Supplementary Information of "Back-action supercurrent diodes"

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Supplementary Note

Current-phae relation symmetries due to supercurrent back-action.

In this section, we will demonstrate that the back-action mechanism on the supercurrent i) will not produce any spontaneous phase, i.e, $I(\phi = 0) = 0$ and ii) rectification of the supercurrent will emerge for any functional form of the effective Josephson coupling $E_J(I)$ with an odd parity. To this aim, we start considering the free energy of a Josephson junction that can be cast into the form of

$$F_J = E_J[1 - \cos(\phi)]$$

with E_J the Josephson coupling and ϕ the phase difference among the superconductors forming the junction. For conventional configurations, E_J depends on factors related to the junction (e.g. material, barrier, etc). From the expression of the free energy, one can directly deduce the supercurrent flowing across the junction as the variation of the free energy with respect to the phase bias (in units of $\frac{2e}{\hbar}$):

$$I(\phi) = \frac{\partial F_J}{\partial \phi} \tag{1}$$

that yields the well-known Josephson relation $I(\phi) = I_c \sin(\phi)$ with $I_c = E_J$ being the critical current, namely, the maximal current that can sustain the junction without any voltage drop. This relation sets out the dc Josephson effect with a supercurrent existing between two superconductors that are coupled through a thin layer.

Now, we introduce the back-action mechanism on the amplitude of the supercurrent. We assume that the coupling among the superconductors depends on the amplitude of the supercurrent flowing across the junction. Hence, one can postulate a free energy of the type:

$$F_J = E_J(I)[1 - \cos(\phi)] \tag{2}$$

where the coupling $E_J(I)$ is a function of the supercurrent I flowing through the superconductors. Before discussing the structure of the coupling, we would like to emphasize that Eq. 2 describes a Josephson system with a transmission probability of Cooper pairs that depends on the amplitude of the supercurrent flowing across the junction. The form of F_J is consistent with the physical requirement that at zero applied phase bias ($\phi = 0$) the free energy is constant (i.e. $F_J(\phi = 0) = 0$) at any value of the supercurrent flowing through the junction. As a consequence, the self-induced supercurrent model fulfills the relation $I(0) = 0$, i.e. there is no spontaneous supercurrent flowing at zero phase bias. Such result can be deduced from the supercurrent expression obtained from Eq. 1 and Eq. 2:

$$I(\phi) = \frac{\partial E_J(I)}{\partial I}\frac{\partial I(\phi)}{\partial \phi}[1 - \cos(\phi)] + E_J(I)\sin\phi \tag{3}$$

There, by making the limit for $\phi \to 0$ on the two sides of the Eq. 3, the $I(0) = 0$ constraint is immediately deduced independently of the functional form for $E_J(I)$.

The second important feature refers to the connection between the back-action mechanism and the possibility of achieving a rectification of the supercurrent, i.e., the maximal positive forward amplitude turns out to be different from the maximal negative backward amplitude of the supercurrent.

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For this issue, we consider the dependence of the Josephson coupling on the supercurrent amplitude in two different cases: i) $E_J(I)$ is an even parity function with respect to I, $E_J(-I) = E_J(I)$. ii) $E_J(I)$ is an odd parity function with respect to I, $E_J(-I) = -E_J(I)$.

In the first scenario, we can observe that for an effective coupling $E_J(I)$ with even parity, the solution for $I(\phi)$ will present a definite and odd parity with respect to ϕ, i.e. $I(\phi) = -I(-\phi)$. This conclusion can be deduced by analyzing the parity of Eq. 3 and constructing the solution in an iterative way with respect to the strength of the back-action amplitude coupling. Hence, for an effective coupling that has an even parity dependence in the supercurrent amplitude, the current phase relation yields a reciprocal transport.

In the second scenario ii), with $E_J(I)$ being an odd parity function with respect to I, the solution of the Eq. 3 will give a current phase relation that does not have a definite parity in the phase bias ϕ and therefore nonreciprocal transport. We can arrive to this conclusion without solving the self-consistent equations from the first derivative of the supercurrent

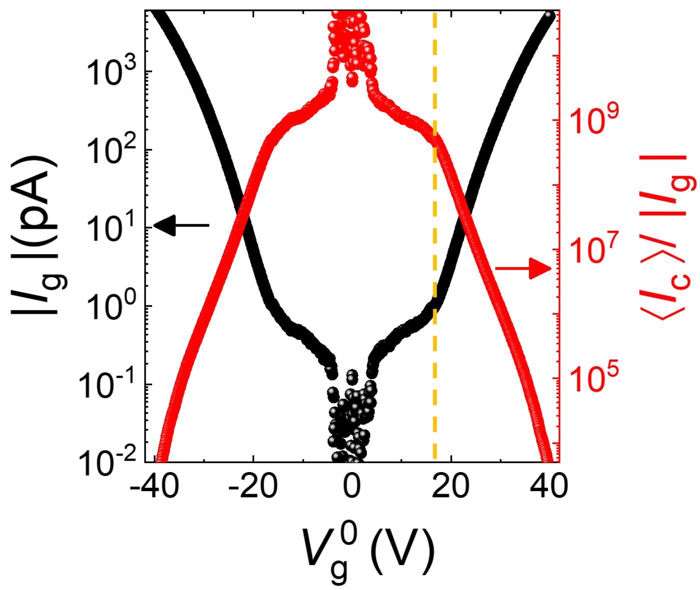
$$\frac{\partial I(\phi)}{\partial \phi} = \frac{I(\phi) - E_J(I)\sin(\phi)}{(1-\cos\phi)\frac{\partial E_J(I)}{\partial I}} \,. \tag{4}$$

obtained from Eq. 3.

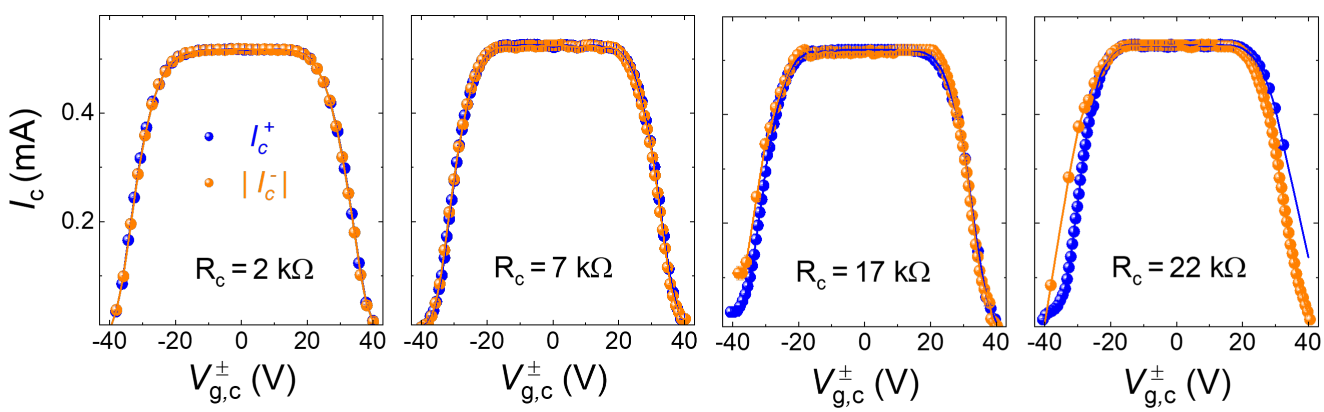
Maximum amplitude for the positive I_c^+ and negative I_c^- supercurrents is given by the condition $\frac{\partial I(\phi)}{\partial \phi} = 0$ (assuming that there are no discontinuities in the current phase relation). Since $I(\phi)$ does not have a definite parity in the variable ϕ, $I_c^+ = I(\phi_+) \neq I_c^- = I(\phi_-)$. This result implies that the supercurrent will be nonreciprocal for any functional form of the effective coupling $E_J(I)$ that includes odd parity terms with respect to the supercurrent amplitude I.

The model can be also extended to non-tunnel superconducting circuits in which the free energy may be more complex with respect to the simple cosine function. In particular, for reciprocal junctions, the cosine function will be substituted by a generic periodic even function, and through simple parity constraints the same conclusions can be extracted.

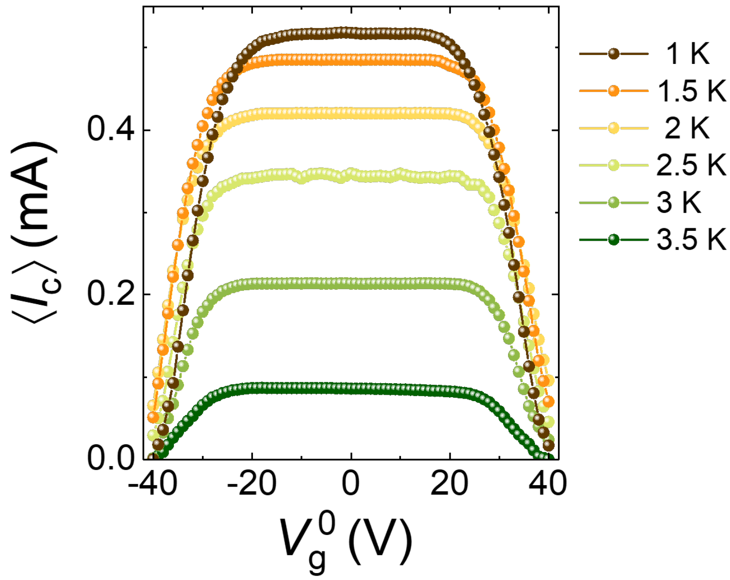
Supplementary Figures



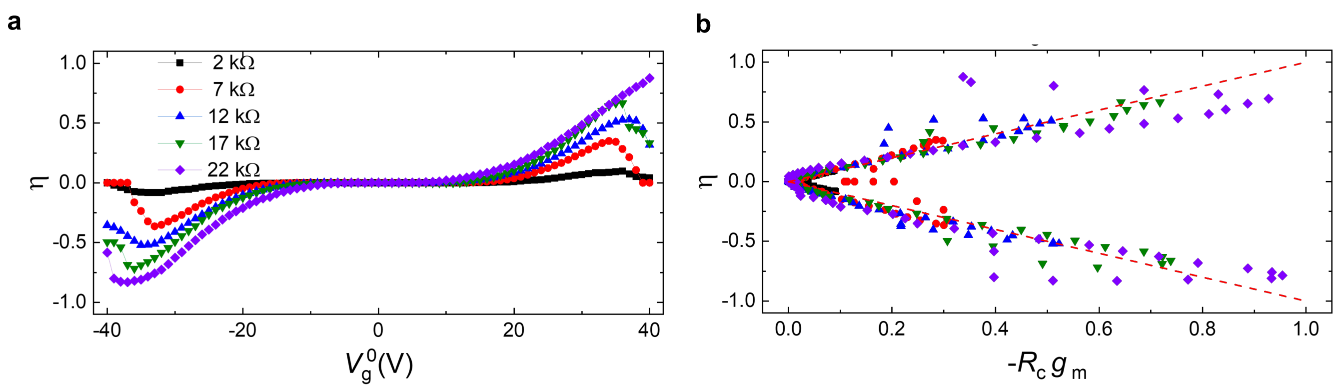
Supplementary Figure 1. **Gate current.** Gate-delivered leakage current to the weak link I_g at T =1 K (black dots). I_g increases from a few pA at the onset of the critical current $\langle I_c \rangle$ suppression (yellow dashed line) to a few nA at full suppression. $\langle I_c \rangle / I_g$ ratio is given by red dots.



Supplementary Figure 2. **Back-action normalization.** Quasi-symmetric behavior of the critical currents obtained by normalizing the gate voltage with the back-action contribution, $V_{g,c}^{\pm} = V_g - I_c^{\pm} R_c$ for different control resistors R_c.



Supplementary Figure 3. **Temperature dependence of the gate-controlled supercurrent.** Average critical current $\langle I_c \rangle$ as a function of the gate voltage V_g^0 for selected values of bath temperature.



Supplementary Figure 4. **Diode rectification efficiency. a,** Diode rectification parameter η as a function of the applied gate voltage V_g^0 for different control resistors R_c. **b,** η as a function of the back-action parameter $\alpha = -R_c g_m$ extracted from **a,** with the same color code. Negative rectification efficiency corresponds to negative gate voltages. Red dashed lines represent the ideal rectification $\eta = \alpha$ given by the linear model.