Supplementary material

Detection of quadratic phase coupling by cross-bicoherence and spectral Granger causality in bifrequencies interactions

Takeshi Abe[1,*], Yoshiyuki Asai[2], Alessandra Lintas[3], Alessandro E.P. Villa[4,*]

(1) AI Systems Medicine Research and Training Center, Graduate School of Medicine and University Hospital, Yamaguchi University, Yamaguchi, Japan

(2) Department of Systems Bioinformatics, Graduate School of Medicine, Yamaguchi University, Yamaguchi, Japan

(3) Neuroheuristic Research Group & HEC-LABEX, University of Lausanne, Quartier UNIL-Chamberonne, CH-1015 Lausanne, Switzerland

(4) Neuroheuristic Research Group & Complexity Sciences Research Group, University of Lausanne, Quartier UNIL-Chamberonne, CH-1015 Lausanne, Switzerland

[*]**Corresponding authors:**
Takeshi Abe, Yamaguchi University
E-mail address: t.abe@yamaguchi-u.ac.jp

Alessandro E.P. Villa, University of Lausanne
E-mail address: alessandro.villa@unil.ch.

S1. Signal-to-noise ratio : SNR

For a signal $x_i(t)$, the average (normalized) power is given by (Grami, 2015):

$$P_{x_i} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x_i(t)|^2 dt.$$

For any independent white noise signal $\xi_i(t)$ following a standard Gaussian distribution (mean $\mu = 0$, standard deviation $\sigma = 1$), the average power is $P_{\xi_i} = 1$. For each channel $X_i(t,\omega)$ of our three-channels model of bifrequencies interaction, the power of the signal (without noise) is $P_{(X_i-\xi_i)}$ and the signal-to-noise ratio (SNR) values can be computed as

$$\mathrm{SNR}_{X_i} = \frac{P_{(X_i-\xi_i)}}{P_{\xi_i}}.$$

Here ω_i represents a specific realization of signal X_i sampled from a uniform distribution on period $[0,2\pi)$. A signal's power and its SNR are positive real-valued random variables, depending on phase ω_i in general.

The signal recorded by channel 1 is defined by:

$$\begin{cases} I_1(t,\omega) &= 2\,\mathrm{asin}\big(\sin\big(2\pi f_1 t + \omega_1\big)\big); \\ X_1(t,\omega) &= I_1(t,\omega) + \xi_1(t). \end{cases} \tag{1}$$

The average power of channel 1 is $P_{(X_1-\xi_1)} = P(2\,\mathrm{asin}\big(\sin\big(2\pi f_1 t + \omega_1\big)\big)$, which yields $P_{(X_1-\xi_1)} = \frac{\pi^2}{3}$ for any $f_1 > 0$ and $\omega_1 \in [0,2\pi)$. Hence,

$$\mathrm{SNR}_{X_1} = \frac{\pi^2}{3}. \tag{2}$$

The signal recorded by channel 2 is defined by:

$$\begin{cases} I_2(t,\omega) &= \begin{cases} 1 & 0 \le (2\pi f_2 t + \omega_2) \bmod 2\pi < \pi; \\ -1 & \text{otherwise} \end{cases} \\ X_2(t,\omega) &= I_2(t,\omega) + \xi_2(t). \end{cases} \tag{3}$$

The average power of channel 2 is $P_{(X_2-\xi_2)} = 1$, because $I_2(t,\omega)^2 = 1$ for any $f_2 > 0$ and $\omega_2 \in [0,2\pi)$. Hence,

$$\mathrm{SNR}_{X_2} = 1. \tag{4}$$

The signal recorded by channel 3 is defined by:

$$\begin{cases} I_3(t,\omega) &= \cos\big(2\pi f_3 x + \omega_3\big) \\ X_3(t,\omega) &= W_{(1,2)} X_1(t,\omega) X_2(t,\omega) + I_3(t,\omega) + \xi_3(t). \end{cases} \tag{5}$$

The average power of channel 3 is computed as follows

$$\begin{aligned} P_{(X_3-\xi_3)} &= P_{(W_{(1,2)} X_1 X_2 + I_3)} \\ &= W_{(1,2)}^2 P_{X_1 X_2} + 2W_{(1,2)} \varepsilon(\omega) + P_{I_3}, \end{aligned} \tag{6}$$

where

$$\varepsilon(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_1(t,\omega) I_2(t,\omega) I_3(t,\omega) dt.$$

Given that $X_1^2 = I_1^2 + 2I_1\xi_1 + \xi_1^2$ and $X_2^2 = I_2^2 + 2I_2\xi_2 + \xi_2^2$, the average power of X_1X_2 is

$$P_{X_1X_2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} (I_1^2 + 2I_1\xi_1 + \xi_1^2)\ (I_2^2 + 2I_2\xi_2 + \xi_2^2) dt. \tag{7}$$

Because $|I_2| = 1$, we have

$$P_{X_1X_2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} (I_1^2 + 2I_1\xi_1 + \xi_1^2)\ (1 + 2I_2\xi_2 + \xi_2^2) dt \tag{8}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} (I_1^2 + 2I_1^2 I_2\xi_2 + I_1^2\xi_2^2 + 2I_1\xi_1 + 4I_1I_2\xi_1\xi_2 + 2I_1\xi_1\xi_2^2 + \xi_1^2 + 2I_2\xi_1^2\xi_2 + \xi_1^2\xi_2^2) dt. \tag{9}$$

The white noise signals $\xi_i(t)$ follow a standard Gaussian distribution (mean $\mu = 0$, standard deviation $\sigma = 1$), such that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_1^2 I_2\xi_2 dt = 0;$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_1\xi_1 dt = 0;$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_1 I_2\xi_1\xi_2 dt = 0;$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_1\xi_1\xi_2^2 dt = 0;$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_2\xi_1^2\xi_2 dt = 0;$$

Then, equation (9) can be rewritten as

$$P_{X_1X_2} = P_{I_1} + 0 + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I_1^2\xi_2^2 dt + 0 + 0 + 0 + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \xi_1^2 dt + 0 + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \xi_1^2\xi_2^2 dt \tag{10}$$

$$= P_{I_1} + P_{I_1\xi_2} + P_{\xi_1} + P_{\xi_1\xi_2}. \tag{11}$$

Notice that $P_{I_1} = \frac{\pi^2}{3}$ according to equation (1) and $P_{I_1\xi_2} = \frac{\pi^2}{3}$ because I_1^2 is independent of ξ_2^2 and $P_{\xi_1} = 1$ by definition and $P_{\xi_1 xi_2} = 1$ because ξ_1^2 and ξ_2^2 are mutually independent. Hence, we obtain

$$P_{X_1X_2} = \frac{\pi^2}{3} + \frac{\pi^2}{3} + 1 + 1$$

$$P_{X_1X_2} = \frac{2\pi^2}{3} + 2. \tag{12}$$

For each $f_3 > 0$, $\varepsilon(\omega) = 0$ with probability 1 with respect to ω's distribution ($\omega_i \in [0, 2\pi)$). According to equation (5), $P_{I_3} = \frac{1}{2}$. Hence, equation (6) can be rewritten as

$$P_{(X_3,\xi_3)} = W_{(1,2)}^2 P_{X_1X_2} + 2W_{(1,2)}\varepsilon(\omega) + P_{I_3}$$

$$= W_{(1,2)}^2 (\frac{2\pi^2}{3} + 2) + 0 + \frac{1}{2}, \tag{13}$$

which leads to the signal-to-noise ratio for channel 3 equal to

$$\mathrm{SNR}_{X_3} = \left(\frac{2}{3}\pi^2 + 2\right) W_{(1,2)}^2 + \frac{1}{2}. \tag{14}$$

References

Grami, A. (2015). *Introduction to Digital Communications*. 604 pp. Academic Press, 1st edition.