

Supplementary information file:

Higher order asymptotic crack-tip fields in simplified strain gradient elasticity

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Derivation of separable form for the higher order crack tip fields

In this file, we show the derivation for representation of strain gradient elasticity general solution for crack problems (38)-(41)¹. This representation is derived by using initial form of general solution obtained in terms of modified Bessel functions of the first kind $I_{n/2}(\bar{r})$ (37).

Let us consider series solution (37) together with definitions for angular functions for the symmetric problem (22):

$$\begin{aligned} u_r &= \sum_{n=-\infty}^{\infty} \left(r^{n/2} (a_n \beta_n \cos(\frac{n}{2} - 1)\theta + b_n \cos(\frac{n}{2} + 1)\theta) \right. \\ &\quad \left. + I_{n/2}(\bar{r}) (c_n \cos(\frac{n}{2} - 1)\theta + d_n \cos(\frac{n}{2} + 1)\theta) \right) \\ u_\theta &= \sum_{n=-\infty}^{\infty} \left(r^{n/2} (a_n \sin(\frac{n}{2} - 1)\theta - b_n \sin(\frac{n}{2} + 1)\theta) \right. \\ &\quad \left. + I_{n/2}(\bar{r}) (c_n \sin(\frac{n}{2} - 1)\theta - d_n \sin(\frac{n}{2} + 1)\theta) \right) \end{aligned} \quad (\text{SI.1})$$

where a_n, b_n, c_n, d_n are the constants a, b, c, d , respectively, that persist in definition for angular functions $f_i(\theta, k)$ (22) and that are related to the order $k = n/2$; constants β_n are given by (43), i.e. this is a definition (24) for $k = n/2$; $\bar{r} = r/l$ is normalized radial coordinate.

Let us use a series representation for modified Bessel functions:

$$I_{n/2}(\bar{r}) = r^{n/2} \sum_{s=0}^{\infty} g_{n,s} r^{2s} \quad (\text{SI.2})$$

where $g_{n,s}$ is given by (42), i.e. it is a definition of the coefficients of Bessel function series $g_s(k)$ (26) with order $k = n/2$.

¹All notations and enumeration of equations are given according to main paper. References for equations used inside this Supplementary information file are denoted with SI prefix (SI.1), etc.

Substituting (SI.2) into (SI.1) we can obtain:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} (a_n \beta_n \cos(\frac{n}{2} - 1)\theta + b_n \cos(\frac{n}{2} + 1)\theta) \\
&+ \sum_{n=-\infty}^{\infty} r^{n/2} \sum_{s=0}^{\infty} g_{n,s} r^{2s} (c_n \cos(\frac{n}{2} - 1)\theta + d_n \cos(\frac{n}{2} + 1)\theta) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} (a_n \sin(\frac{n}{2} - 1)\theta - b_n \sin(\frac{n}{2} + 1)\theta) \\
&+ \sum_{n=-\infty}^{\infty} r^{n/2} \sum_{s=0}^{\infty} g_{n,s} r^{2s} (c_n \sin(\frac{n}{2} - 1)\theta - d_n \sin(\frac{n}{2} + 1)\theta)
\end{aligned} \tag{SI.3}$$

The first terms from the double sums that are defined by $s = 0$ we can combine with the first sums in representation (SI.3) to obtain:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} ((a_n \beta_n + g_{n,0} c_n) \cos(\frac{n}{2} - 1)\theta + (b_n + g_{n,0} d_n) \cos(\frac{n}{2} + 1)\theta) \\
&+ \sum_{n=-\infty}^{\infty} r^{n/2} \sum_{s=1}^{\infty} g_{n,s} r^{2s} (c_n \cos(\frac{n}{2} - 1)\theta + d_n \cos(\frac{n}{2} + 1)\theta) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} ((a_n + g_{n,0} c_n) \sin(\frac{n}{2} - 1)\theta - (b_n + g_{n,0} d_n) \sin(\frac{n}{2} + 1)\theta) \\
&+ \sum_{n=-\infty}^{\infty} r^{n/2} \sum_{s=1}^{\infty} g_{n,s} r^{2s} (c_n \sin(\frac{n}{2} - 1)\theta - d_n \sin(\frac{n}{2} + 1)\theta)
\end{aligned} \tag{SI.4}$$

Then, we can change the order of summation in the double sums as follows:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} ((a_n \beta_n + g_{n,0} c_n) \cos(\frac{n}{2} - 1)\theta + (b_n + g_{n,0} d_n) \cos(\frac{n}{2} + 1)\theta) \\
&+ \sum_{s=1}^{\infty} \sum_{n=-\infty}^{\infty} g_{n,s} r^{2s+n/2} (c_n \cos(\frac{n}{2} - 1)\theta + d_n \cos(\frac{n}{2} + 1)\theta) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} ((a_n + g_{n,0} c_n) \sin(\frac{n}{2} - 1)\theta - (b_n + g_{n,0} d_n) \sin(\frac{n}{2} + 1)\theta) \\
&+ \sum_{s=1}^{\infty} \sum_{n=-\infty}^{\infty} g_{n,s} r^{2s+n/2} (c_n \sin(\frac{n}{2} - 1)\theta - d_n \sin(\frac{n}{2} + 1)\theta)
\end{aligned} \tag{SI.5}$$

Now, let us observe that we can choose the appropriate shift for the summation index n in the double series in Eq. (SI.5) to collect all the terms with the same order with respect to radial coordinate. We can do it introducing shifted index m that is related to n as $n = m - 4s$. Introduction of this shifted index m

instead of n is possible since the series with respect to n are infinite. For given value of s in the external sum, we can choose the shifted order of summation of infinite numbers of terms in the internal sum (this can be also checked by using explicit analysis for the terms with different s and n in double series in Eq. (SI.5)).

Thus, substituting relation $n = m - 4s$ into the double series in Eq. (SI.5) (so that $m = n + 4s$ and $-\infty < m < \infty$ if $-\infty < n < \infty$), we obtain the following modification in the double series:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n \beta_n + g_{n,0} c_n) \cos\left(\frac{n}{2} - 1\right)\theta + (b_n + g_{n,0} d_n) \cos\left(\frac{n}{2} + 1\right)\theta \right) \\
&\quad + \sum_{s=1}^{\infty} \sum_{m=-\infty}^{\infty} g_{m-4s,s} r^{m/2} \left(c_{m-4s} \cos\left(\frac{m}{2} - 1 - 2s\right)\theta + d_{m-4s} \cos\left(\frac{m}{2} + 1 - 2s\right)\theta \right) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n + g_{n,0} c_n) \sin\left(\frac{n}{2} - 1\right)\theta - (b_n + g_{n,0} d_n) \sin\left(\frac{n}{2} + 1\right)\theta \right) \\
&\quad + \sum_{s=1}^{\infty} \sum_{m=-\infty}^{\infty} g_{m-4s,s} r^{m/2} \left(c_{m-4s} \sin\left(\frac{m}{2} - 1 - 2s\right)\theta - d_{m-4s} \sin\left(\frac{m}{2} + 1 - 2s\right)\theta \right)
\end{aligned} \tag{SI.6}$$

Now, we can see that the double series in (SI.6) are presented in separable form with respect to r and θ . For the further simplification, let us note that index m is a summation (dummy) index in (SI.6) and we can re-define it. It is convenient, to define this is index back to $m = n$ (in this case it is just re-definition of the letter). Then, we can change the order of summation in the double series and combine the terms with the same distribution along the radial coordinate $r^{n/2}$. As a result, we obtain:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n \beta_n + g_{n,0} c_n) \cos\left(\frac{n}{2} - 1\right)\theta + (b_n + g_{n,0} d_n) \cos\left(\frac{n}{2} + 1\right)\theta \right. \\
&\quad \left. + \sum_{s=1}^{\infty} g_{n-4s,s} \left(c_{n-4s} \cos\left(\frac{n}{2} - 1 - 2s\right)\theta + d_{n-4s} \cos\left(\frac{n}{2} + 1 - 2s\right)\theta \right) \right) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n + g_{n,0} c_n) \sin\left(\frac{n}{2} - 1\right)\theta - (b_n + g_{n,0} d_n) \sin\left(\frac{n}{2} + 1\right)\theta \right. \\
&\quad \left. + \sum_{s=1}^{\infty} g_{n-4s,s} \left(c_{n-4s} \sin\left(\frac{n}{2} - 1 - 2s\right)\theta - d_{n-4s} \sin\left(\frac{n}{2} + 1 - 2s\right)\theta \right) \right)
\end{aligned} \tag{SI.7}$$

Now, the whole series are presented in separable form and our last task it to simplify the representation for angular distribution. Let us separate the internal series for the terms with coefficients c_{n-4s} and d_{n-4s} and also separately

consider the term with constant d_{n-4} in (SI.7). Then we can write:

$$\begin{aligned}
u_r = \sum_{n=-\infty}^{\infty} r^{n/2} & \left((a_n \beta_n + g_{n,0} c_n) \cos\left(\frac{n}{2} - 1\right)\theta \right. \\
& + (b_n + g_{n,0} d_n) \cos\left(\frac{n}{2} + 1\right)\theta \\
& + \sum_{s=1}^{\infty} g_{n-4s,s} c_{n-4s} \cos\left(\frac{n}{2} - 1 - 2s\right)\theta \\
& + g_{n-4,1} d_{n-4} \cos\left(\frac{n}{2} - 1\right)\theta \\
& \left. + \sum_{s=2}^{\infty} g_{n-4s,s} d_{n-4s} \cos\left(\frac{n}{2} + 1 - 2s\right)\theta \right) \\
u_\theta = \sum_{n=-\infty}^{\infty} r^{n/2} & \left((a_n + g_{n,0} c_n) \sin\left(\frac{n}{2} - 1\right)\theta \right. \\
& - (b_n + g_{n,0} d_n) \sin\left(\frac{n}{2} + 1\right)\theta \\
& + \sum_{s=1}^{\infty} g_{n-4s,s} c_{n-4s} \sin\left(\frac{n}{2} - 1 - 2s\right)\theta \\
& - g_{n-4,1} d_{n-4} \sin\left(\frac{n}{2} - 1\right)\theta \\
& \left. - \sum_{s=2}^{\infty} g_{n-4s,s} d_{n-4s} \sin\left(\frac{n}{2} + 1 - 2s\right)\theta \right)
\end{aligned} \tag{SI.8}$$

Now, it can be seen that the term with constants d_{n-4} can be combined with those that have the same angular distribution $\cos(\frac{n}{2} - 1)\theta$ and $\sin(\frac{n}{2} - 1)\theta$. Also, for the sums with constants d_{n-4s} ($s = 2 \dots \infty$) we can introduce new index: $p = s - 1$ ($s = p + 1$). As a result, from (SI.8), we find:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n \beta_n + g_{n,0} c_n + g_{n-4,1} d_{n-4}) \cos\left(\frac{n}{2} - 1\right)\theta \right. \\
&\quad + (b_n + g_{n,0} d_n) \cos\left(\frac{n}{2} + 1\right)\theta \\
&\quad + \sum_{s=1}^{\infty} g_{n-4s,s} c_{n-4s} \cos\left(\frac{n}{2} - 1 - 2s\right)\theta \\
&\quad \left. + \sum_{p=1}^{\infty} g_{n-4(p+1),p+1} d_{n-4(p+1)} \cos\left(\frac{n}{2} - 1 - 2p\right)\theta \right) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n + g_{n,0} c_n - g_{n-4,1} d_{n-4}) \sin\left(\frac{n}{2} - 1\right)\theta \right. \\
&\quad - (b_n + g_{n,0} d_n) \sin\left(\frac{n}{2} + 1\right)\theta \\
&\quad + \sum_{s=1}^{\infty} g_{n-4s,s} c_{n-4s} \sin\left(\frac{n}{2} - 1 - 2s\right)\theta \\
&\quad \left. - \sum_{p=1}^{\infty} g_{n-4(p+1),p+1} d_{n-4(p+1)} \sin\left(\frac{n}{2} - 1 - 2p\right)\theta \right)
\end{aligned} \tag{SI.9}$$

One can see that internal series with indexes s and p are summed with the same functions of angular coordinate. Thus, taking into account that p is dummy summation index we can re-define $p = s$ and combine both internal series in (SI.9) as follows:

$$\begin{aligned}
u_r &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n \beta_n + g_{n,0} c_n + g_{n-4,1} d_{n-4}) \cos\left(\frac{n}{2} - 1\right)\theta \right. \\
&\quad + (b_n + g_{n,0} d_n) \cos\left(\frac{n}{2} + 1\right)\theta \\
&\quad + \sum_{s=1}^{\infty} (g_{n-4s,s} c_{n-4s} + g_{n-4(s+1),s+1} d_{n-4(s+1)}) \cos\left(\frac{n}{2} - 1 - 2s\right)\theta \Big) \\
u_\theta &= \sum_{n=-\infty}^{\infty} r^{n/2} \left((a_n + g_{n,0} c_n - g_{n-4,1} d_{n-4}) \sin\left(\frac{n}{2} - 1\right)\theta \right. \\
&\quad - (b_n + g_{n,0} d_n) \sin\left(\frac{n}{2} + 1\right)\theta \\
&\quad + \sum_{s=1}^{\infty} (g_{n-4s,s} c_{n-4s} - g_{n-4(s+1),s+1} d_{n-4(s+1)}) \sin\left(\frac{n}{2} - 1 - 2s\right)\theta \Big)
\end{aligned} \tag{SI.10}$$

This result (SI.10) coincides with relations (38), (39), (41) in the main paper up to re-normalization of the constants a_n , b_n , c_n , d_n with respect to 4μ . Definitions for auxiliary constants F_n , L_n , M_n , G_{ns} , H_{ns} in (41) is introduced for the constants of general solution that are collected in the brackets before angular functions of the same periodicity in representation (SI.10). Representation

for the skew-symmetric loading can be obtained by using similar derivations (SI.1)-(SI.10) and definition (40) for angular functions.