare $|\psi_{2,\theta}\rangle\otimes|\psi_{2,\theta}\rangle$ is illustrated in Fig. 2(b). Photon 1 and 2 pass through two PBSs creating the resulting state of $|HH\rangle_{12}$. The step-by-step description of optical elements to generate $|\psi_{2,\theta}\rangle\otimes|\psi_{2,\theta}\rangle$ is

$$\begin{aligned}|HH\rangle_{12}&\xrightarrow[\text{are applied on photon 1 and 2}]{\text{Two HWPs@}\theta/2}(\cos\theta\,|H\rangle_1+\sin\theta\,|V\rangle_1)\otimes(\cos\theta\,|H\rangle_2+\sin\theta\,|V\rangle_2)\\ &\xrightarrow[\text{are applied on photon 1 and 2}]{\text{BD1\&BD2}}(\cos\theta\,|Hh\rangle_{11'}+\sin\theta\,|Vv\rangle_{11'})\otimes(\cos\theta\,|Hh\rangle_{22'}+\sin\theta\,|Vv\rangle_{22'})\\ &\xrightarrow[\text{are applied on path }h\text{ of photon 1 and 2}]{\text{Two HWPs@}45°}|VV\rangle_{12}\otimes[(\cos\theta\,|h\rangle_{1'}+\sin\theta\,|v\rangle_{1'})\otimes(\cos\theta\,|h\rangle_{2'}+\sin\theta\,|v\rangle_{2'})]=|VV\rangle_{12}\otimes|\psi_{2,\theta}\rangle_{1'2'}\\ &\xrightarrow[\text{are applied on both path of photon 1 and 2}]{\text{Two HWPs@}\theta/2}|\psi_{2,\theta}\rangle_{12}\otimes|\psi_{2,\theta}\rangle_{1'2'}\end{aligned}\tag{22}$$

As shown in Eq. (22), the first $|\psi_{2,\theta}\rangle$ is encoded into the polarization degree of freedom (DOF) of both photons, while the second $|\psi_{2,\theta}\rangle$ is encoded on the path DOF of both photons.

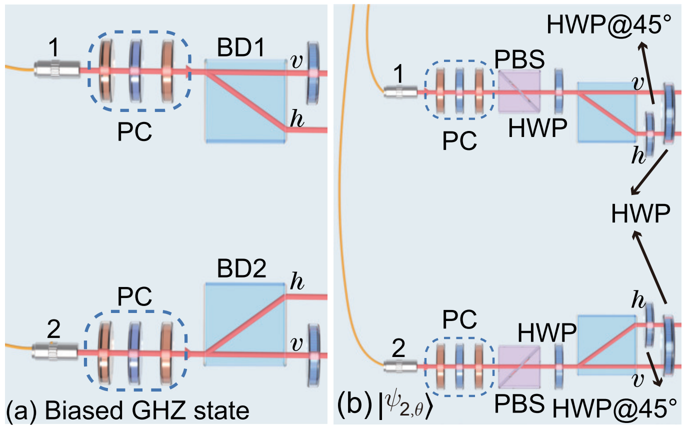

FIG. 2. (a) The setup to generate biased four-qubit GHZ states $|\mathrm{GHZ}_\theta\rangle$. (b) The setup to generate two-copy states $|\psi_{2,\theta}\rangle$. PC: polarization control is a combination of QWP, HWP and QWP which can be set to calibrate the variation of polarization caused by the twist of fibre.

TABLE I. The fidelities of the prepared $\rho_{\mathrm{GHZ}_\theta}$.

θ	F
0	0.9293 ± 0.0004
$\pi/20$	0.9050 ± 0.0006
$2\pi/20$	0.9057 ± 0.0007
$3\pi/20$	0.9331 ± 0.0005
$4\pi/20$	0.9136 ± 0.0006
$5\pi/20$	0.9124 ± 0.0007
$6\pi/20$	0.9155 ± 0.0003
$7\pi/20$	0.8850 ± 0.0007
$8\pi/20$	0.8833 ± 0.0009
$9\pi/20$	0.9155 ± 0.0005
$10\pi/20$	0.9290 ± 0.0004

C. Experimental setups to perform shadow estimation and collective measurements

In shadow estimation, the single-qubit Clifford unitary operation $U_n \in \mathrm{Cl}_2$ followed by a projective measurement on Pauli-Z basis is equivalent to measuring qubit on basis $J_n = U_n^\dagger Z U_n \in \{\pm X, \pm Y, \pm Z\}$. The experimental setup to perform measurement on arbitrary basis $(\alpha|H\rangle + \delta|V\rangle) \otimes (\gamma|h\rangle + \mu|v\rangle)$ is shown in Fig. 3(a), where $\alpha|H\rangle + \delta|V\rangle$ is encoded on polarization DOF and $\gamma|h\rangle + \mu|v\rangle$ is encoded on path DOF. Projection on $\alpha|H\rangle + \delta|V\rangle$ is realized by combination of E-HWP1 (set at angle of θ_1) and E-QWP1 (set at angle of υ_1), while the projection of $\gamma|h\rangle + \mu|v\rangle$ and its orthogonal basis is realized by combination of E-HWP2 (set at angle of θ_2), E-QWP2 (set at angle of υ_2) and a PBS. A step-by-step description of implementation of such projective measurement is described by

$$
\begin{aligned}
(\alpha|H\rangle + \beta|V\rangle) \otimes (\gamma|h\rangle + \delta|v\rangle) &\xrightarrow[\text{on both path}]{\text{E-HWP1@}\omega_1\text{\&E-QWP1@}\vartheta_1} |H\rangle \otimes (\gamma|h\rangle + \delta|v\rangle) \\
&\xrightarrow[\text{on path h}]{\text{HWP@}45^\circ} \gamma|Vh\rangle + \delta|Hv\rangle \\
&\xrightarrow[\text{combine the path } h \text{ and } v]{\text{BD3}} \gamma|V\rangle + \delta|H\rangle \\
&\xrightarrow{\text{HWP@}45^\circ} \gamma|H\rangle + \delta|V\rangle \\
&\xrightarrow{\text{E-HWP2@}\omega_2\text{\&E-QWP2@}\vartheta_2} |H\rangle \\
&\xrightarrow{\text{PBS}} \begin{cases} \text{transmitted,} & \text{results of } (\alpha|H\rangle + \beta|V\rangle) \otimes (\gamma|h\rangle + \delta|v\rangle) \\ \text{reflected,} & \text{results of } (\alpha|H\rangle + \beta|V\rangle) \otimes (\gamma|h\rangle + \delta|v\rangle)^\perp \end{cases},
\end{aligned}
\tag{23}
$$

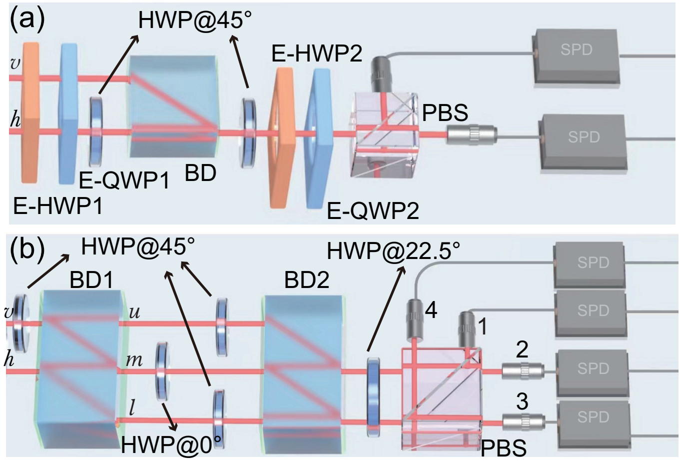

FIG. 3. Illustration of setups to perform (a) shadow estimation and (b) collective measurements.

where $(\gamma\,|h\rangle + \delta\,|v\rangle)^{\perp}$ is the orthogonal state of $\gamma\,|h\rangle + \delta\,|v\rangle$.

The experimental setup to implement BSM between polarization-encoded qubit and path-encoded qubit is shown in Fig. 3(b). Note that the projection on four Bell states is demonstrated with single experimental setting. A detailed description of Fig. 3(b) is

$$\left\{\begin{array}{l} |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|Hv\rangle + |Vh\rangle) \\ |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|Hv\rangle - |Vh\rangle) \\ |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|Hh\rangle + |Vv\rangle) \\ |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|Hh\rangle - |Vv\rangle) \end{array}\right. \xrightarrow[\text{on path } v]{\text{HWP@45}^\circ} \left\{\begin{array}{l} \frac{1}{\sqrt{2}}(|Vv\rangle + |Vh\rangle) \\ \frac{1}{\sqrt{2}}(|Vv\rangle - |Vh\rangle) \\ \frac{1}{\sqrt{2}}(|Hh\rangle + |Hv\rangle) \\ \frac{1}{\sqrt{2}}(|Hh\rangle - |Hv\rangle) \end{array}\right. \xrightarrow[\text{Re-encoding}]{\text{BD1}} \left\{\begin{array}{l} \frac{1}{\sqrt{2}}(|Vu\rangle + |Vm\rangle) \\ \frac{1}{\sqrt{2}}(|Vu\rangle - |Vm\rangle) \\ \frac{1}{\sqrt{2}}(|Hl\rangle + |Hm\rangle) \\ \frac{1}{\sqrt{2}}(|Hl\rangle - |Hm\rangle) \end{array}\right.$$

$$\xrightarrow[\text{HWP@0}^\circ\text{on path } m]{\text{HWPs@45}^\circ\text{on path } u \text{ and } l} \left\{\begin{array}{l} \frac{1}{\sqrt{2}}(|Hu\rangle - |Vm\rangle) \\ \frac{1}{\sqrt{2}}(|Hu\rangle + |Vm\rangle) \\ \frac{1}{\sqrt{2}}(|Vl\rangle + |Hm\rangle) \\ \frac{1}{\sqrt{2}}(|Vl\rangle - |Hm\rangle) \end{array}\right. \xrightarrow[\text{combine path}]{\text{BD2}} \left\{\begin{array}{l} |-\rangle\,|m\rangle \\ |+\rangle\,|m\rangle \\ |+\rangle\,|l\rangle \\ |-\rangle\,|l\rangle \end{array}\right. \xrightarrow[\text{on path } m \text{ and } l]{\text{HWP@22.5}^\circ} \left\{\begin{array}{l} |V\rangle\,|m\rangle \\ |H\rangle\,|m\rangle \\ |H\rangle\,|l\rangle \\ |V\rangle\,|l\rangle \end{array}\right. \xrightarrow{\text{PBS}} \left\{\begin{array}{l} \text{clicks on detector 1} \\ \text{clicks on detector 2} \\ \text{clicks on detector 3} \\ \text{clicks on detector 4} \end{array}\right. . \tag{24}$$