

Novel nonlinear wave transitions and interactions for $(2 + 1)$ -dimensional generalized fifth-order KdV equation

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Abstract

The $(2 + 1)$ -dimensional generalized fifth-order KdV (2GKdV) equation is revisited via combined physical and mathematical methods. By the Hirota perturbation expansion technique and via setting nonzero background wave on the multiple soliton solution of the 2GKdV equation, breather waves are constructed, for which some transformed wave conditions are considered that yield abundant novel nonlinear waves including the X/Y -Shaped (XS/YS), asymmetric M -Shaped (MS), W -Shaped (WS), Space-Curved (SC) and Oscillation M -Shaped (OMS) solitons etc. Furthermore, distinct nonlinear wave molecules and interactional structures involving the asymmetric MS, WS, XS/YS, SC solitons, and breathers, lumps are constructed after considering the corresponding existence conditions. The dynamical properties of the obtained nonlinear molecular waves and interactional structures are revealed via analyzing the trajectory equations along with the change of the phase shifts.

Keywords: $(2+1)$ -dimensional generalized fifth-order KdV equation; Nonlinear molecular wave; Phase shift; Velocity resonance; Inelastic interaction.

1. Introduction

The study of nonlinear wave bounded state structures set off a new research upsurge in more and more fields, such as optical fiber communication, fluid mechanics, biophysics and information science [1, 2, 3]. Investigating the above problems are of great significance because they can provide valuable mathematical physics information and theoretical support for nonlinear phenomena and experimental results analysis in some scientific fields [4, 5, 6, 7]. The types of nonlinear waves can be roughly divided into four categories: soliton, breather, lump and rogue wave. Solitons have many intriguing properties of particles and waves, reflecting common nonlinear characteristics in nature. It is a well-known fact that periodicity is a very basic and important attribute of nonlinear wave propagation, originating from periodic coherence between atoms. The breather perfectly describes such a solution that is periodic in both time and space dimensions [8]. The rogue wave depicts steep wave localized in both space and time dimensions [9]. Lump solution is constructed

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in many integrable models, it is a type of rational exact solution and spatially localized in certain region [10]. The aforementioned nonlinear wave structures can transform each other under the particular conditions, which produce many new wave structures.

Several bounded (coherent) states of solitons including soliton molecules (SMs) and breather molecules (BMs) have been a hot topic owing to they can provide application and extensively theory values in nonlinear physics fields. The velocity resonance involved in this article specifically includes two conditions: the trajectory equations of nonlinear wave components are parallel, and further, their propagation velocities are equal. Thus, multifarious novel kinds of the SMs and BMs have been constructed successively via utilizing velocity resonance. Heretofore, the interactional solutions among different types of nonlinear waves have been deeply investigated, because the collision behaviors can be utilized to accurately imitate the nonlinear behaviors in nature, from the view of this point, the elastic collision and inelastic collision modes, the emergence and degradation behaviors of solitons, the oscillatory and non-oscillatory modes have been carefully studied. Nevertheless, there are few investigations on the interactional structures among novel kinds of nonlinear molecular waves, resonance soliton solutions and breathers, especially the collision-free interaction modes, which arouses our interest to explore this kind solution in another model.

In this paper, we consider the (2+1)-dimensional generalized fifth-order KdV (2GKdV) equation

$$u_t + \alpha u_{5x} + \beta uu_{xxx} + \gamma u_x u_{xx} + \delta u^2 u_x + u_y = 0, \quad (1.1)$$

where α , β , γ and δ are real constant coefficients of the dispersion term and higher-order derivative terms in 2GKdV equation, respectively. Eq. (1.1) demonstrates the behaviors of long wave under the gravity and in a two-dimensional nonlinear lattice in shallow water [11, 12, 13]. Eq. (1.1) could convert into the (1+1)-dimensional Sawada-Kotera equation [14, 15] via removing the derivative term u_y and setting $\alpha = 1, \beta = 15, \gamma = 15$ and $\delta = 45$. There are numerous study that have focused on the 2GKdV equation such as the lump wave and interaction structures of Eq. (1.1) were acquired in Ref. [16], the complexitons and periodic soliton solutions were also studied in Ref. [17]; nonlinear superposition phenomena between lump solitons and other forms of nonlinear localized waves were examined in Ref. [18] and the lump-periodic, breather and two-soliton solutions were presented in Ref. [19].

The article is structured as follows. In Sec. 2, we first revisit the N -soliton solution for the 2GKdV equation via the Hirota perturbation expansion method, then obtain the first-order breather, X-Shaped soliton and lump of the 2GKdV equation. Moreover, we analyze the dynamics of the breather, X-Shaped soliton and lump wave. Next, we present a new class of transformed wave conditions to construct different nonlinear wave structures such as the asymmetric MS soliton, the XS/YS soliton, the OMS soliton, the SC soliton and lattice periodic soliton, etc. In Sec. 3, new types of molecule structures and interactional structures are constructed via the velocity resonance ansatz. We also demonstrate the dynamical properties of the above nonlinear

wave structures. The last section is a brief summary.

2. The X -Shaped soliton, breather, lump and other types of nonlinear waves

According to the Refs. [16, 20], we adopt the following bilinear transformation under the conditions that $\beta = 15\alpha$, $\gamma = 45\alpha$, $\delta = 15\alpha$ and

$$u(x, y, t) = u_0 + u_x, \quad (2.1)$$

where $u_0 = u|_{x,y=\pm\infty}$ is an arbitrary constant. Inserting Eq. (2.1) into the 2GKdV equation and integrating once with respect to x , we derive

$$15\alpha(u_x^3 + u_x u_{xxx} + u_0 u_{xxx}) + 45\alpha(u_0 u_x^2 + u_0^2 u_x) + \alpha u_{5x} + u_t + u_y = 0, \quad (2.2)$$

Adopting the transformation $u = 2(\ln f)_x$, the 2GKdV equation is described in the bilinear form using the Hirota D operator

$$(45u_0^2 \alpha D_x^2 + 15u_0 \alpha D_x^4 + \alpha D_x^6 + D_x D_y + D_x D_t) f \cdot f = 0, \quad (2.3)$$

where D_x , D_y and D_t represent the bilinear derivatives. [21]

We first give the multiple soliton solutions of the 2GKdV equation by the Hirota perturbation expansion method, then the first-order breather solution is constructed by setting partial parameters to be paired-complex conjugate forms [22], and the lump wave is created via the long-wave limit method [23, 24]. The N -soliton solution of the 2GKdV equation is of the form

$$f = f_N = \sum_{\mu \in 0,1} \exp \left(\sum_{i=1}^N \mu_i \tau_i + \sum_{i < j < N} \mu_i \mu_j A_{ij} \right), \quad (2.4)$$

and

$$\begin{aligned} \tau_i &= k_i (x + l_i y + \omega_i t) + \phi_i, \\ \omega_i &= -(\alpha k_i^4 + 45\alpha u_0^2 l_i + 15\alpha u_0 k_i^2 + l_i), \\ e^{A_{ij}} &= \frac{(k_i^2 - k_i k_j + k_j^2)(k_i - k_j)^2 + 9u_0(k_i - k_j)^2}{(k_i^2 + k_i k_j + k_j^2)(k_i + k_j)^2 + 9u_0(k_i + k_j)^2}, \end{aligned} \quad (2.5)$$

where τ_i , A_{ij} represent phases and phase shifts of solitons, respectively. Besides, k_i, l_i, ϕ_i ($i = 1, 2, \dots, N$) are arbitrary real constants; $\sum_{\mu=0,1}$ is the summation over possible combinations of $\mu_i, \mu_j = 0, 1$ ($i, j = 1, 2, \dots, N$); the summation $\sum_{i < j}^N$ should be taken all possible pairs selected from $(i, j = 1, 2, \dots, N)$.

2.1. X -Shaped soliton and first-order breather

From the expression (2.4), the two-soliton of Eq. (1.1) is

$$u = 2(\ln f_2)_x, \quad f_2 = 1 + e^{\tau_1} + e^{\tau_2} + e^{\tau_1 + \tau_2 + A_{12}}, \quad (2.6)$$

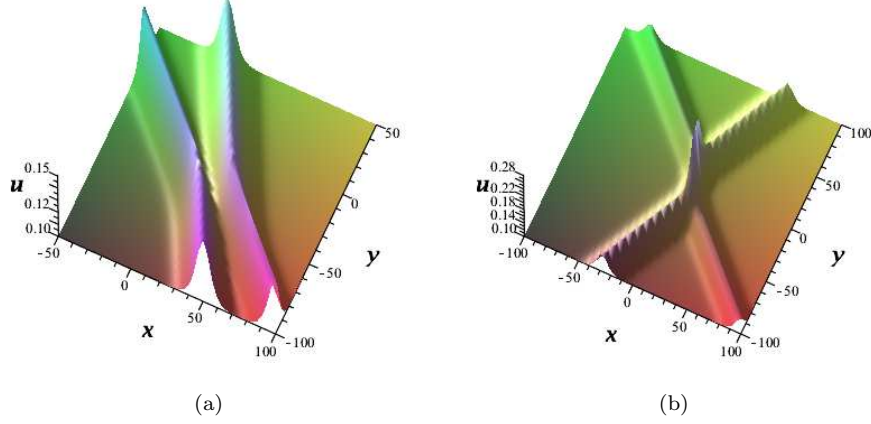


Fig. 1. (a) The X -Shaped soliton given by (2.11) with $k_1 = -0.32$, $k_2 = 0.3$, $l_1 = 0.5$, $l_2 = 1$, $\alpha = 1$, $u_0 = 0.1$, $\phi_1 = 0$, $\phi_2 = 0$, $t = 0$. (b) The X -Shaped soliton given by (2.11) with $k_1 = -0.32$, $k_2 = -0.3$, $l_1 = -0.5$, $l_2 = 1$, $\alpha = 1$, $u_0 = 0.1$, $\phi_1 = 0$, $\phi_2 = 0$, $t = 0$.

where $\tau_1, \tau_2, e^{A_{12}}$ are determined by Eq. (2.5).

To construct the X -Shaped solitons, we further take the above two-soliton solution (2.6) as an example to explore the generation mechanism of X -Shaped soliton solution. By setting $A_{12} \gg 1$ or $A_{12} \ll 1$ in solution (2.6), we derive some X -Shaped solitons. Figure 1 displays the X -Shaped solitons based on solution (2.6). In fact, these X -Shaped solitons can describe different mutual collision modes when $A_{12} = O(1)$, $A_{12} \gg 1$ or $A_{12} \ll 1$. In order to better study the superposition mechanism among nonlinear waves, we start with the first-order breather solution and some parameters are chosen as

$$k_1 = k_2^* = m_1 + in_1, l_1 = l_2^* = p_1 + iq_1, \phi_1 = \phi_2^* = \gamma_1 + i\eta_1, \quad (2.7)$$

where the asterisk $*$ represents the complex conjugation and $m_1, n_1, p_1, q_1, \gamma_1, \eta_1, \lambda_1$ denote arbitrary constants, i is the imaginary unit. Substituting Eq. (2.7) into Eq. (2.6), we suggest the function $f = f(x, y, t)$ having the following appropriate form [25, 26]

$$f_2 \sim e^{\xi_1} + \lambda_1 \cos(\zeta_1) + \lambda_2 e^{-\xi_1}, \quad (2.8)$$

and

$$\begin{aligned} \xi_1 &= m_1 x + (m_1 p_1 - n_1 q_1) y + \omega_{R1} t + \gamma_1, \\ \zeta_1 &= n_1 x + (m_1 q_1 + n_1 p_1) y + \omega_{I1} t + \eta_1, \\ \omega_{R1} &= -\alpha m_1^5 + 10\alpha m_1^3 n_1^2 - 5\alpha m_1 n_1^4 - 45\alpha u_0^2 m_1 p_1 + 45\alpha u_0^2 n_1 q_1 \\ &\quad - 15\alpha u_0 m_1^3 + 45\alpha u_0 m_1 n_1^2 - m_1 p_1 + n_1 q_1, \\ \omega_{I1} &= -5\alpha m_1^4 n_1 + 10\alpha m_1^2 n_1^3 - \alpha n_1^5 - 45\alpha u_0^2 m_1 q_1 - 45\alpha u_0^2 n_1 p_1 \end{aligned}$$

$$\begin{aligned} & -45\alpha u_0 m_1^2 n_1 + 15\alpha u_0 n_1^3 - m_1 q_1 - n_1 p_1, \\ \lambda_2 = & -\frac{n_1^2 (m_1^2 - 3n_1^2 + 9u_0)}{4m_1^2 (3m_1^2 - n_1^2 + 9u_0)} \lambda_1. \end{aligned} \quad (2.9)$$

Based on the definition of hyperbolic function, the solution (2.8) can be written as

$$f_2 \sim 2\sqrt{\lambda_2} \cosh\left(\xi_1 - \frac{1}{2} \ln \lambda_2\right) + \lambda_1 \cos \zeta_1. \quad (2.10)$$

Carrying Eq. (2.10) into the transformation $u(x, y, t) = u_0 + 2\ln(f_2)_{xx}$, the first-order breather solution of the 2GKdV equation can be obtained

$$u = u_0 + \frac{r_1 + r_2 \cosh\left(\xi_1 - \frac{1}{2} \ln \lambda_2\right) \cos \zeta_1 + r_3 \sinh\left(\xi_1 - \frac{1}{2} \ln \lambda_2\right) \sin \zeta_1}{(2\sqrt{\lambda_2} \cosh\left(\xi_1 - \frac{1}{2} \ln \lambda_2\right) + \lambda_1 \cos \zeta_1)^2}, \quad (2.11)$$

with $r_1 = 8\lambda_2 m_1^2 - 2\lambda_1^2 n_1^2$, $r_2 = 4\lambda_1 \sqrt{\lambda_2} (m_1^2 - n_1^2)$, $r_3 = 8\lambda_1 \sqrt{\lambda_2} m_1 n_1$ and u_0 being a constant background wave, which has also been presented in Refs. [27, 28].

From the above simplified expression, we find that the breather solution is composed of hyperbolic functions $H = \{\cosh(\xi_1 - \frac{1}{2} \ln \lambda_2), \sinh(\xi_1 - \frac{1}{2} \ln \lambda_2)\}$ and trigonometric functions $T = \{\cos \zeta_1, \sin \zeta_1\}$, which play essential roles in deciding the key properties of solution (2.11). The functions H and T reveal localization and periodicity of the evolution process, respectively. Thus, the breather is actually a mixed solution component of the solitary-type wave and periodic-type wave. It describes the localized solutions with breathing oscillation behavior on the constant background u_0 . Furthermore, the propagation velocity vector of solitary wave $\mathbf{v}_{\xi_1}$ and propagation velocity vector of periodic wave $\mathbf{v}_{\zeta_1}$ are expressed in the following forms

$$\begin{aligned} \mathbf{v}_{\xi_1} &= (v_{\xi_{1x}}, v_{\xi_{1y}}) = \left(\frac{\omega_{R1}}{m_1}, \frac{\omega_{R1}}{m_1 p_1 - n_1 q_1}\right), \\ \mathbf{v}_{\zeta_1} &= (v_{\zeta_{1x}}, v_{\zeta_{1y}}) = \left(\frac{\omega_{I1}}{n_1}, \frac{\omega_{I1}}{m_1 q_1 + n_1 p_1}\right), \end{aligned} \quad (2.12)$$

where $v_{\xi_{1x}}, v_{\xi_{1y}}$ denote the disseminate speed of solitary wave in the x -axis and y -axis; $v_{\zeta_{1x}}, v_{\zeta_{1y}}$ denote the disseminate speed of period wave in the x -axis and y -axis, respectively.

The periods T_1 of the breather solution (2.11) is given by

$$T_1 = (T_{1x}, T_{1y}) = \left(\frac{2\pi(n_1 q_1 - m_1 p_1)}{q_1(m_1^2 + n_1^2)}, \frac{2\pi m_1}{q_1(m_1^2 + n_1^2)}\right), \quad (2.13)$$

where T_{1x}, T_{1y} represent the periods of the breather solution (2.11) in the x -axis and y -axis directions, respectively.

From (2.13), we obtain the distance d_{min} between the two nearest peaks of solution (2.11)

$$d_{min} = |T_1| = \frac{2\pi}{m_1^2 + n_1^2} \sqrt{\frac{m_1^2 + (n_1 q_1 - m_1 p_1)^2}{q_1^2}}. \quad (2.14)$$

Additionally, the trajectory equations of the solitary-type wave member (SWM) and periodic-type wave member (PWM) in breather solution (2.11) are:

$$L_{s_1} : m_1 x + (m_1 p_1 - n_1 q_1) y + \omega_{R1} t + \gamma_1 = 0, \quad L_{p_1} : n_1 x + (m_1 q_1 + n_1 p_1) y + \omega_{I1} t + \eta_1 = 0.$$

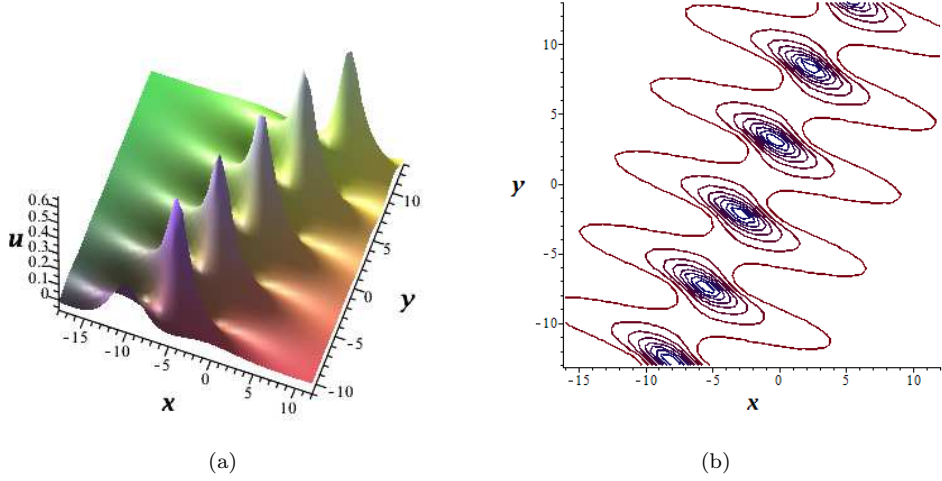


Fig. 2. The breather solution given by (2.11) with $m_1 = 0.3$, $n_1 = 0.3$, $p_1 = 1.5$, $q_1 = 2$, $\lambda_1 = 2$, $\alpha = 1$, $u_0 = -0.01$, $\gamma_1 = 0$, $\eta_1 = 0$, $t = 0$. (a) Spatial structure plot. (b) Contour plot of (a).

Figure 2 shows the breather solution described by Eq. (2.11) in (x, y) -plane. In Fig. 2, the velocities of the solitary-type wave and periodic-type wave are $v_{\xi_1} = (0.5077, -1.0153)$ and $v_{\zeta_1} = (-3.4564, -0.9875)$, and the corresponding trajectory equations are $0.3x - 0.15y + 0.1523t = 0$ and $0.3x + 1.05y - 1.0369t = 0$ respectively. It should be clarified that the analytical condition to the solution (2.11) reads $2\sqrt{\lambda_2} > |\lambda_1|$.

2.2. Lump-type soliton and W-Shaped soliton solutions

In the previous section, we investigate the components of the breather solution and discuss the related dynamical properties. At present, we mainly consider the lump-type wave via the long wave limit technique, namely, when $m_1^2 + n_1^2 \rightarrow 0$, taking the limit of the infinity period. By analyzing Eqs. (2.13) and (2.14), we find that if the condition $m_1^2 + n_1^2 \rightarrow 0$ holds, the period T_1 of solution (2.11) will tend to infinity and the breather solution as shown in Fig. 2 degenerates to a lump-type wave.

For the lump-type wave, we take

$$k_1 = k_2^* = m_1 + in_1, l_1 = l_2^* = p_1 + iq_1, \phi_1 = \phi_2^* = \gamma_1 + i\pi, \quad (2.15)$$

and put $(m_1, n_1) = (\epsilon, \epsilon)$, $\lambda_1 = -2$, $\lambda_2 = 1 + \epsilon^2$. Then, taking the Taylor expansion of (2.8) when $\epsilon = 0$, we obtain

$$u_{\text{lump}} = 2(\ln f_l)_{xx} + u_0 = \frac{8 - 16(x + (p_1 + q_1)y - (p_1 + q_1)t)(x + (p_1 - q_1)y - (p_1 - q_1)t)}{(1 + (x + (p_1 + q_1)y - (p_1 + q_1)t)^2 + (x + (p_1 - q_1)y - (p_1 - q_1)t)^2)^2}, \quad (2.16)$$

where

$$f_l = (x + (p_1 + q_1)y - (p_1 + q_1)t)^2 + (x + (p_1 - q_1)y - (p_1 - q_1)t)^2 + 1. \quad (2.17)$$

Figure 3 exhibits the lump-type wave with one wall-like peak and two small holes, which is localized in all directions in space [29, 30]. This solution can be viewed as a hybrid solution including two-wave solution with variables $\xi'_1 = x + (p_1 + q_1)y - (p_1 + q_1)t$ and $\zeta'_1 = x + (p_1 - q_1)y - (p_1 - q_1)t$. Similar to Eq. (2.12), we can express the speeds of two-wave solution as the forms $v_{\xi'_1} = (-(p_1 + q_1), -1)$, $v_{\zeta'_1} = (-(p_1 - q_1), -1)$. The trajectory equations ($\xi'_1 = 0$, $\zeta'_1 = 0$) are determined accordingly. By utilizing the extremum value theorem of function, we calculate the maximum or minimum value points of the function $u_{\text{lump}} = u(x, y, t_0)$. It shows that the lump solution (2.16) possesses three extreme points at $(0, t_0)$, $(-0.8018, t_0)$ and $(0.8018, t_0)$ with parameters being selected as $p_1 = 0.5$, $q_1 = -2$, $\alpha = 1$, $u_0 = 0$. The maximum value of lump solution (2.16) is 2 at $(0, t_0)$, while the minimum value is -3.0625 at $(\pm 0.8018, t_0)$.

It is interesting that if we introduce a new class of parameter constraint condition that is known as the velocity resonance equation [31, 32, 33], a phenomena that the breather (2.11) converting into a rational W-Shaped solitary (WS) wave will occur when the following condition is satisfied, see Fig. 4.

$$v_{\xi'_1} = v_{\zeta'_1}, \quad i.e., \quad q_1 = 0. \quad (2.18)$$

Under this circumstance, a W-Shaped soliton solution can be expressed in the form

$$u_{ws} = 2(\ln f_w)_{xx} = \frac{4 - 8(x + p_1 y - p_1 t)^2}{(2(x + p_1 y - p_1 t)^2 + 1)^2}, \quad (2.19)$$

where

$$f_w = 2(x + p_1 y - p_1 t)^2 + 1. \quad (2.20)$$

As displayed in Fig. 4, we learn that the WS soliton solution has single peak and double symmetrical valleys, of which the trajectory equation is $x + 0.2(y - t) = 0$.

2.3. Nonlinear wave structures from two-soliton solution (2.11)

In this section, we investigate the transformed wave conditions and the nonlinear wave structures including the W-Shaped (WS) solitary wave, the M-Shaped (MS) solitary wave, the Oscillation M-Shaped (OMS) solitary wave and the Multi-Peak (MP) solitary wave, etc. The associated dynamical properties of the nonlinear wave structures will be studied detailedly.

The previous investigation shows that if the direction of the propagation velocity vector v_{ξ_1} of solitary wave and the propagation velocity vector v_{ζ_1} of periodic wave are parallel, we can obtain different types of nonlinear wave structures. In order to study the structural properties of the breather (2.11), we take advantage of the following condition,

$$v_{\xi_1} \times v_{\zeta_1} = 0, \quad (2.21)$$

where the operator $\times$ denotes the cross product of two velocity vectors, which is equivalent to

$$\begin{vmatrix} \frac{\omega_{R1}}{m_1} & \frac{\omega_{R1}}{m_1 p_1 - n_1 q_1} \\ \frac{\omega_{I1}}{n_1} & \frac{\omega_{I1}}{m_1 q_1 + n_1 p_1} \end{vmatrix} = 0. \quad (2.22)$$

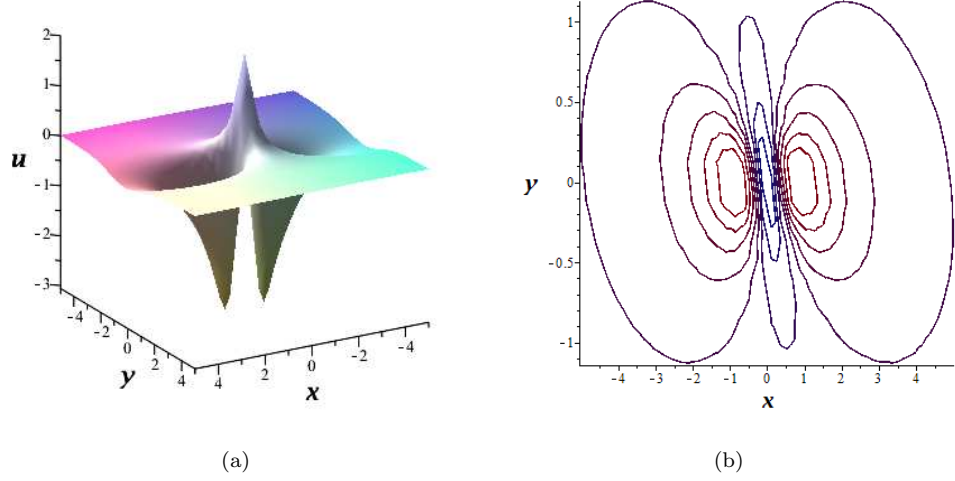


Fig. 3. The lump solution given by (2.16) with $p_1 = 0.5$, $q_1 = -2$, $\alpha = 1$, $u_0 = 0$, $t = 0$. (a) Spatial structure plot. (b) Contour plot of (a).

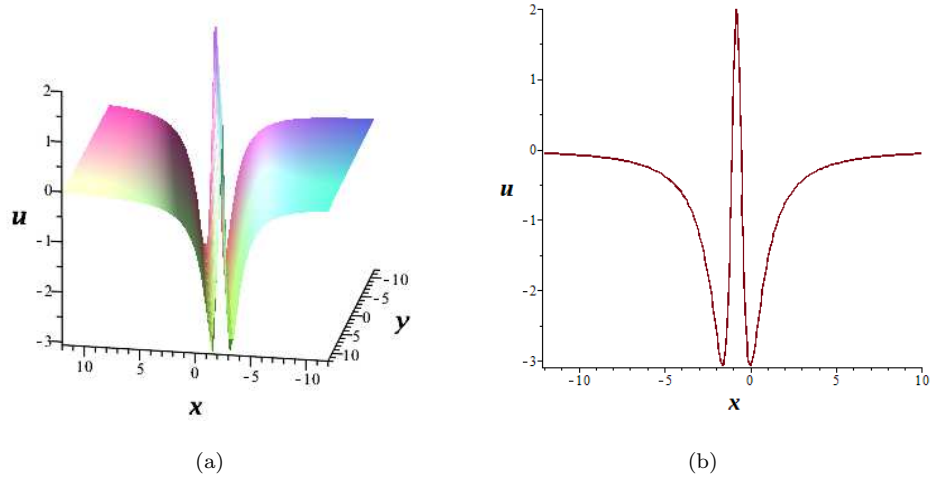


Fig. 4. The WS-soliton solution given by (2.19) with $p_1 = 0.2$, $\alpha = 1$, $u_0 = 0$, $t = 0$. (a) Spatial structure plot. (b) Contour plot of (a).

That is to say, the subtle parameter condition should be met

$$q_1 = 0. \quad (2.23)$$

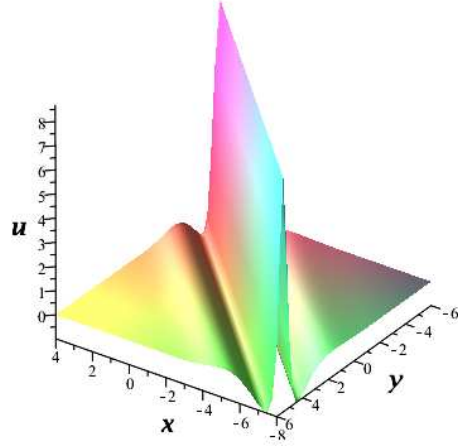
From the above analysis, we learn that the breather solution (2.11) component of the functions H and T possesses uniform velocity direction. The tangent of the velocity direction is expressed in the form: $\beta_1 = \arctan(p_1)$, where β_1 is the angle between the counterclockwise direction of two velocity vectors and the positive direction of the y -axis. Hence, it is summarized that the parameter p_1 is responsible for the velocity direction of different nonlinear wave structures. A series of nonlinear localized wave structures could be constructed via the specific mechanism (2.21)-(2.23). Figure 5 displays several types of nonlinear localized waves. The W -Shaped (WS) soliton solution with single peak and double symmetric valleys ($L_{s_1} = x + y + 1.9725t = 0$ and $L_{p_1} = 1.1x + 1.1y + 5.6704t = 0$) is shown in Fig. 5 (a). Next, we demonstrate three cases:

- (i) When $\frac{n_1}{m_1} \approx 1$, an asymmetric M -Shaped (MS) soliton solution has double peaks and a valley ($L_{s_1} = 0.3x + 0.3y - 0.2967t = 0$ and $L_{p_1} = 0.2x + 0.2y - 0.1952t = 0$), is exhibited in Fig. 5 (b);
- (ii) With the increase of value $\frac{n_1}{m_1}$ gradually, the MS solution will convert into the Oscillation M -Shaped (OMS) soliton solution ($\frac{n_1}{m_1} = 3.5$), the trajectory equations $L_{s_1} = 0.5x + 0.6y + 6.5152t = 0$ and $L_{p_1} = 1.75x + 2.1y + 1.065t = 0$, is presented in Fig. 5(c);
- (iii) When $\frac{n_1}{m_1} = 10$, a Multi-Peak (MP) soliton solution will be obtained as shown in Fig. 5(d), where the trajectory equations are $L_{s_1} = 0.1x + 0.12y + 0.2495t = 0$ and $L_{p_1} = x + 1.2y - 2.5955t = 0$.

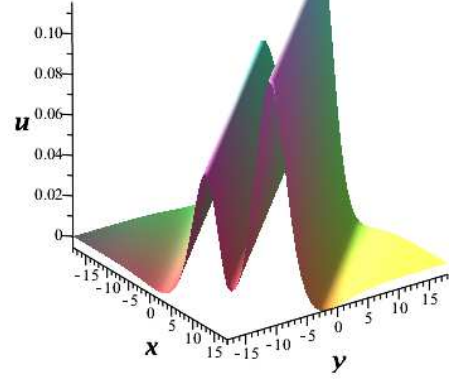
To proceed, we give an analysis about the essential physical quantity, phase shift, which has profound impact on the dynamical behaviors of the obtained nonlinear waves. In this paper, the concrete expressions of phase shifts of solution (2.11) are given by

$$\begin{aligned} \phi_{\xi_1}(t) &= |\mathbf{v}_{\xi_1}|t = \frac{\omega_{R1}t}{m_1(m_1p_1 - n_1q_1)} \sqrt{(m_1p_1 - n_1q_1)^2 + m_1^2}, \\ \phi_{\zeta_1}(t) &= |\mathbf{v}_{\zeta_1}|t = \frac{\omega_{I1}t}{n_1(m_1q_1 + n_1p_1)} \sqrt{(m_1q_1 + n_1p_1)^2 + n_1^2}, \end{aligned} \quad (2.24)$$

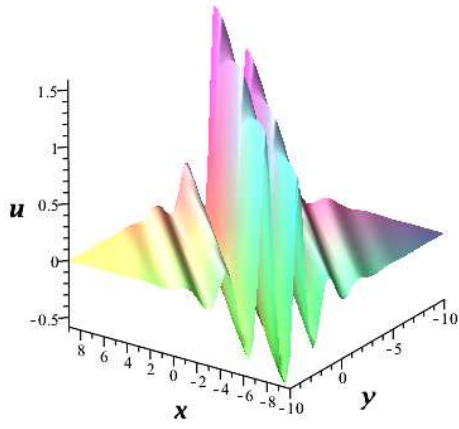
where $\phi_{\xi_1}(t)$, $\phi_{\zeta_1}(t)$ represent the phase shifts of the SWM and PWM, respectively, $|\mathbf{v}_{\xi_1}|$, $|\mathbf{v}_{\zeta_1}|$ denote the modulus of velocity vectors of the SWM and PWM, ω_{R1} , ω_{I1} are determined by Eq. (2.9). According to Eqs. (2.24), we conclude that the phase shifts of SWM and PWM are not equal ($\phi_{\xi_1}(t) \neq \phi_{\zeta_1}(t)$) and will alter with time varying. As a result, we obtain different nonlinear wave interactional structures at different times t because the interaction regions have changed that caused by the transformation of the trajectory equation. Figure 6 demonstrates the process of the asymmetric MS solitary wave converting into the WS solitary wave when $t : -1 \rightarrow 0$ and the reverse process when $t : 0 \rightarrow 1$.



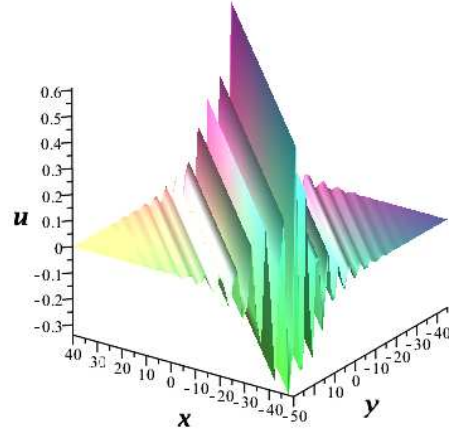
(a)



(b)



(c)



(d)

Fig. 5. Spatial structure plots of the different nonlinear waves described by (2.11) and $\gamma_1 = 0$, $\eta_1 = 0$, $\alpha = 1$, $t = 0$. (a) The WS soliton solution and $m_1 = 1$, $n_1 = 1.1$, $p_1 = 1$, $\lambda_1 = 12$, $u_0 = -0.02$. (b) The MS soliton solution and $m_1 = 0.3$, $n_1 = 0.2$, $p_1 = 1$, $\lambda_1 = 2$, $u_0 = -0.01$. (c) The OMS soliton solution and $m_1 = 0.5$, $n_1 = 1.75$, $p_1 = 1.2$, $\lambda_1 = 2$, $u_0 = 0.5$. (d) The MP soliton solution and $m_1 = 0.1$, $n_1 = 1$, $p_1 = 1.2$, $\lambda_1 = 2$, $u_0 = 0.3$.

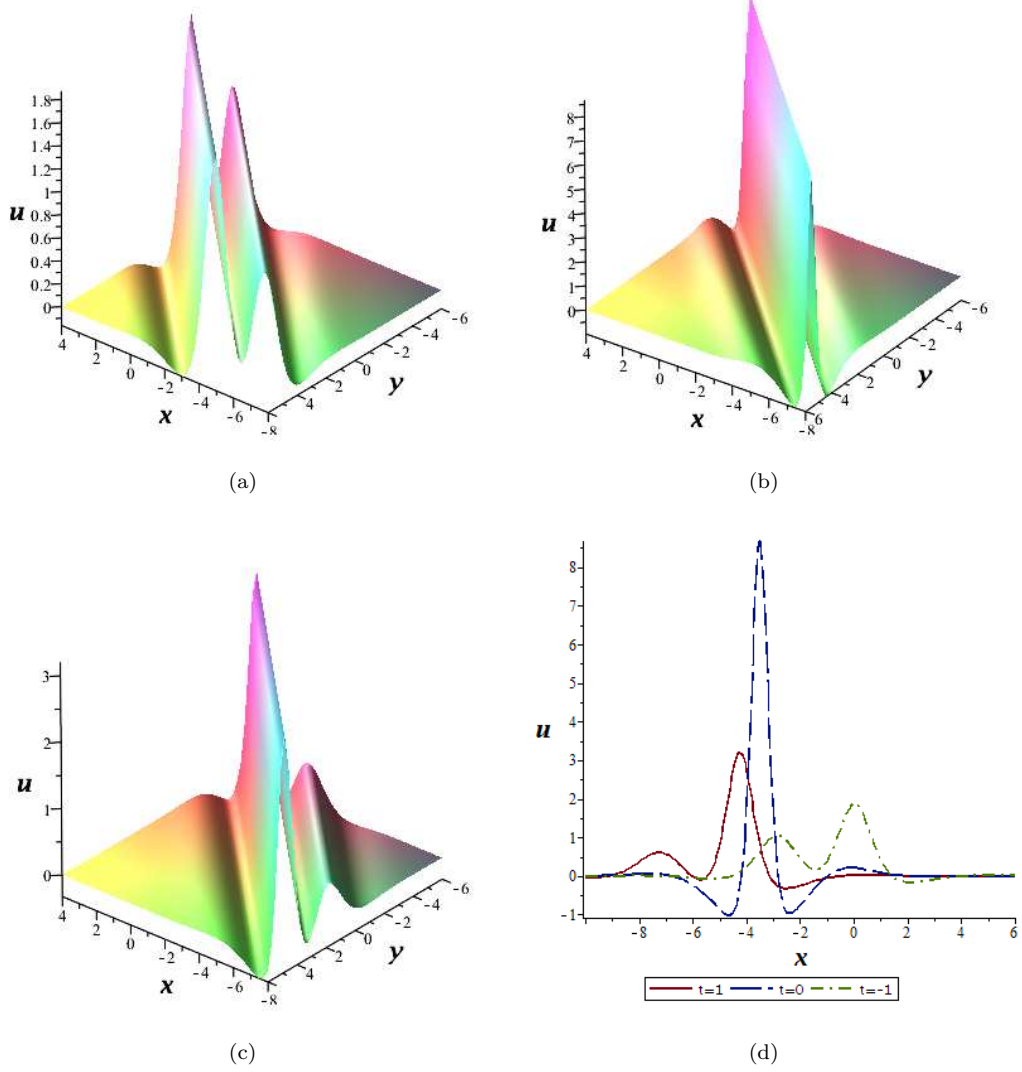


Fig. 6. The evolution diagram of the MS soliton solution with time and $m_1 = 1$, $n_1 = 1.1$, $p_1 = 1$, $q_1 = 0$, $\lambda_1 = 12$, $u_0 = -0.02$, $\alpha = 1$. (a) The MS soliton solution when $t = -1$. (b) The WS soliton solution when $t = 0$. (c) The MS soliton solution when $t = 1$. (d) The cross-sectional plot of (a), (b) and (c) when $y = 1$.

If the solution f_2 is expressed in the form

$$f_2 \sim e^{\tilde{\xi}_1} + \lambda_1 \cos(\tilde{\zeta}_1) + \lambda_2 e^{-\tilde{\xi}_1}, \quad (2.25)$$

where

$$\begin{aligned} \tilde{\xi}_1 &= a_1(x + b_1 y + c_1 t), \quad \tilde{\zeta}_1 = a_2(x + b_2 y + c_2 t), \\ c_1 &= -a_1^4 \alpha + 10a_1^2 a_2^2 \alpha - 5a_2^4 \alpha - 15\alpha a_1^2 u_0 + 45\alpha a_2^2 u_0 - 45\alpha b_1 u_0^2 - b_1, \\ c_2 &= -5a_1^4 \alpha + 10a_1^2 a_2^2 \alpha - a_2^4 \alpha - 45\alpha a_1^2 u_0 + 15\alpha a_2^2 u_0 - 45\alpha b_2 u_0^2 - b_2, \\ \lambda_2 &= -\frac{a_2^2(a_1^2 - 3a_2^2 + 9u_0)}{4a_1^2(3a_1^2 - a_2^2 + 9u_0)} \lambda_1, \end{aligned} \quad (2.26)$$

and $a_i, b_i (i = 1, 2)$ are arbitrary real constants.

The solution (2.11) is readily rewritten as the following form

$$\tilde{u} = u_0 + \frac{r_1 + r_2 \cosh\left(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2\right) \cos \tilde{\zeta}_1 + r_3 \sinh\left(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2\right) \sin \tilde{\zeta}_1}{\left(2\sqrt{\lambda_2} \cosh\left(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2\right) + \lambda_1 \cos \tilde{\zeta}_1\right)^2}, \quad (2.27)$$

with $r_1 = 8\lambda_2 a_1^2 - 2\lambda_1^2 a_2^2$, $r_2 = 4\lambda_1 \sqrt{\lambda_2} (a_1^2 - a_2^2)$, $r_3 = 8\lambda_1 \sqrt{\lambda_2} a_1 a_2$ and u_0 being a constant background wave.

Similarly to expression (2.12), the propagation velocity vector $\mathbf{v}_{\tilde{\xi}_1}$ of solitary wave and the propagation velocity vector $\mathbf{v}_{\tilde{\zeta}_1}$ of periodic wave can be expressed in the following forms

$$\mathbf{v}_{\tilde{\xi}_1} = \left(a_1 c_1, \frac{a_1 c_1}{b_1}\right), \quad \mathbf{v}_{\tilde{\zeta}_1} = \left(a_2 c_2, \frac{a_2 c_2}{b_2}\right). \quad (2.28)$$

If the condition $\mathbf{v}_{\tilde{\xi}_1} \times \mathbf{v}_{\tilde{\zeta}_1} = 0$ holds, namely, $b_1 = b_2$, $c_1 = c_2$, and further letting the parameter a_1 vanish, the corresponding periodic wave solution is given in the form

$$u'' = u_0 - \frac{2\lambda_1 a_2^2 (\lambda_1 + (1 + \lambda_2) \cos \tau)}{(1 + \lambda_2 + \lambda_1 \cos \tau)^2}, \quad (2.29)$$

where $u_0 = \frac{2a_2^2}{15}$, $\tau = a_2(x + b_2 y + (a_2^4 \alpha - \frac{4}{5} \alpha b_2 a_2^4 - b_2)t)$. The existence condition for the periodic wave is $\left|\frac{\lambda_1}{\lambda_2 + 1}\right| < 1$. we notice that the solution (2.29) will convert into different types of periodic structures under variant constraints about the parameters λ_1, λ_2 .

- (i) When $0 < \left|\frac{\lambda_1}{\lambda_2 + 1}\right| < 1$, a sine-like periodic wave is exhibited;
- (ii) When $\left|\frac{\lambda_1}{\lambda_2 + 1}\right| = \frac{1}{2}$, a soliton lattice with *U*-Shaped bottom is derived;
- (iii) When $\frac{1}{2} < \left|\frac{\lambda_1}{\lambda_2 + 1}\right| < 1$, a soliton lattice with *W*-Shaped bottom is constructed.

3. Mixed solutions including *Y*-Shaped and Space-Curved solitons and other nonlinear molecular waves

In this section, the aim is to investigate the interaction structures and the nonlinear molecular waves based on the mixed solution and the fourth-order soliton under the specific velocity resonance mechanism. We discuss the elastic and inelastic collision modes among the nonlinear localized wave structures. Then, we give further explanation for the reasons of inelastic collisions by the trajectory equation and the phase shift analysis.

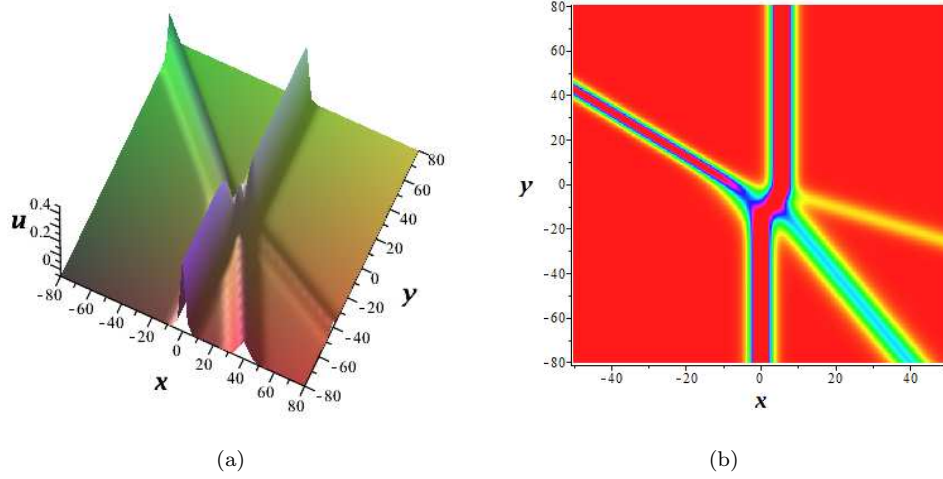


Fig. 7. Spatial structure plots of the mixed solution given by (3.1) with $\alpha = 1$, $u_0 = -0.05$, $t = 0$. (a) The interactional structure between the Y-type soliton and single soliton with $k_1 = 0.5$, $k_2 = 0.3$, $l_1 = 0.5$, $l_2 = 1$, $l_3 = 0$, $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$. (b) Contour plot of (a).

3.1. Interaction structures from the mixed solution

Next, we focus on the Y-type soliton of the 2GKdV equation. Taking $N = 3$, $A_{12} = 0$ in Eqs. (2.4) and (2.5), the relevant resonant Y-type soliton is presented by the following expression,

$$u = 2(\ln f_3)_x, \quad f_3 = 1 + e^{\tau_1} + e^{\tau_2} + e^{\tau_3} + e^{\tau_1 + \tau_3 + A_{13}} + e^{\tau_2 + \tau_3 + A_{23}}, \quad (3.1)$$

and the additional constrains $k_2 = \frac{1}{2}k_1 \pm \frac{1}{2}\sqrt{-3k_1^2 - 36u_0}$, $u_0 < -\frac{1}{12}k_1^2$ should be met. Figure 7 shows the mutual collision between the Y-shaped soliton and single soliton. The Y-shaped solitons are formed because of the resonance effect between solitons, which describe the fusion and fission phenomena between solitons. When $A_{12} = 0$, the fission phenomenon represents the splitting of one soliton into two solitons with the same structure. The fusion phenomenon represents two structurally similar solitons merge into one single soliton.

By introducing complex conjugation relationships to partial parameters, the well known Hirota multiple soliton solutions can be expanded to obtain the following form of mixed solution. Based on this type of solution, we can construct rich nonlinear localized waves and their interaction structures [34]. Let the detailed mixed solution expression f_3 be of the form

$$\begin{aligned} f_3 \sim & 1 - \lambda_1 \cos(\zeta_1) + \lambda_2 e^{2\xi_1} + e^{\tau_3} (1 - \lambda_1 e^{\xi_1} (L_{R1} \cos(\zeta_1) - L_{I1} \sin(\zeta_1))) \\ & + \lambda_2 e^{\tau_3} e^{2\xi_1} (L_{R1}^2 + L_{I1}^2), \end{aligned} \quad (3.2)$$

with

$$\tau_3 = k_3(x + l_3 y + \omega_3 t) + \phi_3, \quad \omega_3 = -(\alpha k_3^4 + 45\alpha u_0^2 l_3 + 15\alpha u_0 k_3^2 + l_3),$$

$$\begin{aligned}
\lambda_2 &= -\frac{n_1^2(m_1^2 - 3n_1^2 + 9u_0)}{4m_1^2(3m_1^2 - n_1^2 + 9u_0)}\lambda_1, \\
L_{R1} &= \text{Re}(e^{A_{13}}), \quad L_{I1} = \text{Im}(e^{A_{13}}),
\end{aligned} \tag{3.3}$$

where ξ_1, ζ_1 are described in expression (2.26). Substituting Eq. (3.2) into Eq. (2.1) leads to the interactional structure consisted of breather and soliton, which presents an elastic collision mode due to the shape of the interactional structure remaining unchanged, as shown in Fig. 9(a). By applying $(m_1, n_1) = (\epsilon, \epsilon)$, $\lambda_1 = 2$, $\lambda_2 = 1 + \epsilon^2$ and the Taylor Eq. (3.2) as $\epsilon = 0$. We obtain an interactional wave structure including a lump and a soliton via appropriate parameter selections, as exhibited in Fig. 9 (b).

3.2. Interaction structures from the fourth-order solution

For the aim of constructing the space-curved resonant solitons, we assume that $\exp(A_{js}) \rightarrow 0$ is true if and only if $k_j = k_s$, $(0 < j < s \leq 4)$ [35]. Introducing $k_1 = k_2$, $k_3 = k_4$ to the multiple soliton solutions Eqs. (2.4) and (2.5) when $N = 4$ leads to the elimination of all or part of the corresponding terms $\exp(A_{js}) \rightarrow 0$, $(0 < j < s \leq 4)$, and f_4 in $u = 2(\ln f_4)_x$ is of the form

$$f_4 = 1 + e^{\tau_1} + e^{\tau_2} + e^{\tau_3} + e^{\tau_4} + e^{\tau_1+\tau_3+A_{13}} + e^{\tau_1+\tau_4+A_{14}} + e^{\tau_2+\tau_3+A_{23}} + e^{\tau_2+\tau_4+A_{24}}. \tag{3.4}$$

If Eq. (3.4) additional satisfies the following parameters paired-complexification relationships

$$k_3 = k_4^* = m_2 + in_2, \quad l_3 = l_4^* = p_2 + iq_2, \quad \phi_3 = \phi_4^* = \gamma_2 + i\eta_2, \tag{3.5}$$

where $m_2, n_2, p_2, q_2, \gamma_2, \eta_2$ are real constants, the interaction between a space-curved soliton and a breather can be obtained. Figure 8 shows the space-curved resonant soliton interaction including the other space-curved resonant soliton/breather. Similar to the way of obtaining the first-order breather, we can derive the second-order breather solution with condition (3.5). The fourth-order soliton solution is given as

$$\begin{aligned}
f_4 \sim & 1 + \lambda_1 e^{\xi_1} \cos(\zeta_1) + \lambda_2 e^{2\xi_1} + \lambda_3 e^{\xi_2} \cos(\zeta_2) + \lambda_4 e^{2\xi_2} + \lambda_2 \lambda_4 e^{2\xi_1+2\xi_2} (L_{R1}^2 + L_{I1}^2)(L_{R2}^2 + L_{I2}^2) \\
& + \frac{\lambda_1 \lambda_3}{2} e^{\xi_1+\xi_2} (L_{R1} \cos(\zeta_1 + \zeta_2) - L_{I1} \sin(\zeta_1 + \zeta_2)) \\
& + \frac{\lambda_1 \lambda_3}{2} e^{\xi_1+\xi_2} (L_{R2} \cos(\zeta_1 - \zeta_2) - L_{I2} \sin(\zeta_1 - \zeta_2)) \\
& + \lambda_2 \lambda_3 e^{2\xi_1+\xi_2} ((L_{R1} L_{R2} + L_{I1} L_{I2}) \cos(\zeta_2) - (L_{I1} L_{R2} - L_{R1} L_{I2}) \sin(\zeta_2)) \\
& + \lambda_1 \lambda_4 e^{\xi_1+2\xi_2} ((L_{R1} L_{R2} - L_{I1} L_{I2}) \cos(\zeta_1) - (L_{I1} L_{R2} + L_{R1} L_{I2}) \sin(\zeta_1)),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\xi_2 &= m_2 x + (m_2 p_2 - n_2 q_2) y + \omega_{R2} t + \gamma_2, \\
\zeta_2 &= n_2 x + (m_2 q_2 + n_2 p_2) y + \omega_{I2} t + \eta_2, \\
\omega_{R2} &= -\alpha m_2^5 + 10\alpha m_2^3 n_2^2 - 5\alpha m_2 n_2^4 - 45\alpha u_0^2 m_2 p_2 + 45\alpha u_0^2 n_2 q_2 \\
&\quad - 15\alpha u_0 m_2^3 + 45\alpha u_0 m_2 n_2^2 - m_2 p_2 + n_2 q_2,
\end{aligned}$$

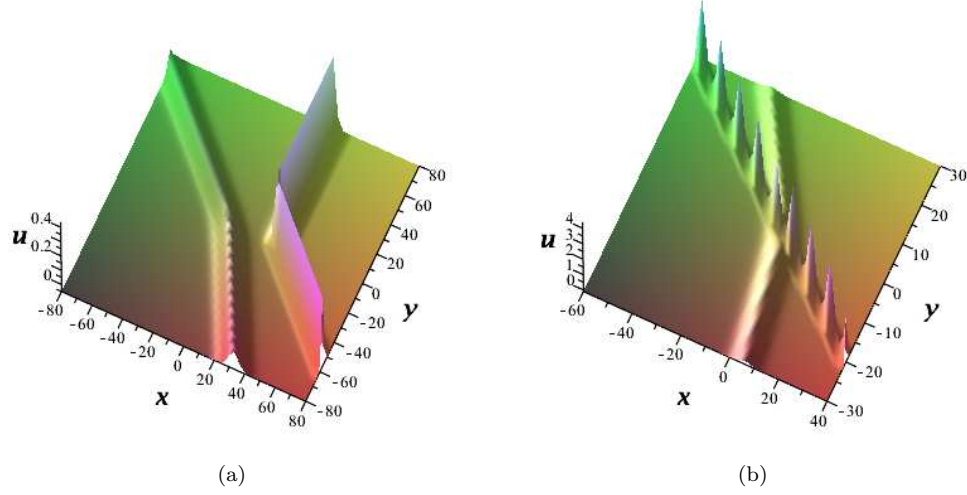


Fig. 8. (a) The interactional structure between two space-curved resonance line soliton $k_1 = k_2 = 0.5$, $k_3 = k_4 = -1$, $l_1 = 0.5$, $l_2 = 1$, $l_3 = 0$, $l_4 = 1$, $\alpha = 1$, $u_0 = -0.05$, $\phi_1 = \phi_2 = 0$, $\phi_3 = \phi_4 = 20$. (b) The interactional structure between a space-curved resonance line soliton and a breather $k_1 = k_2 \approx 0.667$, $l_1 = 0.25$, $l_2 = 1$, $m_2 = 3$, $n_2 = 0.005$, $p_2 = 2$, $q_2 = 0.005$, $\alpha = 1$, $u_0 = -0.1$, $\phi_1 = \phi_2 = 1$, $\gamma_2 = 2$, $\eta_2 = 0.1$.

$$\begin{aligned}
\omega_{12} &= -5\alpha m_2^4 n_2 + 10\alpha m_2^2 n_2^3 - \alpha n_2^5 - 45\alpha u_0^2 m_2 q_2 - 45\alpha u_0^2 n_2 p_2 \\
&\quad - 45\alpha u_0 m_2^2 n_2 + 15\alpha u_0 n_2^3 - m_2 q_2 - n_2 p_2, \\
\lambda_4 &= -\frac{n_2^2 (m_2^2 - 3n_2^2 + 9u_0)}{4m_2^2 (3m_2^2 - n_2^2 + 9u_0)} \lambda_3, \\
L_{R2} &= \text{Re}(e^{A_{14}}), \quad L_{I2} = \text{Im}(e^{A_{14}}),
\end{aligned} \tag{3.7}$$

where ξ_1 , ζ_1 , λ_2 are defined by Eq. (2.26), and L_{R1} , L_{I1} are defined by Eq. (3.3). The essential difference between the second-order breather and the first-order breather is that the former has two sets of velocity vectors $\{(\mathbf{v}_{\xi_1}, \mathbf{v}_{\zeta_1}), (\mathbf{v}_{\xi_2}, \mathbf{v}_{\zeta_2})\}$. The expressions of $(\mathbf{v}_{\xi_2}, \mathbf{v}_{\zeta_2})$ are similar to Eq. (2.12), which are given by

$$\begin{aligned}
\mathbf{v}_{\xi_2} &= \left(\frac{\omega_{R2}}{m_2}, \frac{\omega_{R2}}{m_2 p_2 - n_2 q_2} \right), \\
\mathbf{v}_{\zeta_2} &= \left(\frac{\omega_{I2}}{n_2}, \frac{\omega_{I2}}{m_2 q_2 + n_2 p_2} \right).
\end{aligned} \tag{3.8}$$

For the convenience of discussion, we define the slop of the trajectory equation where the two breathers are located as follows

$$\Lambda_i = -\frac{m_i}{m_i p_i - n_i q_i}, \quad i = 1, 2. \tag{3.9}$$

Figure 10 (a) displays the ordinary second-order breather solution under the condition that

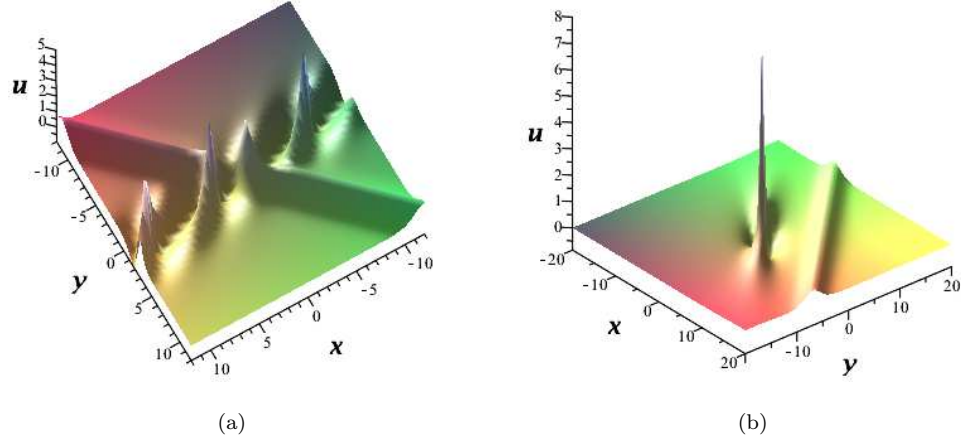


Fig. 9. Spatial structure plots of the mixed solution given by (3.2) with $\alpha = 1$, $t = 0$. (a) The interactional structure between the breather and the soliton with $m_1 = 0.25$, $n_1 = 1$, $p_1 = -1$, $q_1 = 1$, $a_3 = 1.5$, $b_3 = 1$, $\lambda_1 = 2$, $\gamma_3 = 0$. (b) The interactional structure between the lump and the soliton with $p_1 = 1$, $q_1 = 0.5$, $a_3 = -0.5$, $b_3 = 1$, $\gamma_3 = -13$.

$\xi_1 \neq \xi_2$, $\zeta_1 \neq \zeta_2$ and $\Lambda_1 \neq \Lambda_2$. Figure 10 (b) shows the particular second-order breather structure, i.e. the breather molecule [36, 37, 38] under the conditions $\xi_1 \neq \xi_2$, $\zeta_1 \neq \zeta_2$ and $\Lambda_1 = \Lambda_2$.

If we consider the following fixed conditions, i.e., let the parameters satisfy the conditions

$$v_{\xi_1} \times v_{\zeta_1} = 0, \quad v_{\xi_2} \times v_{\zeta_2} \neq 0, \quad \Lambda_1 = \Lambda_2, \quad (3.10)$$

namely,

$$q_1 = 0, \quad -\frac{1}{p_1} = -\frac{m_2}{m_2 p_2 - n_2 q_2}, \quad (3.11)$$

then the second-order breather solution (3.6) transforms into a coherent structure (also called collision-free modes) of breather and soliton [39]. From Fig. 11, it is observed that the propagation direction of the breather and soliton are parallel, and the breather and soliton do not interfere with each other. On the other hand, we consider the following case, say, the directions of two sets of propagation velocities are parallel

$$v_{\xi_1} \times v_{\zeta_1} = 0, \quad v_{\xi_2} \times v_{\zeta_2} = 0, \quad (3.12)$$

Under this case, if $\Lambda_1 \neq \Lambda_2$ ($p_1 \neq p_2$), the molecule structure between two M -Shaped soliton (2MS) solutions is given in Fig. 11. On the contrary, if $\Lambda_1 = \Lambda_2$ ($p_1 = p_2$), together with the above conditions (3.12), we can obtain a variety of nonlinear wave molecules that are composed of bounded state of different kinds of nonlinear waves and are viewed as collision-free modes. The components of molecules do not affect each other during the process of propagation. We show

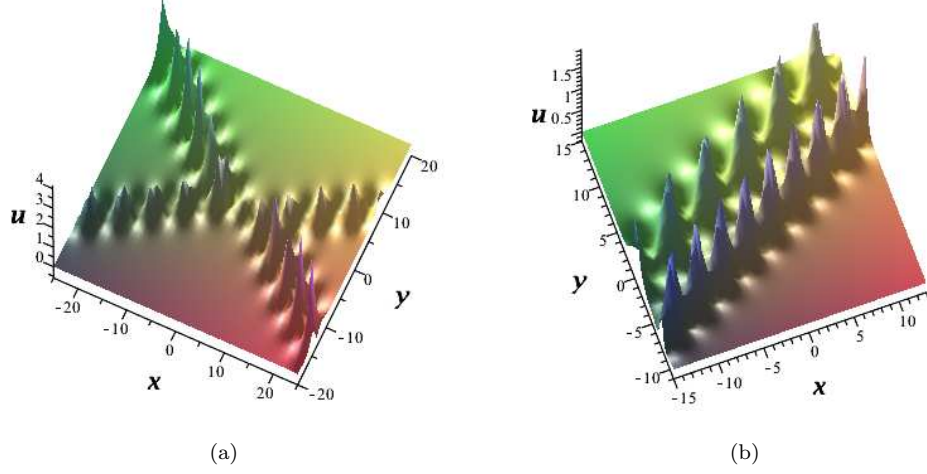


Fig. 10. Spatial structure plots of the second-order breather described by (3.6) with $p_1 = 0$, $p_2 = 0$, $\alpha = 1$, $\lambda_1 = \lambda_2 = 2$, $\eta_1 = \eta_2 = 0$, $u_0 = 0.1$, $t = 0$. (a) The ordinary second-order breather with $m_1 = 0.65$, $n_1 = -1$, $q_1 = 0.85$, $m_2 = 0.5$, $n_2 = 1$, $q_2 = 1$, $\delta_1 = \delta_2 = 0$. (b) The special second-order breather with $m_1 = 0.5$, $n_1 = 1$, $q_1 = 1$, $m_2 = 0.75$, $n_2 = 1.5$, $q_2 = 1$, $\delta_1 = 6$, $\delta_2 = 0$.

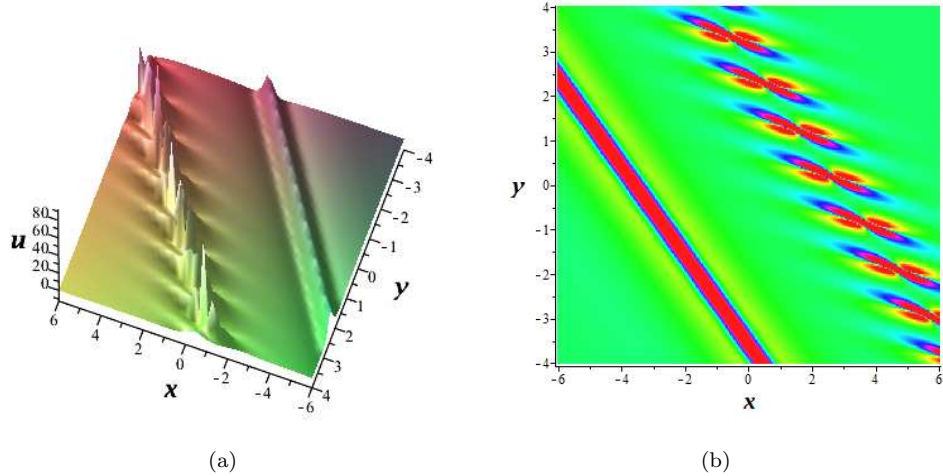


Fig. 11. The breather-soliton molecule (BSM) with $m_1 = n_1 = 1$, $p_1 = 1$, $q_1 = 0$, $m_2 = n_2 = 1.5$, $p_2 = 3$, $q_2 = 2$, $\alpha = 1$, $\lambda_1 = 0.75$, $\lambda_3 = 2$, $\delta_1 = 5$, $\delta_2 = 0$, $\eta_1 = \eta_2 = 0$, $u_0 = 0$, $t = 0$. (a) Spatial structure plot. (b) Density plot of (a).

four kinds of molecules structures including the *W*-Shaped soliton and *M*-Shaped soliton (WSMS) molecule, the *M*-Shaped soliton and *M*-Shaped soliton (MSMS) molecule, the *W*-Shaped solitray and *W*-Shaped solitary (WSWS) molecule and the *M*-Shaped solitary and periodic soliton (MSP) molecule, which are shown in Fig. 12 (a)-12 (d).

Different from the previous expressions of phase shifts (2.24), the phase shifts of various interaction structures in Sec. 3 are determined not only by time change but also the collision of different types waves, which are given as follows

$$\begin{aligned}\phi_{\xi_i}(t) &= |\mathbf{v}_{\xi_i}|t + \text{Re}(Z_{ij}) = \frac{\omega_{Ri}t}{m_i(m_ip_i - n_iq_i)}\sqrt{(m_ip_i - n_iq_i)^2 + m_i^2} + \text{Re}(Z_{ij}), \\ \phi_{\zeta_i}(t) &= |\mathbf{v}_{\zeta_i}|t + \text{Im}(Z_{ij}) = \frac{\omega_{Li}t}{n_i(m_iq_i + n_ip_i)}\sqrt{(m_iq_i + n_ip_i)^2 + n_i^2} + \text{Im}(Z_{ij}), \\ Z_{ij} &= \frac{(k_i^2 - k_ik_j + k_j^2)(k_i - k_j)^2 + 9u_0(k_i - k_j)^2}{(k_i^2 + k_ik_j + k_j^2)(k_i + k_j)^2 + 9u_0(k_i + k_j)^2}, \quad (i = 1, 2, j = 3, 4)\end{aligned}\quad (3.13)$$

Figure 13 displays an evolution process of the 2MS molecule solution under the combined action of time and collision. The phase shift of the 2MS molecule solution is different in the evolution process, and their trajectory equations also change, which results in the change of the external waveform. When $t = -0.01$, *W*-Shaped soliton (S_{r2}) converts into *W*-Shaped soliton (S_{b2}) with higher amplitude after interaction. When $t = 0$, the amplitude of *M*-Shaped soliton (S'_{r1}, S'_{r2}) becomes larger, and the waveforms of both change accordingly. When $t = 0.2$, compared with $t = 0.1$, the amplitudes of the two peaks of *M*-Shaped solitons (S''_{r2}) exchanged. The amplitudes of the peaks of *M*-Shaped solitons (S''_{b1}) are greatly increased, and the amplitudes of the peaks of *M*-Shaped solitons (S''_{b2}) are slightly decreased.

4. Concluding Remarks

We investigate the 2GKdV equation to unveil its abundant dynamical properties of the related nonlinear waves while setting on constant background wave. By introducing the paired-complex parameter restrictions and the velocity resonance conditions, we derive the first/second-order breather solutions, lump solution, and various kinds of nonlinear waves. Furthermore, the analytic expressions of the obtained nonlinear wave structures are given, and the associated dynamical characteristics are analyzed. By using the concept of trajectory equation of breather members (SWM and PWM) and investigating the expressions of the phase shifts, we point out the reason why the shape and amplitude of the nonlinear wave change with time. For the mixed solution and the fourth-order soliton solution, we obtain abundant interactional structures and nonlinear wave molecule solutions composed of the WSMS, MSMS, WSWS and MSP molecules. We also unearth the essence of the inelastic collision among the different types of localized waves and nonlinear wave molecule via decomposing the phase shifts into time variation part and collision part. The combined methods for investigating nonlinear wave structures to the 2GKdV equation are also suitable to study the related nonlinear wave phenomena in other nonlinear models.

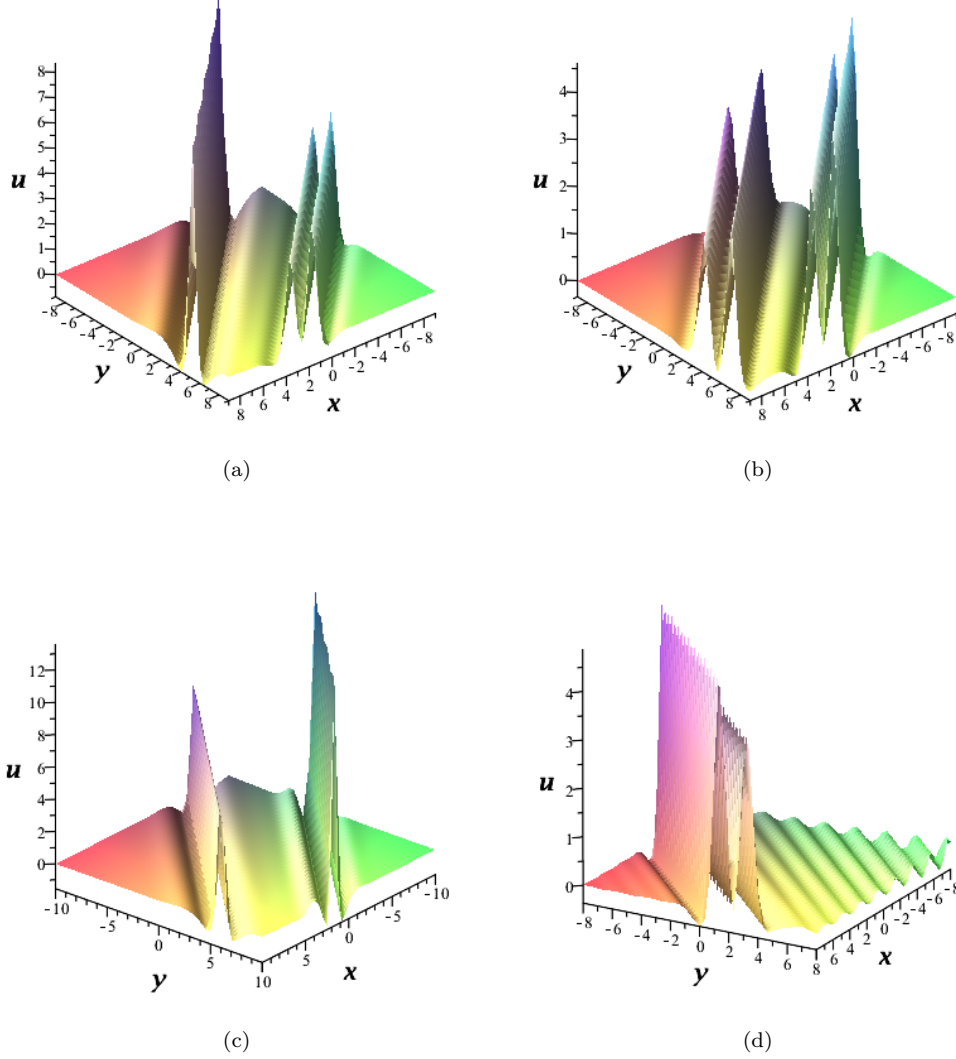


Fig. 12. Spatial structure plots of nonlinear wave molecules and $p_1 = p_2 = -1$, $\eta_1 = 10$, $\eta_2 = 0$, $\alpha = 1$, $u_0 = -0.1$, $t = 0$. (a) The WSMS molecule with $m_1 = 1.8$, $n_1 = 1.6$, $m_2 = 1$, $n_2 = 1$, $\lambda_1 = 10$, $\lambda_3 = -1$, $\delta_1 = 10$, $\delta_2 = 0$. (b) The MSMS molecule solution with $m_1 = 1.8$, $n_1 = 1.6$, $m_2 = 1.5$, $n_2 = 1$, $\lambda_1 = 10$, $\lambda_3 = 10$, $\delta_1 = 10$, $\delta_2 = 0$. (c) The WSWs molecule solution with $m_1 = 1.5$, $n_1 = 1.5$, $m_2 = 1$, $n_2 = 1$, $\lambda_1 = 10$, $\lambda_3 = -1$, $\delta_1 = 10$, $\delta_2 = 0$. (d) The MSP molecule solution with $m_1 = 1.8$, $n_1 = 1.6$, $m_2 = 0.12$, $n_2 = 3$, $\lambda_1 = 2$, $\lambda_3 = -1$, $\delta_1 = -10$, $\delta_2 = 0$.

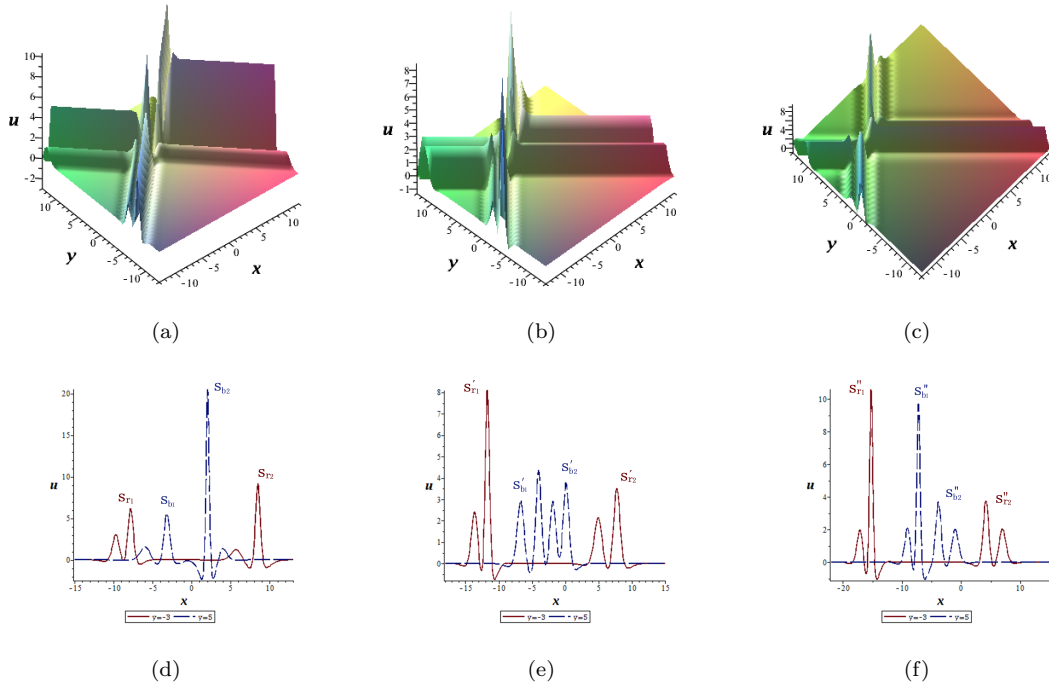


Fig. 13. The evolution diagram of the 2MS molecule interactional structure with time and $m_1 = 1.8$, $n_1 = 1.6$, $p_1 = -1$, $q_1 = 0$, $m_2 = 1.5$, $n_2 = 1$, $p_2 = 1$, $q_2 = 0$, $\lambda_1 = 5$, $\lambda_3 = 2$, $\delta_1 = 10$, $\delta_2 = 0$, $\eta_1 = 10$, $\eta_2 = 0$, $\alpha = 1$, $u_0 = -0.1$. (a) $t = -0.01$. (b) $t = 0.1$. (c) $t = 0.2$. (d), (e), (f) are the cross-sectional plots of (a), (b) and (c) when $y = -3$, $y = 5$, respectively.

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References

- [1] H Bailung, S K Sharma, Y Nakamura. Observation of peregrine solitons in a multicomponent plasma with negative ions. *Physical Review Letters*, 2011, 107: 255005.
- [2] X M Liu, X K Yao, Y D Cui. Real-time observation of the buildup of soliton molecules. *Physical Review Letters*, 2018, 121: 023905.
- [3] J G Rao, Y Cheng, P Kuppuswamy, et al. *PT*-symmetric nonlocal Davey-Stewartson I equation: Soliton solutions with nonzero background. *Physica D: Nonlinear Phenomena*, 2020, 401: 132180.
- [4] Y V Kartashov, B A Malomed, L Torner. Solitons in nonlinear lattices. *Review of Modern Physics*, 2011, 83 (1): 248-305.
- [5] J W Yang, Y T Gao, Y J Feng, et al. Solitons and dromion-like structure in an inhomogeneous optical fiber. *Nonlinear Dynamics*, 2017, 87: 851-862.
- [6] Y L Ma. Interaction and energy transition between the breather and rogue wave for a generalized nonlinear Schrödinger system with two higher-order dispersion operators in optical fibers. *Nonlinear Dynamics*, 2019, 97: 95-105.
- [7] F Yuan. The dynamics of the smooth positon and b-positon solutions for the NLS-MB equations. *Nonlinear Dynamics*, 2020, 102: 1761-1771.
- [8] Y S Tao, J S He. Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation. *Physical Review E*, 2012, 85: 026601.
- [9] X E Zhang, Y Chen, X Y Tang. Rogue wave and a pair of resonance stripe solitons to KP equation. *Computers and Mathematics with Applications*, 2018, 76: 1938-1949.
- [10] D J Kaup. The lump solutions and the Bäcklund transformation for the three-dimensional three-wave resonant interaction. *Journal of Mathematical Physics*, 1981, 22: 1176-1181.
- [11] F W Sun, W Gao. *N*-soliton solution of the (2+1)-dimensional generalized fifth-order KdV equation (in Chinese). *Journal of North China University of Technology*, 2014, 26 (3): 47-52.
- [12] A M Wazwaz. The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations. *Applied Mathematics and Computation*, 2007, 184: 1002-1014.

- [13] S D Bilige, C L Temuer. An extended simplest equation method and its application to several forms of the fifth-order KdV equation. *Applied Mathematics and Computation*, 2010, 216: 3146-3152.
- [14] A H Bilge. Singular travelling wave solutions of the fifth-order KdV, Sawada-Kotera and Kaup equations. *Journal of Physics A: Mathematical and General*, 1996, 29 (16): 4967-4975.
- [15] W Wei, R X Yao, S Y Lou. Abundant traveling wave structures of (1+1)-dimensional Sawada-Kotera equation: few cycle solitons and soliton molecules. *Chinese Physics Letters*, 2020, 37 (10): 100501.
- [16] J Q Lü, S Bilige, T Chaolu. The study of lump solution and interaction phenomenon to (2+1)-dimensional generalized fifth-order KdV equation. *Nonlinear Dynamics*, 2018, 91: 1669-1676.
- [17] M Tahir, A U Awan. The study of complexitons and periodic solitary-wave solutions with fifth-order KdV equation in (2+1)-dimensions. *Modern Physics Letters B*, 2019, 33 (33): 1950411.
- [18] W Tan, J Liu. Superposition behaviour between lump solutions and different forms of N -solitons ($N \rightarrow \infty$) for the fifth-order Korteweg-de Vries equation. *Pramana*, 2020, 94 (1): 36.
- [19] A Yusuf, T A Sulaiman. Dynamics of lump-periodic, breather and two-wave solutions with long wave in shallow water under gravity and 2D nonlinear lattice. *Communication Nonlinear Science and Numerical Simulation*, 2021, 99: 105846.
- [20] J G Liu. Lump-type solutions and interaction solutions for the (2+1)-dimensional generalized fifth-order KdV equation. *Applied Mathematics Letters*, 2018, 86: 36-41.
- [21] R Hirota. *The Direct Method in Soliton Theory*. Cambridge University Press. 2004, Vol.155.
- [22] Y Li, R X Yao, Y R Xia, et al. Plenty of novel interaction structures of soliton molecules and asymmetric solitons to (2+1)-dimensional Sawada-Kotera equation. *Communications in Nonlinear science and Numerical Simulation*, 2021, 100: 105843
- [23] M J Ablowitz, J Satsuma. Solitons and rational solutions of nonlinear evolution equations. *Journal of Mathematical Physics*, 1978, 19: 2180-2186.
- [24] Q Chen, Z Qi, J Chen, et al. Resonant line wave soliton solutions and interaction solutions for (2+1)-dimensional nonlinear wave equation. *Results in Physics*, 2021: 104480.
- [25] Z D Dai, Z J Liu, D L Li. Exact periodic solitary-wave solution for KdV equation. *Chinese Physics Letters*, 2018, 25(5), 1531-1533.
- [26] C J Wang, H Fang, X X Tang. State transition of lump-type waves for the (2+1)-dimensional generalized KdV equation. *Nonlinear Dynamics*, 2019, 95: 2943-2961.

- [27] X B Wang, S F Tian, C Y Qin, et al. Dynamics of the breathers, rogue waves and solitary waves in the $(2+1)$ -dimensional Ito equation. *Applied Mathematics Letters*, 2017, 68: 40-47.
- [28] X Zhang, L Wang, C Liu, et al. High-dimensional nonlinear wave transtions and their mechanisms. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2020, 30: 113107.
- [29] W X Ma. Abundant lumps and their interaction solutions of $(3+1)$ -dimensional linear PDEs. *Journal of Geometry and Physics*, 2018, 133: 10-16.
- [30] R X Yao, Y L Shen, Z B Li. Lump solutions and bilinear Bäcklund transformation for the $(4+1)$ -dimensional Fokas equation. *Mathematical Sciences*, 2020, 14: 301-308.
- [31] S Y Lou. Soliton molecules and asymmetric solitons in three fifth order systems via velocity resonance. *Journal of Physics Communications*, 2020, 4 (4): 041002.
- [32] X Y Yang, R Fan, B Li. Soliton molecules and some novel interaction solutions to the $(2+1)$ -dimensional B-type Kadomtsev–Petviashvili equation. *Physica Scripta*, 2020, 95 (4): 045213.
- [33] Z W Yan, S Y Lou. Special types of solitons and breather molecules for a $(2+1)$ -dimensional fifth-order KdV equation. *Communications in Nonlinear science and Numerical Simulation*, 2020, 91: 105425.
- [34] F Yuan, Y Cheng, J S He. Degeneration of the breathers in the Kadomtsev–Petviashvili I equation. *Communications in Nonlinear Science and Numerical Simulation*, 2020, 83: 105027.
- [35] Z Q Qi, Z Zhang, B Li. Space-Curved resonant line solitons in a genelized $(2+1)$ -dimensional fifth-order KdV system. *Chinese Physics Letters*, 2021, 38(6), 060501.
- [36] G Xu, A Gelash, A Chabchoub, V Zakharov, et al. Breather wave molecules. *Physical Review Letters*, 2019, 122: 084101.
- [37] M Jia, J Lin, S Y Lou. Soliton and breather molecules in few-cycle-pulse optical model. *Nonlinear Dynamics*, 2020, 100 (4): 3745-3757.
- [38] Y P Cui, L Wang, H Gegen. M-breather, M-lump, breather molecules and their interaction solutions for a $(2+1)$ -dimensional KdV equation. *Physica Scripta*, 2021, 96: 095211.
- [39] Z Zhang, X Y Yang, B Li. Novel soliton molecules and breather-positon on zero background for the complex modified KdV equation. *Nonlinear Dynamics*, 2020, 100: 1551.