Methods

In this work, we derive formula for RDS methods. For the convenience of description, the basic definitions are defined as follows.

Basic Definition 1: $\mathbf{x}^{\mathbf{DV}} = \left(x_1^{DV}, x_2^{DV}, \cdots x_j^{DV}, \cdots x_m^{DV}\right)$, a vector, which is composed of all design variables. The number of all design variables is *m*.

Basic Definition 2: $\mathbf{x}^{\mathbf{Pareto}} = \left[\mathbf{x_1^P}; \mathbf{x_2^P}; \cdots \mathbf{x_j^P}; \cdots; \mathbf{x_k^P}\right]$, a matrix, also called non-inferior solution set, which is composed of all Pareto Front Set. The number of all the solutions is *k*.

Basic Definition 3: $\mathbf{x_j^P} = \left(x_1^{P\_j}, x_2^{P\_j}, \cdots x_j^{P\_j}, \cdots x_m^{P\_j}\right)$, a vector representing the *j*th solution in the Pareto Front Set. In this set of solutions, the value of the design variable x_j^{DV} is $x_j^{P\_j}$.

Basic Definition 4: $\mathbf{x_i} = \left(x_{i\_1}, x_{i\_2}, \cdots, x_{i\_j}, \cdots, x_{i\_n}\right)$, a vector composes of design variable related to the *i*th constraint condition.

Basic Definition 5: x_{i_j}, $x_{i\_j} \in \mathbf{x_i}$, a scalar, which represents the *j*th design variable in the vector of $\mathbf{x_i}$.

Basic Definition 6: $g_i = f_i\left(\mathbf{x_i}\right)$, a function $f_i\left(\cdot\right)$, which represents the *i*th constraint condition. The number of design variables in g_i is n.*Basic Definition 7*: $\mathbf{x}^{\mathbf{RDS}} = \left(x_1^{RDS}, x_2^{RDS}, \cdots x_m^{RDS}\right)$, a vector parameter representing the optimal solution screened by the RDS method from the Pareto Front Set $\mathbf{x}^{\mathbf{Pareto}}$, $\mathbf{x}^{\mathbf{RDS}} \subset \mathbf{x}^{\mathbf{Pareto}}$.

1.Rise-Dimension Transformation of Design Variables

We use the KDE method[28] to construct the PDF of the design variable x_{i_j} with historical data as samples.

Definition 1 (Historical Data): The historical data of the design variable x_{i_j} refers to its *Sample Database* formed by the actual value of x_{i_j} after a previous decision.

When describing more than two design variables' historical data (e.g., x_{i_i}, x_{i_j} and x_{i_k}), the actual values of these design variables should appear in the same optimization model. Otherwise, the calculation of the correlation coefficient is invalid.

Definition 2 (Sample Database): The sample database of x_{i_j} can be expressed as $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$, and the expansion form of this vector as $\tilde{\mathbf{x}}_{\mathbf{i\_j}} = \left[\tilde{x}_{i\_j}^{(1)}, \tilde{x}_{i\_j}^{(2)}, \cdots, \tilde{x}_{i\_j}^{(s)}, \cdots, \tilde{x}_{i\_j}^{(m)}\right]^T$, where $\tilde{x}_{i\_j}^{(s)}$ represents the actual value of the *s*th sample. The total number of samples is *m*.

Definition 3 (Sample Database's mean & Standard Deviation Value): The sample database of $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$'s mean value $\tilde{x}_{i\_j}^{mean}$ and standard deviation value $\tilde{x}_{i\_j}^{std}$ can be expressed as

$$\begin{cases} \tilde{x}_{i\_j}^{mean} = \dfrac{1}{m}\left[\tilde{x}_{i\_j}^{(1)} + \tilde{x}_{i\_j}^{(2)} + \cdots + \tilde{x}_{i\_j}^{(m)}\right] \\ \tilde{x}_{i\_j}^{std} = \dfrac{1}{m-1}\sqrt{\sum_{s=1}^{m}\left(\tilde{x}_{i\_j}^{(s)} - \tilde{x}_{i\_j}^{mean}\right)^2} \end{cases}$$

Definition 4 (Nominal Value): In the sample database of $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$, the nominal value of x_{i_j} can be expressed as $\overline{x}_{i\_j}$.The nominal value $\overline{x}_{i\_j}$ is a scalar representing the theoretical value initially selected for the design variable of x_{i_j}. The vector formed by the nominal values of all design variables $\mathbf{x_i}$ is represented by $\overline{\mathbf{x}}_{\mathbf{i}}$.

By using the sample database $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$, a set of data can be constructed that satisfies the "confidence level" of x_{i_j} as $1-\alpha$ $(0<\alpha<1)$, and the "confidence interval"[29] as $\left(\underline{\theta}, \overline{\theta}\right) = \left(\min\left(\tilde{\mathbf{x}}_{\mathbf{i\_j}}\right), \max\left(\tilde{\mathbf{x}}_{\mathbf{i\_j}}\right)\right)$. The kernel density estimator of x_{i_j} can be expressed as Eq.(1).

$$\hat{f}_{i\_j}\left(x_{i\_j}\right) = \frac{1}{m}\sum_{s=1}^{m} K_h\left(x_{i\_j} - \tilde{x}_{i\_j}^{(s)}\right) \tag{1}$$

where $K_h(\cdot)$ is a kernel function. h is a positive hyper-parameter called "bandwidth".

m is equal to the number of samples.

Theorem 1.[30] Assume that f is continuous at x and that $h \to 0$ and $mh \to \infty$ as $m \to \infty$. Then

$$\hat{f}(x) \xrightarrow{P} f(x)$$

According to *Assumption 1* and *Theorem 1*, if the sample size of historical data is large enough, we may safely consider $\hat{f}_{i\_j}(x_{i\_j})$ as the PDF of x_{i_j}.

*Lemma 1.*The first moment of $\hat{f}_{i\_j}(x_{i\_j})$ is equal to the first moment of $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$.

The proofs of Lemma 1: The first moment with $\hat{f}_{i\_j}(x_{i\_j})$ as the PDF can be expressed as

$$\begin{aligned}
&\int_{-\infty}^{+\infty} x \cdot \hat{f}_{i_j}(x)dx = \frac{1}{mh}\sum_{s=1}^{m}\int_{-\infty}^{+\infty} x \cdot K\left(\frac{x-\tilde{x}_{i_j}^{(s)}}{h}\right)dx \\
&= \frac{1}{mh}\sum_{s=1}^{m}\left[h\int_{-\infty}^{+\infty}\left(hu+\tilde{x}_{i_j}^{(s)}\right)K(u)du\right] = \frac{1}{m}\sum_{s=1}^{m}\left[\int_{-\infty}^{+\infty}(hu)K(u)du+\int_{-\infty}^{+\infty}\left(\tilde{x}_{i_j}^{(s)}\right)K(u)du\right] \\
&= \frac{1}{m}\sum_{s=1}^{m}\left[h\int_{-\infty}^{+\infty}(u)K(u)du+\left(\tilde{x}_{i_j}^{(s)}\right)\int_{-\infty}^{+\infty}K(u)du\right] = \frac{1}{m}\sum_{s=1}^{m}\left[h\cdot 0+\left(\tilde{x}_{i_j}^{(s)}\right)\cdot 1\right] = \frac{1}{m}\sum_{s=1}^{m}\left[\tilde{x}_{i_j}^{(s)}\right]
\end{aligned}$$

*Lemma 2.*The second moment of $\hat{f}_{i\_j}(x_{i\_j})$ converges to the second moment of $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$.

The proofs of Lemma 2: The second moment with $\hat{f}_{i\_j}(x_{i\_j})$ as the PDF can be expressed as

$$\begin{aligned}
&\int_{-\infty}^{+\infty} x^2 \cdot \hat{f}_{i_j}(x)dx \\
&= \frac{1}{mh}\sum_{s=1}^{m}\int_{-\infty}^{+\infty} x^2 \cdot K\left(\frac{x-\tilde{x}_{i_j}^{(s)}}{h}\right)dx = \frac{1}{mh}\sum_{s=1}^{m}\left[h\int_{-\infty}^{+\infty}\left(\tilde{x}_{i_j}^{(s)}+hu\right)^2 K(u)du\right] \\
&= \frac{1}{m}\sum_{s=1}^{m}\left[\int_{-\infty}^{+\infty}\left(\left(\tilde{x}_{i_j}^{(s)}\right)^2 K(u)+2\tilde{x}_{i_j}^{(s)}huK(u)+h^2u^2K(u)^2\right)du\right] \\
&= \frac{1}{m}\sum_{s=1}^{m}\left[\left(\tilde{x}_{i_j}^{(s)}\right)^2+0+h^2k_{21}\right] \to \frac{1}{m}\sum_{s=1}^{m}\left[\left(\tilde{x}_{i_j}^{(s)}\right)^2\right]
\end{aligned}$$

The design variable's CDF can be expressed as Eq.(2).

$$\hat{F}_{i\_j}(x)=\int_{-\infty}^{x}\hat{f}_{i\_j}(x_{i\_j})dx_{i\_j} \tag{2}$$

According to *Assumption 2*, the design variable x_{i_j} is independent of time, the optimization model designer's cognition of the design variable remains unchanged, and the design variable's nominal value is determined. Therefore, the CDF of $\overline{x}_{i\_j}$ is constant, which can be expressed as Eq.(3).

$$\hat{F}_{i\_j}(\overline{x}_{i\_j})=\int_{-\infty}^{\overline{x}_{i\_j}}\hat{f}_{i\_j}(x_{i\_j})dx_{i\_j}=\text{constant} \tag{3}$$

2.Equal Cumulative Probability Transformation of Design Variable's PDF

According to *Lemma 1* and *Lemma 2*, $\hat{f}_{i\_j}(x_{i\_j})$ can reflect the character of the first and second moment of the design variable $\tilde{\mathbf{x}}_{\mathbf{i\_j}}$. In addition, the Gaussian distribution function is composed of the mean (first moment), standard deviation (second moment), and correlation coefficient matrix (1-dimension Gaussian distribution is excluded). Here we introduce how to transform $\hat{f}_{i\_j}(x_{i\_j})$ into a Gaussian distribution function and calculate the converted correlation coefficient matrix. According to the *Basic Definition 6*, there are *n* numbers of design variables ($x_{i\_1},x_{i\_2},\cdots,x_{i\_n}$) in the constraint condition g_i. According to *Assumption 2*, for any two design variables x_{i_i} and x_{i_j}, the historical data can be used to calculate their "time-invariant-linear" correlation coefficient ρ_{ix_ij} as Eq.(4), which is also called the "Pearson Correlation Coefficient"[31].

$$\rho_{ix\_ij}=\frac{\text{cov}(\tilde{\mathbf{x}}_{\mathbf{i\_i}},\tilde{\mathbf{x}}_{\mathbf{i\_j}})}{\sigma_{\tilde{\mathbf{x}}_{\mathbf{i\_i}}}\sigma_{\tilde{\mathbf{x}}_{\mathbf{i\_j}}}} \tag{4}$$

Therefore, the design variable $\mathbf{x_i}$'s correlation coefficient matrix $\boldsymbol{\rho}_{\mathbf{ix}}$ in the constraint condition g_i can be expressed as Eq.(5).

$$\boldsymbol{\rho}_{\mathbf{ix}} = \begin{bmatrix} \rho_{ix\_11}, \rho_{ix\_12}, \cdots, \rho_{ix\_1j}, \cdots, \rho_{ix\_1n} \\ \rho_{ix\_21}, \rho_{ix\_22}, \cdots, \rho_{ix\_2j}, \cdots, \rho_{ix\_2n} \\ \vdots\ , \ \vdots \quad , \ddots, \rho_{ix\_ij}\ , \ \cdots, \rho_{ix\_in} \\ \rho_{ix\_n1}, \rho_{ix\_n2}, \cdots, \rho_{ix\_nj}, \ \cdots, \rho_{ix\_nn} \end{bmatrix} \tag{5}$$

We construct a Gaussian distribution vector $\mathbf{y_i} = \left(y_{i\_1}, y_{i\_2}, \cdots, y_{i\_n}\right)$, and establish the relationship between $\mathbf{x_i}$ and $\mathbf{y_i}$ though the Equal Cumulative Probability Transformation (ECP Transformation)[32] as Eq.(6) .

$$\begin{cases} \Phi_{\sigma_{i\_j}}\left(y_{i\_j}\right) = \hat{F}_{i\_j}\left(x_{i\_j}\right) \\ y_{i\_j} = \Phi^{-1}_{\sigma_{y\_i\_j}}\left(\hat{F}_{i\_j}\left(x_{i\_j}\right)\right), j = 1, 2, \cdots, n \\ \boldsymbol{\rho}_{\mathbf{iy}} = \left(\rho_{iy\_ij}\right)_{n\times n} \end{cases} \tag{6}$$

where $\Phi_{\sigma_{y\_i\_j}}\left(y_{i\_j}\right)$ and $\Phi^{-1}_{\sigma_{y\_i\_j}}\left(F_{i\_j}\left(x_{i\_j}\right)\right)$ are the CDF and the inverse CDF of the Gaussian distribution variable with the mean value of 0 and the standard deviation of $\sigma_{y\_i\_j}$, respectively. $\boldsymbol{\rho}_{\mathbf{iy}}$ is the correlation coefficient of $\mathbf{y_i}$.

$\sigma_{y\_i\_j}$ is a hyper parameter. Here we give one of the commonly used expression. Let the standard deviation of y_{i_j} corresponding to x_{i_j} to be 1, that is $\sigma_{y\_i\_j} = 1$. The standard deviation of other parameter y_{i_k} corresponding to x_{i_k} can be expressed as Eq.(7).

$$\sigma_{y\_i\_k} = \tilde{x}^{std}_{i\_k} \big/ \tilde{x}^{std}_{i\_j} \tag{7}$$

If all $\sigma_{y\_i\_j}$ are defined as 1, Eq.(6) is called NATAF[33] Transformation. Eq.(8) can be obtained according to Eq.(6).

$$\begin{cases} (a) \quad \hat{F}_{i\_j}\left(x_{i\_j}\right) = \Phi_{\sigma_{i\_j}}\left(y_{i\_j}\right) \\ (b) \quad x_{i\_j} = \hat{F}^{-1}_{i\_j}\left[\Phi_{\sigma_{y\_i\_j}}\left(y_{i\_j}\right)\right] \\ (c) \quad y_{i\_j} = \Phi^{-1}_{\sigma_{y\_i\_j}}\left[\hat{F}_{i\_j}\left(x_{i\_j}\right)\right] \end{cases} \tag{8}$$

The nominal value $\bar{x}_{i\_j}$ of the design variable x_{i_j} is transformed into $\bar{y}_{i\_j}$. In constraint condition g_i , the vector expression in which CDF of all nominal values $\bar{\mathbf{x}}_{\mathbf{i}}$ is transformed into $\overline{\mathbf{y}}_{\mathbf{i}}$, and its expression is shown in Eq.(9).

$$\overline{\mathbf{y}}_{\mathbf{i}} = \mathbf{\Phi}^{-1}_{\boldsymbol{\sigma}_{\mathbf{y\_i}}}\left[\mathbf{F}_{\mathbf{i}}\left(\mathbf{x}_{\mathbf{i}}\right)\right] \tag{9}$$

Eq.(10) can be obtained according to perform a differential transformation on Eq.(8-a).

$$\hat{f}_{i\_j}\left(x_{i\_j}\right)dx_{i\_j} = \phi_{\sigma_{y\_i\_j}}\left(y_{i\_j}\right)dy_{i\_j} \tag{10}$$

where $\phi_{\sigma_{y\_i\_j}}(\cdot)$ is the PDF of the Gaussian distribution variable with the mean value of 0 and the standard deviation of $\sigma_{y\_i\_j}$.

Equation (11) can be obtained by modifying Eq. (10)[34].

$$\left.\frac{dx_{i\_j}}{dy_{i\_j}}\right|_{y^*} = \left.\frac{\partial x_{i\_j}}{\partial y_{i\_j}}\right|_{y^*} = \frac{\phi_{i\_j}\left(y_i^*\right)}{\hat{f}_{i\_j}\left(x_i^*\right)} \tag{11}$$

Since the CDF of both $\Phi_{\sigma_{y\_i\_j}}\left(y_{i\_j}\right)$ and $\hat{F}_{i\_j}\left(x_{i\_j}\right)$ are monotonically increasing functions, Eq.(11) and Eq.(8-a) is a one-to-one correspondence. The transformation $x_{i\_j} \leftrightarrow y_{i\_j}$ is also one-to-one correspondence. The Joint Probability Distribution (JPD) of $\mathbf{x}_{\mathbf{i}}$ can be written as Eq. (12)[35].

$$f_{\mathbf{X}_i}\left(\mathbf{x}_i\right) = \det \mathbf{J}_{\mathbf{yx}} \phi_n\left(\mathbf{y}_i, \mathbf{0}, \boldsymbol{\sigma}_{\mathbf{y\_i}}, \boldsymbol{\rho}_{\mathbf{iy}}\right) \tag{12}$$

where $f_{\mathbf{X}_i}\left(\mathbf{x}_i\right)$ is the JPD of $\mathbf{x}_{\mathbf{i}}$. $\det \mathbf{J}_{\mathbf{yx}}$ is the Jacobian Determinant of Eq.(8-c). $\boldsymbol{\sigma}_{\mathbf{y\_i}}$ is the vector representation of $(\sigma_{y\_i\_1};\sigma_{y\_i\_2};\cdots;\sigma_{y\_i\_j})$.

According to Eq.(8-a), the expansion form of $\det \mathbf{J}_{\mathbf{yx}}$ can be obtained as Eq.(13).

$$\det \mathbf{J}_{\mathbf{yx}} = \prod_{i=1}^{n} \frac{\hat{f}_{i\_j}\left(x_{i\_j}\right)}{\phi_{\sigma_{y\_i\_j}}\left(y_{i\_j}\right)} \tag{13}$$

Therefore, Eq.(12) can be expanded to Eq. (14)[36].

$$f_{\mathbf{X}_i}\left(\mathbf{x}_i\right) = \frac{\hat{f}_{i\_1}\left(x_{i\_1}\right)\times\cdots\hat{f}_{i\_n}\left(x_{i\_n}\right)\times\phi_n\left(\mathbf{y}_i, \mathbf{0}, \boldsymbol{\sigma}_{\mathbf{i}}, \boldsymbol{\rho}_{\mathbf{iy}}\right)}{\phi_1\left(y_{i\_1}, 0, \sigma_{y\_i\_1}\right)\times\cdots\phi_1\left(y_{i\_n}, 0, \sigma_{y\_i\_n}\right)} \tag{14}$$

In Eq.(12) and Eq.(14), $\phi_{\mathbf{n}}\left(\mathbf{y}_{\mathbf{i}}, \mathbf{0}, \boldsymbol{\sigma}_{\mathbf{y\_i}}, \boldsymbol{\rho}_{\mathbf{iy}}\right)$ is an n-dimension Gaussian distribution function with the correlation coefficient matrix as $\boldsymbol{\rho}_{\mathbf{iy}}$, the mean value as 0, and the standard deviation matrix as $\boldsymbol{\sigma}_{\mathbf{y\_i}}$. The expanded expression of $\phi_{\mathbf{n}}\left(\mathbf{y}_{\mathbf{i}},\mathbf{0},\boldsymbol{\sigma}_{\mathbf{y\_i}},\boldsymbol{\rho}_{\mathbf{iy}}\right)$ can be written as Eq.(15).

$$\phi_{\mathbf{n}}\left(\mathbf{y}_{\mathbf{i}},\mathbf{0},\boldsymbol{\sigma}_{\mathbf{y\_i}},\boldsymbol{\rho}_{\mathbf{iy}}\right)=\frac{\mathbf{1}}{\left(\mathbf{2\pi}\right)^{\mathbf{n/2}}\left|\Sigma_{\mathbf{y_i}}\right|^{\mathbf{1/2}}}\mathbf{exp}\left[-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{y}_{\mathbf{i}}\,\Sigma_{\mathbf{y_i}}^{-1}\,\mathbf{y}_{\mathbf{i}}^{\mathbf{T}}\right] \tag{15}$$

where $\Sigma_{\mathbf{y_i}}$ is the covariance matrix of $\mathbf{y_i}$ .The expanded expression of $\Sigma_{\mathbf{y_i}}$ can be written as Eq.(16).

$$\Sigma_{\mathbf{y_i}}=\begin{bmatrix} \rho_{iy\_11}\cdot\sigma_{y\_i\_1}^{2} & \cdots & \rho_{iy\_1n}\cdot\sigma_{y\_i\_1}\sigma_{y\_i\_n} \\ \rho_{iy\_12}\cdot\sigma_{y\_i\_1}\sigma_{y\_i\_2} & \cdots & \rho_{iy\_2n}\sigma_{y\_i\_2}\sigma_{y\_i\_n} \\ \vdots & \ddots & \vdots \\ \rho_{iy\_1n}\sigma_{y\_i\_1}\sigma_{y\_i\_n} & \cdots & \rho_{iy\_nn}\cdot\sigma_{y\_i\_n}^{2} \end{bmatrix} \tag{16}$$

According to Eq.(6) and Eq.(14), we may obtain the correlation coefficient matrix $\boldsymbol{\rho}_{\mathbf{iy}}$ as Eq. (17)[37].

$$\begin{aligned} \rho_{iy\_ij} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{x_{i\_i}-\mu_{x_{i\_i}}}{\sigma_{x_{i\_i}}}\right)\left(\frac{x_{i\_j}-\mu_{x_{i\_j}}}{\sigma_{x_{i\_j}}}\right)\cdot f_{\mathbf{X_i}}\cdot dx_{i\_i}\,dx_{i\_j} \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{\hat{F}_{i\_i}^{-1}\left(\Phi\left(y_{i\_i}\right)-\mu_{x_{i\_i}}\right)}{\sigma_{x_{i\_i}}}\right)\left(\frac{\hat{F}_{i\_j}^{-1}\left(\Phi\left(y_{i\_j}\right)-\mu_{x_{i\_j}}\right)}{\sigma_{x_{i\_j}}}\right)\cdot\varphi_2\cdot dy_{i\_i}dy_{i\_j} \end{aligned} \tag{17}$$

3.Independent Convert of Design Variable's PDF

We use Orthogonal Transformation[38] to convert the covariance matrix $\Sigma_{\mathbf{y_i}}$ into Diagonal Matrix $\Sigma_{\mathbf{z_i}}$. The element of $\Sigma_{\mathbf{z_i}}$ can be expressed as λ_{i1}, λ_{i2},⋯, λ_{in} , which is the eigenvalue of $\Sigma_{\mathbf{y_i}}$.

$$Let:\mathbf{z}_{\mathbf{i}}^{'}=\mathbf{A}_{\mathbf{i}}\times\mathbf{y}_{\mathbf{i}}^{\mathbf{T}}$$

where $\mathbf{A_i}$ is an orthogonal matrix, and its column vector is the eigenvector of $\Sigma_{\mathbf{y_i}}$.

The vector $\mathbf{z}_{\mathbf{i}}^{'}=\left(z_{i\_1}^{'},z_{i\_2}^{'},\cdots,z_{i\_n}^{'}\right)$ is a set of uncorrelated Gaussian distribution functions. The covariance $\Sigma_{\mathbf{z}_{\mathbf{i}}^{'}}$ of $\mathbf{z_i}$ can be expressed as Eq.(18).

$$\Sigma_{\mathbf{z}_\mathbf{i}'} = \boldsymbol{\lambda}_\mathbf{i} = \begin{bmatrix} \lambda_{i1} & & & \\ & \lambda_{i2} & & \\ & & \ddots & \\ & & & \lambda_{in} \end{bmatrix} = \mathbf{A}_\mathbf{i}^\mathbf{T} \times \Sigma_{\mathbf{y}_\mathbf{i}} \times \mathbf{A}_\mathbf{i} \tag{18}$$

Thus, a set of uncorrelated Gaussian distribution parameters $\mathbf{z}_\mathbf{i} = \left(z_{i\_1}, z_{i\_2}, \cdots, z_{i\_n}\right)$ is obtained. The conversion formula of $\mathbf{z}_\mathbf{i}$ and $\mathbf{y}_\mathbf{i}$ can be expressed as Eq.(19).

$$\mathbf{z}_\mathbf{i} = \sqrt{\boldsymbol{\lambda}_\mathbf{i}}^{-1} \mathbf{z}_\mathbf{i}' = \sqrt{\boldsymbol{\lambda}_\mathbf{i}}^{-1} \times \mathbf{A}_\mathbf{i}^\mathbf{T} \times \mathbf{y}_\mathbf{i}^\mathbf{T} = \mathbf{G}_\mathbf{i} \times \mathbf{y}_\mathbf{i}^\mathbf{T} \tag{19}$$

According to Eq.(19), the *j*th element z_{i_j} in $\mathbf{z}_\mathbf{i}$ can be written as Eq.(20).

$$z_{i\_j} = \left(\mathbf{G}_\mathbf{i}\right)_j \times \mathbf{y}_\mathbf{i}^\mathbf{T} \tag{20}$$

where $\left(\mathbf{G}_\mathbf{i}\right)_j$ represents the *j*th row in the matrix of $\sqrt{\boldsymbol{\lambda}_\mathbf{i}}^{-1} \times \mathbf{A}_\mathbf{i}^\mathbf{T}$.

According to *Assumption* 3, Eq.(15), Eq.(19), and the properties of the uncorrelated Gaussian distribution of $\mathbf{z}_\mathbf{i}$, we have completed the conversion from a correlation Gaussian distribution function $\phi_n\left(\mathbf{y}_\mathbf{i}, \mathbf{0}, \boldsymbol{\sigma}_\mathbf{i}, \boldsymbol{\rho}_\mathbf{iy}\right)$ to n numbers of 1-dimension independent Gaussian distribution function $\phi_1\left(\mathbf{z}_\mathbf{i}, \mathbf{0}, \boldsymbol{\sigma}_\mathbf{z\_i}\right)$. This means that the JPD of the design variables are equal to the product of the PDF of each design variables. Therefore, we can obtain Eq.(21).

$$\begin{cases} (a)\ \ \phi_n\left(\mathbf{y}_i, \mathbf{0}, \boldsymbol{\sigma}_\mathbf{y\_i}, \boldsymbol{\rho}_\mathbf{iy}\right) = \prod_{j=1}^{n} \phi_1\left(z_{i\_j}, 0, \sigma_{z\_i\_j}\right) \\ (b)\ \ \sigma_{z\_i\_j} = \sqrt{\mathbf{G}_\mathbf{i\_j}^2 \times \boldsymbol{\sigma}_\mathbf{y\_i}^2} \end{cases} \tag{21}$$

Eq.(21-a) describes the relationship between $\mathbf{y}_\mathbf{i}$ and $\mathbf{z}_\mathbf{i}$. Eq.(21-b) describes the standard deviation $\sigma_{z\_i\_j}$ of $\phi_1\left(z_{i\_j}, \sigma_{z\_i\_j}\right)$ based on Eq.(20). All of the parameters $\sigma_{z\_i\_j}$ in Eq.(21-b) can be represented by vector $\boldsymbol{\sigma}_\mathbf{z\_i}$.

4.Rise-Dimension Transformation of constraint condition

As described above, in constraint condition g_i, the design variables are converted to $\phi_1\left(\mathbf{z}_\mathbf{i}, \mathbf{0}, \boldsymbol{\sigma}_\mathbf{z\_i}\right)$. According to *Assumption* 4, the calculation result of g_i is also converted from the certainty value $f_i\left(\mathbf{x_i}\right)$ to the PDF $g_{i\_PDF}\left(\phi_1\left(\mathbf{z_i},\mathbf{0},\boldsymbol{\sigma}_\mathbf{i}\right)\right)$, which is defined in Eq.(22).

$$g_{i\_PDF}\left(\phi_1\left(\mathbf{z_i},\mathbf{0},\boldsymbol{\sigma}_\mathbf{z\_i}\right)\right)=f_i\left(\phi_1\left(z_{i\_1},\sigma_{z\_i\_1}\right),\cdots,\phi_1\left(z_{i\_n},\sigma_{z\_i\_n}\right)\right) \tag{22}$$

The formal solution of Eq.(22), named ***g-solution***, can be written in terms of a Dirac Delta Function as Eq. (23)[39].

$$g_{i\_PDF}\left(\phi_1\left(\mathbf{z_i},\mathbf{0},\boldsymbol{\sigma}_\mathbf{z\_i}\right)\right)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{j=1}^{n}\phi_1\left(z_{i\_j},\sigma_{i\_j}\right)\delta\left(s-f_i\left(\mathbf{z_i}\right)\right)dz_{i\_1}dz_{i\_2}\cdots dz_{i\_n} \tag{23}$$

The definition of the Dirac Delta Function[40] is shown in Eq.(24).

$$\begin{cases}\delta\left(s\right)=\begin{cases}+\infty, s=0\\ 0, s\neq 0\end{cases}\\ \int_{-\infty}^{+\infty}\delta\left(s\right)ds=1\end{cases} \tag{24}$$

Equation (23) is multiple integral issues and involves the generalized integration of s in the Dirac Delta Function. The calculation is complex, and the numerical analysis method is usually used for calculation. However, if the constraint condition's expression is a linear combination of design variable $\mathbf{x_i}$, such as Eq.(25), then the expression of $g_{i\_PDF}\left(\phi_1\left(\mathbf{z_i},\boldsymbol{\sigma}_\mathbf{z\_i}\right)\right)$ can be directly written as Eq.(26) .

$$f_i\left(\phi_1\left(z_{i\_1},\sigma_{z\_i\_1}\right),\cdots,\phi_1\left(z_{i\_n},\sigma_{z\_i\_n}\right)\right)=\sum_{j=1}^{n}c_j\cdot\phi_1\left(z_{i\_j},\sigma_{z\_i\_j}\right) \tag{25}$$

$$g_{i\_PDF}\left(\phi_1\left(\mathbf{z_i},\boldsymbol{\sigma}_\mathbf{z\_i}\right)\right)=\phi_1\left(w,0,\sqrt{\sum_{j=1}^{n}c_j^2\sigma_{z\_i\_j}^2}\right) \tag{26}$$

5.Screen the Pareto Front Solution Set

In constraint condition g_i, for different Pareto Front sets $\mathbf{x^{Pareto}}$, the mean value of $\phi_1\left(\mathbf{z_i},\boldsymbol{\sigma}_\mathbf{z\_i}\right)$ is a variable parameter, and the standard deviation of $\phi_1\left(\mathbf{z_i},\boldsymbol{\sigma}_\mathbf{z\_i}\right)$ is unchanged. If the formal solution of Eq.(22) is expressed as Eq.(23), for the jth solution, $\mathbf{x_j^P}$, in the Pareto Front Set, the vector formed by its mean value $\boldsymbol{\mu}_\mathbf{j\_z}^\mathbf{REAL}$ can be defined as

Eq.(27).

$$\boldsymbol{\mu}_{\mathbf{j\_z}}^{\mathbf{REAL}} = \left(\mathbf{G_i}\right)_j \times \boldsymbol{\mu}_{\mathbf{j\_y}}^{\mathbf{REAL}} = \left(\mathbf{G_i}\right)_j \times \left(\mathbf{x_j^P} - \overline{\mathbf{y}}_{\mathbf{i}}\right)^T \tag{27}$$

where the specific expression of $\left(\mathbf{G_i}\right)_j$ is stated in **Eq.(20)**, and the particular expression of $\overline{\mathbf{y}}_{\mathbf{i}}$ is stated in **Eq.(9)**.

Equation (27) can be understood in two steps. Step1, according to Eq.(3), the CDF of nominal value $\overline{x}_{i\_j}$ is the same in the historical data and in the Pareto Front where the screening non-inferior solutions needs to be made. By the ECP Transformation of Eq.(6), the PDF of x_{i_j} is mapped to the Gaussian distribution function, with a mean value of 0. In the Pareto Front Set, $x_{i\_j} = x_{i\_j}^{P\_j}$ (which means in the *j*th non-inferior solutions, the design value of x_{i_j} equals $x_{i\_j}^{P\_j}$) is equivalent to moving the whole PDF to the right of the number axis by $x_{i\_j}^{P\_j}$ units. According to *Assumption 2*, suppose that the mean value of the Gaussian distribution function corresponding to the non-inferior solution $\mathbf{x_j^P}$ is $\boldsymbol{\mu}_{\mathbf{j\_y}}^{\mathbf{REAL}}$. We may safely obtain Eq.(28), which means the real mean value of the Gaussian distribution function corresponding to each non-inferior solution is a function of the offset between the non-inferior and the nominal parameter.

$$\mathbf{0} + \mathbf{x_j^P} = \overline{\mathbf{y}} + \boldsymbol{\mu}_{\mathbf{j\_y}}^{\mathbf{REAL}} \Rightarrow \boldsymbol{\mu}_{\mathbf{j\_y}}^{\mathbf{REAL}} = \mathbf{x_j^P} - \overline{\mathbf{y}} \tag{28}$$

Step2, according to *Assumption 3*, since the design variables are also linear related, it is difficult to obtain their PDF through $g_i = f_i\left(\mathbf{x_i}\right)$ directly. Therefore, we use Eq.(20) to perform a linear transformation on the design variables so that the PDF after the conversion are independent of each other.

If the formal solution of Eq.(22) is expressed as Eq.(26), which means $f_i\left(\cdot\right)$ is a linear function expression, for the *j*th solution $\mathbf{x_j^P}$ in the Pareto Front Set, the vector formed by its mean value μ_j^{real} can be expressed as Eq.(29).

$$\begin{cases} \mu_j^{REAL} = f_i(\boldsymbol{\omega}) \\ \boldsymbol{\omega} = (\mathbf{G_i})_j \times \left(\mathbf{x_j^P} - \overline{\mathbf{y}}_\mathbf{i}\right)^T \end{cases} \tag{29}$$

where $\boldsymbol{\omega}$ is a vector representing the mean value of the normal PDF corresponding to each independent design variable in g_i, and $f_i(\cdot)$ is the functional representation of g_i.

According to *Assumption 2*, for the design variable x_{i_j}, the characteristics of all the historical data and Pareto Front's PDF are the same except for the mean value. In other words, compared with historical data, different non-inferior solutions are equivalent to moving the entire PDF on the number axis, and the variance has not been changed. Therefore, the vector formed by the design variables' standard deviation value $\boldsymbol{\sigma}_\mathbf{j\_z}^\mathbf{REAL}$ can be expressed as Eq.(30).

$$\boldsymbol{\sigma}_\mathbf{j\_z}^\mathbf{REAL} = \boldsymbol{\sigma}_\mathbf{z\_i} \tag{30}$$

where the specific expression of $\boldsymbol{\sigma}_\mathbf{z\_i}$ is stated in Eq.(21).

According to *Assumption 4*, the constraint conditions are independent of each other. Therefore, it is possible to calculate the probability of simultaneously satisfying all constraint conditions by multiplying the feasibility probabilities of each individual constraint condition, as shown in Eq.(31-a). By plugging all non-inferior solutions into Eq.(31-a), we can obtain the probabilities of different non-inferior solutions satisfying the constraint conditions. According to the definition of RDS, the solution with the highest probability value is selected as the final decision, as shown in Eq.(31-b).

$$\begin{cases} (a)\ \Re\left(\mathbf{x}_{\mathbf{j}}^{\mathbf{p}}\right) = \left(\prod_{i} 100 \times g_{i\_RDS}\left(\boldsymbol{\mu}_{\mathbf{j\_z}}^{\mathbf{REAL}}, \boldsymbol{\sigma}_{\mathbf{j\_z}}^{\mathbf{REAL}}\right)\right)^{\frac{1}{i}} \\ (b)\ \mathbf{x}^{\mathbf{RSD}} = \max\left\{\Re\left(\mathbf{x}^{\mathbf{Pareto}}\right)\right\} \end{cases} \tag{31}$$