1. Appendix A

The coefficients of characteristic equation $\lambda^6 + p_1\lambda^5 + p_2\lambda^4 + p_3\lambda^3 + p_4\lambda^2 + p_5\lambda + p_6 = 0$, in Proposition 3.1 are obtained as follows.

$$p_1 = B + C + \xi + \gamma + 6\,m, \tag{1.1}$$

$$p_2 = (\xi + \gamma + 5m)\,(B + C) + B\,C + \xi\,\gamma + 5\,\xi\,m + 5\gamma\,m + 15\,m^2 + F, \tag{1.2}$$

$$\begin{aligned}
p_3 = {} & (\xi+m)(\gamma+m)(B+m) + (\xi+m)(\gamma+m)(\alpha_1+m) + (\xi+m)(\gamma+m)(\alpha_2+m) \\
& + (\xi+m)(\gamma+m)(\alpha_3+m) + ((\gamma+m)(B+m)(\alpha_1+m) - \beta\,\phi\,\delta_1\,S_0\,\alpha_1\,\eta_1) \\
& + ((\gamma+m)(B+m)(\alpha_2+m) - \beta\,\phi\,\delta_1\,S_0\,\alpha_2\,\eta_2)) + (\alpha_1+m)(\alpha_2+m)(\alpha_3+m) \\
& + ((\gamma+m)(B+m)(\alpha_3+m) - \beta\,\phi\,\delta_1\,S_0\,\alpha_3\,\eta_3) + (B+m)(\alpha_1+m)(\alpha_2+m) \\
& + (B+m)(\alpha_1+m)(\alpha_3+m),
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
p_4 = {} & (\xi+m)(\gamma+m)(B+m)(\alpha_1+m) + (\xi+m)(\gamma+m)(B+m)(\alpha_2+m) \\
& + (\xi+m)(\gamma+m)(B+m)(\alpha_3+m) + (\xi+m)(B+m)(\alpha_1+m)(\alpha_2+m) \\
& + (\xi+m)(B+m)(\alpha_1+m)(\alpha_3+m) + (\xi+m)(B+m)(\alpha_2+m)(\alpha_3+m) \\
& + (\gamma+m)(B+m)(\alpha_1+m)(\alpha_2+m) + (\gamma+m)(B+m)(\alpha_2+m)(\alpha_3+m) \\
& + (B+m)(\alpha_1+m)(\alpha_2+m)(\alpha_3+m) + (\gamma+m)(\alpha_1+m)(\alpha_2+m)(\alpha_3+m) \\
& + (\xi+m)(\alpha_1+m)(\alpha_2+m)(\alpha_3+m) + (\xi+m)(\gamma+m)(\alpha_1+m)(\alpha_2+m) \\
& + (\xi+m)(\gamma+m)(\alpha_1+m)(\alpha_3+m) + (\xi+m)(\gamma+m)(\alpha_2+m)(\alpha_3+m) \\
& - \beta\,\phi\,\delta_1\,S_0\alpha_1\eta_1\,(\xi+m) - \beta\,\phi\,\delta_1 S_0\,\alpha_2\eta_2\,(\xi+m) \\
& - \beta\,\phi\,\delta_1 S_0\alpha_3\eta_3\,(\xi+m) - \beta\,\gamma\,\phi\,\delta_2\,S_0\alpha_1\eta_1 - \beta\,\gamma\,\phi\,\delta_2\,S_0\,\alpha_2\eta_2 \\
& - \beta\,\gamma\,\phi\,\delta_2\,S_0\,\alpha_3\eta_3 - \beta\,\delta_1\,S_0\,\phi\,\alpha_1\eta_1(\alpha_2+m) - \beta\,\delta_1\,S_0\,\phi\,\alpha_1\eta_1(\alpha_3+m) - \beta\,\delta_1 \\
& \times S_0\,\phi\,\alpha_2\eta_2(\alpha_1+m) - \beta\,\delta_1\,S_0\,\phi\,\alpha_2\eta_2(\alpha_3+m) - \beta\,\delta_1\,S_0\,\phi\,\alpha_3\eta_3(\alpha_1+m) \\
& - \beta\,\delta_1\,S_0\,\phi\,\alpha_3\eta_3(\alpha_2+m),
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
p_5 = {} & (\xi+m)(\gamma+m)(\alpha_1+m)(\alpha_2+m)(\alpha_3+m)+(\xi+m)(\gamma+m)(B+m)\\
& \times(\alpha_1+m)(\alpha_2+m)+(\xi+m)(\gamma+m)(B+m)(\alpha_1+m)(\alpha_3+m)+(\xi+m)\\
& \times(B+m)(\alpha_1+m)(\alpha_2+m)(\alpha_3+m)+(\xi+m)(\gamma+m)(B+m)(\alpha_2+m)\\
& \times(\alpha_3+m)+(\xi+m)(B+m)(\alpha_1+m)(\alpha_2+m)(\alpha_3+m)-\beta\,\delta_2\,S_0\,\gamma\,\phi\,\alpha_1\,\eta_1\\
& \times(\alpha_2+m)-\beta\,\delta_2\,S_0\,\phi\,\gamma\,\alpha_1\,\eta_1(\alpha_3+m)-\beta\,\delta_2\,S_0\,\gamma\,\phi\,\alpha_2\,\eta_2(\alpha_1+m)\\
& -\beta\,\delta_2\,S_0\,\gamma\,\phi\,\alpha_2\,\eta_2(\alpha_3+m)-\beta\,\delta_2\,S_0\,\gamma\,\phi\,\alpha_3\,\eta_3(\alpha_1+m)-\beta\,\delta_2\,S_0\,\gamma\,\phi\,\alpha_3\,\eta_3(\alpha_2+m)\\
& -\beta\,\delta_1\,S_0\,\phi\,\alpha_1\,\eta_1(\xi+m)(\alpha_2+m)-\beta\,\delta_1\,S_0\,\phi\,\alpha_1\,\eta_1(\xi+m)(\alpha_3+m)\\
& -\beta\,\delta_1\,S_0\,\phi\,\alpha_1\,\eta_1(\alpha_2+m)(\alpha_3+m)-\beta\,\delta_1\,S_0\,\phi\,\alpha_2\,\eta_2(\xi+m)(\alpha_1+m)\\
& -\beta\,\delta_1\,S_0\,\phi\,\alpha_2\,\eta_2(\alpha_1+m)(\alpha_3+m)-\beta\,\delta_1\,S_0\,\phi\,\alpha_3\,\eta_3(\xi+m)(\alpha_1+m)\\
& -\beta\,\delta_1\,S_0\,\phi\,\alpha_3\,\eta_3(\xi+m)(\alpha_2+m)-\beta\,\delta_1\,S_0\,\phi\,\alpha_3\,\eta_3(\alpha_1+m)(\alpha_2+m)\\
& -\beta\,\delta_1\,S_0\,\phi\,\alpha_2\,\eta_2(\xi+m)(\alpha_3+m),
\end{aligned} \tag{1.5}$$

$$p_6 = (m+\alpha_1)\,(m+\alpha_2)\,(m+\alpha_3)\,(\gamma+m)\,(\xi+m)\,(B+m)\,(1-R_0). \tag{1.6}$$

Parameters B, D, E and G are defined in (3.3)-(3.6) and

$$\begin{aligned}
C &= \alpha_1+\alpha_2+\alpha_3,\\
S_0 &= \frac{L}{m},\\
F &= \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3.
\end{aligned}$$

The basic reproduction number R_0 can be written as follows:

$$\begin{aligned}
R_0 = {} & \frac{\beta\,\delta_1 S_0\,\phi\,\alpha_1\,\eta_1}{(\alpha_1+m)(\gamma+m)(B+m)}+\frac{\beta\,\delta_1 S_0\,\phi\,\alpha_2\,\eta_2}{(\alpha_2+m)(\gamma+m)(B+m)}\\
& +\frac{\beta\,\delta_1 S_0\,\phi\,\alpha_3\,\eta_3}{(\alpha_3+m)(\gamma+m)(B+m)}+\frac{\beta\,\delta_2 S_0\,\phi\,\gamma\,\alpha_1\,\eta_1}{(\alpha_1+m)(\gamma+m)(B+m)(\xi+m)}\\
& +\frac{\beta\,\delta_2 S_0\,\phi\,\gamma\,\alpha_2\,\eta_2(m^2+\alpha_1\,m+\alpha_3\,m)}{(\alpha_1+m)(\alpha_2+m)(\alpha_3+m)(\gamma+m)(B+m)(\xi+m)}\\
& +\frac{\beta\,\delta_2 S_0\,\phi\,\gamma\,\alpha_3\,\eta_3}{(\alpha_3+m)(\gamma+m)(B+m)(\xi+m)}.
\end{aligned}$$

Since $R_0 < 1$, we have

$$\frac{\beta\,\delta_1 S_0\,\phi\,\alpha_1\,\eta_1}{(\alpha_1+m)(\gamma+m)(B+m)} < 1,$$
$$\frac{\beta\,\delta_1 S_0\,\phi\,\alpha_2\,\eta_2}{(\alpha_2+m)(\gamma+m)(B+m)} < 1$$
$$\frac{\beta\,\delta_1 S_0\,\phi\,\alpha_3\,\eta_3}{(\alpha_3+m)(\gamma+m)(B+m)} < 1.$$

Hence $p_3 > 0$. Similar to the above discussion, it can be shown that $p_4 > 0$, $p_5 > 0$. It is clear, the inequality $\triangle_1 = p_1 > 0$ holds. $\triangle_3$ and $\triangle_5$ are obtained from the following relationship:

$$\begin{aligned}
\triangle_3 &= p_3(p_1p_2 - p_3) + p_1(p_5 - p_1p_4),\\
\triangle_5 &= {p_1}^2p_2p_5p_6 + {p_1}^2p_3p_4p_6 - {p_1}^2{p_4}^2p_5\\
&\quad - p_1{p_2}^2{p_5}^2 - p_1p_2{p_3}^2p_6 + p_1p_2p_3p_4p_5\\
&\quad - {p_1}^2{p_6}^2 + p_1p_2p_5p_6 - 2\,p_1p_3p_5p_6\\
&\quad + 2\,p_1p_4{p_5}^2 + p_2p_3{p_5}^2 + {p_3}^3p_6 - {p_3}^2p_4p_5\\
&\quad - p_3p_5p_6 - {p_5}^3.
\end{aligned}$$

By using simple but long calculations, the inequalities $\triangle_3 > 0$ and $\triangle_5 > 0$ hold. To prove this proposition, we have used Liénard and Chipart and Routh-Hurwitz stability criterion which are stated below.

Definition 1.1. *The Routh-Hurwitz stability criterion states that for a system having a characteristic equation as follows:*

$$P_n(\lambda) = p_0\lambda^n + p_1\lambda^{n-1} + \ldots + p_{n-1}\lambda + p_n.$$

The Matrix

$$H_n = \begin{pmatrix}
p_1 & p_0 & 0 & 0 & 0 & 0 & \vdots & 0\\
p_3 & p_2 & p_1 & p_0 & 0 & 0 & \vdots & 0\\
p_5 & p_4 & p_3 & p_2 & p_1 & p_0 & \vdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & p_n
\end{pmatrix}.$$

is called the Hurwitz matrix, associated with $P_n(\lambda)$. The main diagonal of the matrix contains elements $p_1, p_2, ..., p_n$. The first column contains numbers with odd indices $p_1, p_3, p_5, ...$ In each row, the index of each following number (counting from left to right) is less than the index of its predecessor. All other coefficients with indices greater than n or less than 0 are replaced by zeros. To be asymptotically stable of system, all the principal minors of the H_n must be *positive where the principle minors are as follows:*

$$\Delta_1 = |p_1|$$
$$\Delta_2 = \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix}$$
$$\Delta_3 = \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{vmatrix}$$
$$\vdots$$
$$\Delta_n = |H_n|.$$

Note 2.1: It can still be difficult to calculate these determinants; however, Liénard and Chipart showed that if all the coefficients in the characteristic equation are positive ($a_i > 0$ for $i = 1, \ldots, n$) then the system is asymptotically stable if either

- $\Delta_1 > 0, \Delta_3 > 0, ...$

or

- $\Delta_2 > 0, \Delta_4 > 0, ...$