Supplementary Information for "Free particles beyond fermions and bosons"

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This Supplementary Information fills in several technical details omitted in the main text. In Sec. S1 we define a Hermitian inner product on the state space which guarantees the Hermiticity of Hamiltonians and physical observables. In Sec. S2 we present the detailed construction of the state space and prove the linear independence of the basis states constructed in Eq. (11). In Sec. S3 we provide mathematical details on the 1D spin model defined in Eqs. (17-19).

S1. PROOF OF HERMITICITY

We now prove our claim in the main text that the Hermitian conjugate † can be consistently defined on the states and operators such that $e^\dagger_{ij} = \hat{e}_{ji}$, for $1 \le i, j \le N$, which guarantees Hermiticity of Hamiltonians and unitarity of quantum time evolution. To define the Hermitian conjugate † of operators, we need to define a Hermitian inner product $\langle \dots | \dots \rangle$ on the state space, and then the Hermitian conjugate of an operator $\hat{O}$ is defined as $\langle \Psi|\hat{O}^\dagger\Phi\rangle \equiv \langle \hat{O}\Psi|\Phi\rangle$, for any states $|\Psi\rangle, |\Phi\rangle$. In the following we first show that such an inner product can be consistently defined on the state space such that the induced Hermitian conjugate † satisfies $e^\dagger_{ij} = \hat{e}_{ji}$, $\forall i, j$, and then give a more explicit definition of this inner product.

To begin, we first notice that, with the CRs in Eq. (9), the set of operators $\{\hat{e}_{ij} + \hat{e}_{ji}, i(\hat{e}_{ij} - \hat{e}_{ji})|1 \le i, j \le N\}$ spans a closed Lie algebra $\mathfrak{u}_N \cong \mathfrak{su}_N \oplus \mathfrak{u}_1$ (over the field of real numbers $\mathbb{R}$), where the $\mathfrak{u}_1$ part is $n \equiv \sum_{i=1}^N \hat{e}_{ii}$, and the $\mathfrak{su}_N$ part is spanned by $\{\hat{e}_{ij} + \hat{e}_{ji}, i(\hat{e}_{ij} - \hat{e}_{ji})|1 \le i < j \le N\}$ along with $\{\hat{e}_{ii} - \hat{e}_{i+1,i+1}|1 \le i \le N-1\}$. We now invoke the following theorem whose proof can be found in Ref. [1]

Theorem 1. *For each representation ρ of a compact semisimple real Lie algebra $\mathfrak{g}$ on a finite dimensional $\mathbb{C}$-vector space V, there exists a Hermitian inner product on V such that all $\rho(x)(x \in \mathfrak{g})$ are Hermitian.*

Since $\mathfrak{su}_N$ is a compact semisimple real Lie algebra, Thm. 1 guarantees the existence of a Hermitian inner product such that $\{\hat{e}_{ij} + \hat{e}_{ji}, i(\hat{e}_{ij} - \hat{e}_{ji})|1 \le i < j \le N\}$ and $\{\hat{e}_{ii} - \hat{e}_{i+1,i+1}|1 \le i \le N-1\}$ are all Hermitian, as long as the state space is finite dimensional. But even if the state space is infinite dimensional, we will see later that the full state space can always be decomposed as a direct sum of finite dimensional irreducible representations (irreps) of $\mathfrak{su}_N$, and Thm. 1 still applies to each irrep. As for the $\mathfrak{u}_1$ part, since n is proportional to the identity operator in each irrep, with the proportionality constant being the total particle number, it follows that n is also Hermitian. Since Thm. 1 implies that $\hat{e}_{ij} + \hat{e}_{ji} = \hat{e}^\dagger_{ij} + \hat{e}^\dagger_{ji}$ and $i(\hat{e}_{ij} - \hat{e}_{ji}) = -i(\hat{e}^\dagger_{ij} - \hat{e}^\dagger_{ji})$, it follows that $\hat{e}^\dagger_{ij} = \hat{e}_{ji}$, for all $1 \le i, j \le N$.

The Hermitian inner product on the state space can be defined more explicitly as follows. The state space constructed in Eqs. (10-12) decomposes into a direct sum of different particle number sectors, and $\hat{n}$ is proportional to identity in each sector. Each particle number sector further decomposes into a direct sum of irreps of $\mathfrak{su}_N$. We set $\langle\Psi|\Phi\rangle = 0$ if $|\Psi\rangle, |\Phi\rangle$ lie in different irreps (i.e., inequivalent irreps or different copies of equivalent irreps) of $\mathfrak{su}_N$. In this way, n is automatically Hermitian (indeed, it is real and diagonal) and the problem reduces to defining $\langle \dots | \dots \rangle$ within each irrep of $\mathfrak{su}_N$.

We now show that within each irrep, the inner product between any two states is uniquely determined (up to a multiplicative factor) by the requirement $\hat{e}^\dagger_{ij} = \hat{e}_{ji}$, $\forall i, j$. We show this in the framework of highest weight theory [2]. For every finite-dimensional irrep V of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, there exists a unique (up to a multiplicative constant) highest weight vector $|\Lambda\rangle$ (a $|\Lambda\rangle$ that is annihilated by all positive root operators $\hat{e}_\alpha|\Lambda\rangle = 0$), and all other weight vectors $|\Lambda'\rangle$ in V can be constructed by applying negative root operators on $|\Lambda\rangle$, i.e. $|\Lambda'\rangle = \prod_\alpha \hat{e}_{-\alpha}|\Lambda\rangle$, where the product is over some ordered set of positive roots. For the case of $\mathfrak{su}_N$, positive root operators are $\{\hat{e}_{ij}|1 \le i < j \le N\}$, negative root operators are $\{\hat{e}_{ji}|1 \le i < j \le N\}$, while $\{\hat{e}_{ii} - \hat{e}_{i+1,i+1}|1 \le i \le N-1\}$ spans the Cartan subalgebra. Without loss of generality we can assume $\langle\Lambda|\Lambda\rangle = 1$. Then for any two weight vectors $|\Lambda_1\rangle, |\Lambda_2\rangle \in V$, their inner product can be calculated as

$$\begin{aligned}\langle\Lambda_1|\Lambda_2\rangle &= \prod_{\alpha,\beta} \langle\Lambda|\hat{e}^\dagger_{-\beta}\hat{e}_{-\alpha}|\Lambda\rangle \\ &= \langle\Lambda| \prod_{\alpha,\beta} \hat{e}_\beta \hat{e}_{-\alpha}|\Lambda\rangle, \end{aligned} \tag{S1}$$

and the last line can be calculated by using the CRs between $\hat{e}_\beta$ and $\hat{e}_{-\alpha}$ (to move all the positive root operators $\hat{e}_\beta$ to the right). Notice that there may be several different ways to represent $|\Lambda_{1,2}\rangle$ in the form $\prod_\alpha \hat{e}_{-\alpha}|\Lambda\rangle$, and consequently there are different ways to compute the same inner product $\langle\Lambda_1|\Lambda_2\rangle$. Thm. 1 guarantees that all the different ways of computing $\langle\Lambda_1|\Lambda_2\rangle$ give the same result.

S2. MATHEMATICAL DETAILS ON STRUCTURE OF STATE SPACE

In this section we present the detailed construction of the state space on a rigorous mathematical level, and prove several claims we made in the main text. Specifically, we will (1) rigorously define the state space and the action of $\hat{\psi}^{\pm}_{i,b}$ on it; (2) prove that the basis states constructed in Eq. (11) are linearly independent [Note the difference between the objectives (1) and (2) and that of Sec. S1: the latter studies the structure of the state space as a representation of the Lie algebra generated by the $\hat{e}_{ij}$s, which tells us the action of $\hat{e}_{ij}$ on the state space. However, results of Sec. S1 do not define the action of $\hat{\psi}^{\pm}_{i,b}$, and they do not tell us how to determine the structure of the full state space (e.g. the exclusion statistics and the partition function) from the R-matrix. The objectives (1) and (2) in this section solve these issues rigorously.]. We will also show the relation between the second and the first quantization formulation.

We begin by introducing some notations. Denote by $\mathcal{X}_{R,N}$ the unital associative algebra over $\mathbb{C}$ generated by $\{\hat{\psi}^{\pm}_{i,b}|1 \le i \le N, 1 \le b \le m\}$ modulo all the relations in Eq. (6). Define $\mathcal{X}^{+}_{R,N}$ as the (unital) subalgebra of $\mathcal{X}_{R,N}$ generated by all the creation operators $\{\hat{\psi}^{+}_{i,b}|1 \le i \le N, 1 \le b \le m\}$, and similarly $\mathcal{X}^{-}_{R,N}$ the (unital) subalgebra of $\mathcal{X}_{R,N}$ generated by all the annihilation operators $\{\hat{\psi}^{-}_{i,b}|1 \le i \le N, 1 \le b \le m\}$.

An important observation is that the algebra $\mathcal{X}_{R,N}$ can be obtained from $\mathcal{X}_{R,1}$ as

$$\mathcal{X}_{R,N} \cong \mathcal{X}_{\Pi\boxtimes R,1}, \tag{S2}$$

where $\Pi \boxtimes R$ is the direct product R-matrix defined as

$$(\Pi \boxtimes R)^{AB}_{CD} \equiv \Pi^{ij}_{kl} R^{ab}_{cd}, \tag{S3}$$

where we group the spatial index $i = 1, 2, \dots, N$ and the internal index a in $\hat{\psi}^{\pm}_{i,a}$ into a single collective index: $A = (i, a), B = (j, b), C = (k, c)$ and $D = (l, d)$. Π acts on the spatial part defined as $\Pi^{ij}_{kl} = \delta_{il}\delta_{jk}$, and R acts on the internal part. It is straightforward to check that $\Pi\boxtimes R$ constructed this way also satisfies the YBE Eq. (5), and Eq. (S2) can be checked by comparing the defining CRs of both sides. For this reason, in Secs. S2 A and S2 B we focus on the algebra $\mathcal{X}_R \equiv \mathcal{X}_{R,1}$, but keep in mind that any claim we make on $\mathcal{X}_R$ applies equally well to $\mathcal{X}_{R,N}$ by using the product R-matrix $\Pi \boxtimes R$. We will omit the mode labels i, j and simply write $\hat{\psi}^{\pm}_{a}$ when there is no confusion.

A. Existence and uniqueness of vacuum state from physical requirements

For the theory to make physical sense, the spectrum of the total particle number operator $\hat{n}$ should be bounded from below. This means that there exists at least one state $|n_{\min}\rangle$ with the smallest eigenvalue $n_{\min}$ of $\hat{n}$. Since $\hat{\psi}^{-}_{a}$ decreases the eigenvalue of $\hat{n}$ by 1, the minimality of $n_{\min}$ requires that $\hat{\psi}^{-}_{a}|n_{\min}\rangle = 0, \forall a$, since otherwise $\hat{\psi}^{-}_{a}|n_{\min}\rangle$ would be an eigenstate of $\hat{n}$ with eigenvalue $n_{\min} - 1 < n_{\min}$. Therefore $\hat{n}|n_{\min}\rangle = \sum_a \hat{\psi}^{+}_{a}\hat{\psi}^{-}_{a}|n_{\min}\rangle = 0$, i.e., $n_{\min} = 0$. We call this state the vacuum state, denoted by $|0\rangle$.

It can be proven that in an irrep V of $\mathcal{X}_R$, the vacuum state must be unique. Here is a sketch of the proof by contradiction: assume there exists two linearly independent vacuum states, say $|0\rangle, |0'\rangle \in V$. Then $V_0 = \mathcal{X}^{+}_{R}|0\rangle$ would be invariant under the action of $\mathcal{X}_R$. To prove this, it is enough to show that V_0 is invariant under all the generators $\hat{\psi}^{\pm}_{a}$ of $\mathcal{X}_R$: $\hat{\psi}^{+}_{a}$ leaves V_0 invariant since $\hat{\psi}^{+}_{a}\mathcal{X}^{+}_{R} \subseteq \mathcal{X}^{+}_{R}$, while $\hat{\psi}^{-}_{a}\mathcal{X}^{+}_{R} \subseteq \mathcal{X}^{+}_{R}\hat{\psi}^{-}_{a} + \mathcal{X}^{+}_{R}$ according to the first relation in Eq. (6), so $\hat{\psi}^{-}_{a}V_0 \subseteq V_0$. Therefore, V_0 is a subrepresentation of V. Furthermore, $|0'\rangle \notin V_0$ since the only state in V_0 annihilated by $\hat{n}$ is $|0\rangle$. Therefore, V_0 is a proper subrepresentation of V, contradicting the irreducibility of V.

B. The state space generated by $|0\rangle$ and $\{\hat{\psi}^{+}_{a}\}$

The algebra $\mathcal{X}_R$ is the special case of the quantum Weyl algebras (QWAs) $A_m(R)$ studied in Ref. [3] with $q = 1$, by identifying x_a with $\hat{\psi}^{+}_{a}$ and ∂_a with $\hat{\psi}^{-}_{a}$, and Thm. 1.5 in Ref. [3] provides the rigorous mathematical foundation for the construction of state space:

Theorem 2. *(Thm. 1.5 in Ref. [3]) There is a vector space isomorphism $\mathbb{C}_R\langle\hat{\psi}^{+}_{a}\rangle \otimes \mathbb{C}_R\langle\hat{\psi}^{-}_{a}\rangle \cong \mathcal{X}_R$, where $\mathbb{C}_R\langle\hat{\psi}^{+}_{a}\rangle$ is the unital associative algebra generated by $\{\hat{\psi}^{+}_{b}|1 \le b \le m\}$, subject to the second relations in Eq. (6), and similarly $\mathbb{C}_R\langle\hat{\psi}^{-}_{a}\rangle$ is the unital associative algebra generated by $\{\hat{\psi}^{-}_{b}|1 \le b \le m\}$, subject to the third relations in Eq. (6).*

This theorem extends the simpler fact that, as a vector space, $\mathcal{X}_R$ is spanned by $\mathcal{X}^{+}_{R}\otimes\mathcal{X}^{-}_{R}$, since for any monomial of $\hat{\psi}^{+}_{1}, \dots, \hat{\psi}^{+}_{m}, \hat{\psi}^{-}_{1}, \dots, \hat{\psi}^{-}_{m}$ in $\mathcal{X}_R$ (e.g. $\hat{\psi}^{-}_{a}\hat{\psi}^{+}_{b}\hat{\psi}^{-}_{c}\hat{\psi}^{+}_{d}$), one can always use the first relation in Eq. (6) to "normal order" all $\hat{\psi}^{+}_{a}$s to the left and $\hat{\psi}^{-}_{a}$s to the right, leading to a sum of terms, each with at most m creation and m annihilation operators. The non-trivial aspect of this theorem is that $\mathcal{X}^{+}_{R} \cong \mathbb{C}_R\langle\hat{\psi}^{+}_{a}\rangle$, and $\mathcal{X}^{-}_{R} \cong \mathbb{C}_R\langle\hat{\psi}^{-}_{a}\rangle$, i.e. the relations in the first and third lines of Eq. (6) do not imply any additional relations on the $\hat{\psi}^{+}_{a}$s other than the second line in Eq. (6). See Ref. [3] for a detailed proof.

We now construct the state space as the representation space of $\mathcal{X}_R$, defined as the canonical left $\mathcal{X}_R$-module $\mathfrak{V} = \mathcal{X}_R/[\sum_a \mathcal{X}_R\hat{\psi}^{-}_{a}]$ (i.e. the left $\mathcal{X}_R$-module generated by a vacuum state $|0\rangle$ satisfying the relation $\hat{\psi}^{-}_{a}|0\rangle = 0$ for all a). Then Thm. 2 immediately implies that (see the comment at the end of Sec. 1 in Ref. [3]), as a vector space, $\mathfrak{V} = \mathcal{X}_R|0\rangle = \mathcal{X}_R^+|0\rangle \cong \mathbb{C}_R\langle\hat{\psi}_a^+\rangle$. Furthermore, Thm. 3.2 of Ref. [3] proves that the representation $\mathfrak{V}$ of $\mathcal{X}_R$ is irreducible, and our discussion in Sec. S2 A implies that it is the only irrep of $\mathcal{X}_R$ with the spectrum of n bounded from below.

In the following we find a basis for the state space $\mathfrak{V} \equiv \mathcal{X}_R^+|0\rangle$. We use the eigenvalues of the particle number operator $\hat{n}$ to decompose $\mathfrak{V}$ into a direct sum of eigenspaces of $\hat{n}$: $\mathfrak{V} = \bigoplus_{n\geq 0}\mathfrak{V}_n$. Each subspace $\mathfrak{V}_n$ is spanned by states with fixed particle number

$$\mathfrak{V}_n = \mathrm{span}\{\hat{\psi}_{a_1}^+\hat{\psi}_{a_2}^+\dots\hat{\psi}_{a_n}^+|0\rangle | 1\leq a_j\leq m, j=1,2,\dots,n\}. \tag{S4}$$

Notice, however, due to the CR in Eq. (6), the states defined in the RHS of Eq. (S4) are linearly dependent. For example, the state $\hat{\psi}_a^+\hat{\psi}_b^+|0\rangle$ is the same as $\sum_{c,d} R_{ab}^{cd}\hat{\psi}_c^+\hat{\psi}_d^+|0\rangle$. A linearly independent basis for $\mathfrak{V}_n$ is established by the following theorem:

Theorem 3. *The states $\{|n,\alpha\rangle\}_{\alpha=1}^{d_n}$ defined by Eqs. (10-12) (for the case $N=1$) form a complete, linearly independent basis for $\mathfrak{V}_n$.*

We now sketch the proof of Thm. 3. Following our discussion in the previous paragraph, the n-particle space $\mathfrak{V}_n$ of a single mode [defined in Eq. (S4)] can be identified with $\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle$, the subspace of $\mathbb{C}_R\langle\hat{\psi}_a^+\rangle$ spanned by all degree n monomials in $\hat{\psi}_a^+$s. So it remains to be proven that $\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle$ is isomorphic (as a vector space) to the space of solutions $\Psi_{a_1\dots a_n}$ to Eq. (10). For convenience, we define a product vector space $\mathfrak{A}\equiv\mathfrak{a}^{\otimes n}$, where $\mathfrak{a}$ is an m-dimensional vector space with basis $\{v_1,v_2,\dots,v_m\}$. The tensor R_{cd}^{ab} defines a linear map R in the product space $\mathfrak{a}\otimes\mathfrak{a}$ as $R(v_c\otimes v_d)=\sum_{ab}R_{cd}^{ab}v_a\otimes v_b$, and this action is extended to $\mathfrak{a}^{\otimes n}$ as

$$R_{j,j+1} = \underset{(1)}{\mathbb{1}}\otimes\dots\otimes\underset{(j-1)}{\mathbb{1}}\otimes\underset{(j,j+1)}{R}\otimes\underset{(j+2)}{\mathbb{1}}\otimes\dots\otimes\underset{(n)}{\mathbb{1}}. \tag{S5}$$

Furthermore, we can associate a tensor $\Psi_{a_1\dots a_n}$ to a vector in $\mathfrak{a}^{\otimes n}$ through $\Psi=\sum_{a_1\dots a_n}\Psi_{a_1\dots a_n}v_{a_1}\otimes v_{a_2}\otimes\dots\otimes v_{a_n}$. Then Eq. (10) is equivalent to

$$R_{j,j+1}\Psi = \Psi\ (\text{in } \mathfrak{a}^{\otimes n}),\ j=1,2,\dots,n-1. \tag{S6}$$

In short, we need to prove that $\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle$ is isomorphic (as a vector space) to the common eigenspace (with eigenvalue $+1$) of all $R_{j,j+1}$. We have, as a vector space,

$$\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle \cong \frac{\mathfrak{a}^{\otimes n}}{\bigoplus_{j=1}^{n-1}(\mathbb{1}-R_{j,j+1})\mathfrak{a}^{\otimes n}}, \tag{S7}$$

since $\mathbb{C}_R\langle\hat{\psi}_a^+\rangle$ is, by definition, isomorphic to the quotient of the tensor algebra $T(\mathfrak{a})$ over the quadratic relations $R(\mathfrak{a}\otimes\mathfrak{a})=(\mathfrak{a}\otimes\mathfrak{a})$ [4], where $\mathfrak{a}$ is an m-dimensional vector space. We now prove the following lemma:

Lemma 4. *Let $R_1,R_2,\dots,R_k$ be Hermitian matrices satisfying $R_j^2=\mathbb{1}, j=1,2,\dots,k$, acting on a vector space V. Then*

$$\mathrm{span}\{|\psi\rangle\in V \mid R_j|\psi\rangle=|\psi\rangle, 1\leq j\leq k\} \cong \frac{V}{\bigoplus_{j=1}^{k}(\mathbb{1}-R_j)V}. \tag{S8}$$

Proof. For a Hilbert space $\mathcal{H}$, and a subspace $\mathcal{H}_1\subseteq\mathcal{H}$, the quotient space $\mathcal{H}/\mathcal{H}_1$ is isomorphic to the orthogonal complement $\mathcal{H}_1^\perp$. Therefore we have

$$\begin{aligned}\frac{V}{\bigoplus_{j=1}^{k}(\mathbb{1}-R_j)V} &= [\bigoplus_{j=1}^{k}(\mathbb{1}-R_j)V]^\perp \\ &= \bigcap_{j=1}^{k}[(\mathbb{1}-R_j)V]^\perp \\ &= \bigcap_{j=1}^{k}[(\mathbb{1}+R_j)V],\end{aligned} \tag{S9}$$

where in the second line we used $(V_1\oplus V_2)^\perp = V_1^\perp\cap V_2^\perp$, and in the third line we used the fact that R_j is Hermitian and all its eigenvalues are ± 1. Note that $(\mathbb{1}+R_j)/2$ is the projector to the eigenspace of R_j with eigenvalue $+1$, therefore, the last line of Eq. (S9) is the same as the LHS of Eq. (S8). □

Although the R-matrices in our models are not always Hermitian, there always exists a Hermitian inner product on the space $\mathfrak{A}=\mathfrak{a}^{\otimes n}$ with respect to which the matrices $\{R_{j,j+1}\}_{j=1}^{n-1}$ are all Hermitian. This is because any finite-dimensional representation of a finite group is isomorphic to a unitary representation (Theorem 4.6.2 in Ref. [5]), and in our case the matrices $\{R_{j,j+1}\}_{j=1}^{n-1}$ generate the finite group S_n (notice that if $R_{j,j+1}$ is unitary then it is Hermitian since $R_{j,j+1}^2=\mathbb{1}$). Therefore, Lemma 4 still applies, implying that the RHS of Eq. (S7) is isomorphic to the common eigenspaces of $\{R_{j,j+1}\}_{j=1}^{n-1}$ defined by Eq. (10). This concludes the proof of Thm. 3.

C. Many particle state space

We now prove that the states defined in Eqs. (10-12) form a linearly independent basis for the many particle state space $\mathcal{X}_{R,N}^+|0\rangle$ for any positive integer N. We need the following lemma:

Lemma 5. *There is a vector space isomorphism $\mathcal{X}_{R,N}\cong\mathcal{X}_R^{\otimes N}$. In particular, $\mathcal{X}_R$ is isomorphic to the subalgebra of $\mathcal{X}_{R,N}$ generated by $\{\hat{\psi}_{i,a}^\pm|1\leq a\leq m\}$, for any $a\in\{1,2,\dots,N\}$.*

Proof. By Thm. 2, we have (as vector spaces) $\mathcal{X}_R\cong\mathbb{C}_R\langle\hat{\psi}_a^+\rangle\otimes\mathbb{C}_R\langle\hat{\psi}_a^-\rangle$, and $\mathcal{X}_{R,N}\cong\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^+\rangle\otimes\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^-\rangle$, so we only need to prove that (as a vector space) $\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^+\rangle\cong\mathbb{C}_R\langle\hat{\psi}_a^+\rangle^{\otimes N}$. This can be proven by induction on N, where the induction step $N\to N+1$ can be proven in a similar way as Thm. 2. Alternatively, it is straightforward to show that $h_{\Pi\boxtimes R}(x) = h_R(x)^N$, and since for every R-matrix, $\dim \mathbb{C}_R\langle\hat{\psi}^+_a\rangle = h_R(1)$, we have $\dim \mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}^+_a\rangle = h_R(1)^N = \dim \mathbb{C}_R\langle\hat{\psi}^+_a\rangle^{\otimes N}$, so as a vector space, $\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}^+_a\rangle \cong \mathbb{C}_R\langle\hat{\psi}^+_a\rangle^{\otimes N}$. $\square$

While Lemma 5 seems natural, the non-trivial part is that the CRs (6) involving any other modes $\hat{\psi}^\pm_{j,a}$ (with $j \neq i$) do not give rise to any additional algebraic relations on $\{\hat{\psi}^\pm_{i,a} | 1 \le a \le m\}$. This is a rigorous justification that different modes are mutually independent. Lemma 5 along with Thm. 3 immediately imply that the states defined in Eqs. (10-12) form a linearly independent basis for $\mathcal{X}^+_{R,N}|0\rangle$.

D. The relation between the second and the first quantization formulation

We now show the relation between the second quantized formulation of parastatistics and the first quantized wavefunction formulation. To this end we first show the relation between the R-matrix R^{ab}_{cd} and the coefficients $(R_j)^I_J$ appearing in Eq. (3). Let the index I (and similarly for J) be a collection of n auxiliary indices $I = (a_1, a_2, \ldots, a_n)$ labeling the basis states of a product vector space $\mathfrak{A} \equiv \mathfrak{a}^{\otimes n}$ (the internal space of wavefunctions), where the basis of $\mathfrak{a}$ is $\{v_1, v_2, \ldots, v_m\}$. Now let $(R_j)^I_J$ be the matrix element of the linear mapping $R_{j,j+1}$ defined in Eq. (S5). With this choice of R_j, Eq. (3) becomes (take $n = 3$ and $j = 1$ for example)

$$\Psi^{a_1a_2a_3}(x_2, x_1, x_3) = \sum_{b_1,b_2} R^{a_1a_2}_{b_1b_2}\Psi^{b_1b_2a_3}(x_1, x_2, x_3). \tag{S10}$$

Then all the relations in Eq. (4) reduce to Eq. (5). An isomorphism between the space of n-particle wavefunctions in the first quantization formulation and the subspace of n-particle states in the second quantization formulation is defined as follows: each n-particle wavefunction $\Psi^I(x_1, \ldots, x_n)$ satisfying Eq. (3) [with the R-matrix in Eq. (S5)] corresponds to the n-particle state

$$|\Psi\rangle = \sum_{I,x_1,\ldots,x_n} \Psi^I(x_1, \ldots, x_n)\hat{\psi}^+_{x_1,a_1} \ldots \hat{\psi}^+_{x_n,a_n}|0\rangle, \tag{S11}$$

That Eq. (S11) indeed defines an isomorphism between the two vector spaces can be seen as follows. Note that Eq. (10) and Eq. (11) (for the case $N = 1$) with the R-matrix $\Pi\boxtimes R$ is the same as Eq. (3) and Eq. (S11) with the R-matrix R, respectively. Then Thm. 3 applied to the algebra $\mathcal{X}_{\Pi\boxtimes R} \cong \mathcal{X}_{R,N}$ shows that a linearly independent basis for the space of n-particle wavefunctions satisfying Eq. (3) correspond to a linearly independent basis for the n-particle subspace $\mathfrak{V}_n$ of the Fock space $\mathcal{X}^+_{\Pi\boxtimes R}|0\rangle \cong \mathcal{X}^+_{R,N}|0\rangle$ via the relation Eq. (S11), thereby establishing the isomorphism.

S3. DETAILS ON THE 1D SPIN MODEL AND THE MPO JWT

In this section we provide mathematical details on the 1D spin model defined in Eqs. (17-19).

A. Model definition for an arbitrary R-matrix

We first define the local spin operators $\{\hat{x}^\pm_{i,a}, \hat{y}^\pm_{i,a}\}^m_{a=1}$ that appears in the Hamiltonian in Eq. (17), for any given R-matrix. They are constructed to satisfy the following algebraic relations (we omit the site label i since they all act locally on the same site)

$$\begin{aligned}
\hat{y}^-_a\hat{y}^+_b &= \sum_{c,d} R^{ac}_{bd}\hat{y}^+_c\hat{y}^-_d + \delta_{ab} \\
\hat{y}^+_a\hat{y}^+_b &= \sum_{c,d} R^{cd}_{ab}\hat{y}^+_c\hat{y}^+_d \\
\hat{y}^-_a\hat{y}^-_b &= \sum_{c,d} R^{ba}_{dc}\hat{y}^-_c\hat{y}^-_d \\
\hat{x}^-_a\hat{x}^+_b &= \sum_{c,d} R^{ca}_{db}\hat{x}^+_c\hat{x}^-_d + \delta_{ab} \\
\hat{x}^+_a\hat{x}^+_b &= \sum_{c,d} R^{dc}_{ba}\hat{x}^+_c\hat{x}^+_d \\
\hat{x}^-_a\hat{x}^-_b &= \sum_{c,d} R^{ab}_{cd}\hat{x}^-_c\hat{x}^-_d \\
[\hat{x}^+_a, \hat{y}^+_b] &= [\hat{x}^-_a, \hat{y}^-_b] = 0.
\end{aligned} \tag{S12}$$

While these CRs superficially resemble the CRs between paraparticle operators in Eq. (6), the difference is that the spin operators here are strictly local in that they commute on different sites, and therefore are in principle realizable, while the paraparticle operators are generally non-local operators. These CRs are shown graphically in Fig. S2.

We now define a local Hilbert space and a matrix representation of these local spin operators. Notice that the first three CRs in Eq. (S12) are exactly the same as in Eq. (6) for $N = 1$, therefore we can take $\mathfrak{V} = \mathcal{X}^+_R|0\rangle \cong \mathbb{C}_R\langle\hat{\psi}^+_a\rangle$ to be the local Hilbert space of the spin model [6], where the action of $\{\hat{y}^\pm_a\}^m_{a=1}$ is defined by $\hat{y}^\pm_a|\Psi\rangle = \hat{\psi}^\pm_a|\Psi\rangle$, $\forall|\Psi\rangle \in \mathfrak{V}$. The action of $\{\hat{x}^+_a\}^m_{a=1}$ on $\mathfrak{V}$ is defined by $\hat{x}^+_a(\hat{\psi}^+_{b_1}\hat{\psi}^+_{b_2}\ldots\hat{\psi}^+_{b_n}|0\rangle) = \hat{\psi}^+_{b_1}\hat{\psi}^+_{b_2}\ldots\hat{\psi}^+_{b_n}\hat{\psi}^+_a|0\rangle \in \mathcal{X}^+_R|0\rangle$, for any $\hat{\psi}^+_{b_1}\hat{\psi}^+_{b_2}\ldots\hat{\psi}^+_{b_n}|0\rangle \in \mathcal{X}^+_R|0\rangle$. The action of $\{\hat{x}^-_a\}^m_{a=1}$ on $\mathfrak{V}$ is defined by $\hat{x}^-_a(\hat{\psi}^+_{b_1}\hat{\psi}^+_{b_2}\ldots\hat{\psi}^+_{b_n}|0\rangle) = \hat{x}^-_a\hat{x}^+_{b_n}\hat{x}^+_{b_{n-1}}\ldots\hat{x}^+_{b_1}|0\rangle$, where it is understood that the RHS is simplified by moving $\hat{x}^-_a$ all the way to the right using the fourth relation in Eq. (S12); for example, $\hat{x}^-_a(\hat{\psi}^+_b\hat{\psi}^+_c|0\rangle) = \hat{x}^-_a\hat{x}^+_c\hat{x}^+_b|0\rangle = (\delta_{ac}\hat{x}^+_b + \sum_d R^{da}_{bc}\hat{x}^+_d)|0\rangle = (\delta_{ac}\hat{\psi}^+_b + \sum_d R^{da}_{bc}\hat{\psi}^+_d)|0\rangle \in \mathcal{X}^+_R|0\rangle$. It is straightforward to check that the action of $\{\hat{x}^\pm_a, \hat{y}^\pm_a\}^m_{a=1}$ defined this way satisfy all the relations in Eq. (S12).

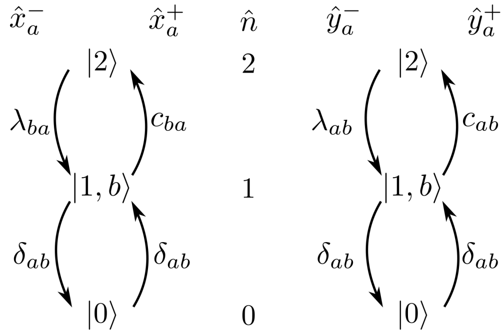

FIG. S1. The action of the operators $\hat{x}_a^\pm, \hat{y}_a^\pm$ and $\hat{n}$ on the basis states $|0\rangle$, $\{|1,b\rangle\}_{b=1}^m$, $|2\rangle$, for the spin model corresponding to the R-matrix in Ex. 4.

Furthermore, we note that this definition implies the additional CRs

$$\sum_a \hat{x}_a^+ \hat{x}_a^- = \sum_a \hat{y}_a^+ \hat{y}_a^- \equiv \hat{n},$$
$$[\hat{n}, \hat{x}_a^\pm] = \pm \hat{x}_a^\pm, \quad [\hat{n}, \hat{y}_a^\pm] = \pm \hat{y}_a^\pm. \tag{S13}$$

The Hilbert space of the whole system with N sites in total is $\mathfrak{V}^{\otimes N}$, and $\{\hat{x}_{i,a}^\pm, \hat{y}_{i,a}^\pm\}_{a=1}^m$ act locally on the i-th factor space as described above.

We can give the matrix elements of $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ more explicitly for the R-matrices given in Tab. I. One finds that for the R-matrices in Ex. 1 and Ex. 2, the corresponding spin models are simply m decoupled chains of 1D XY models (up to an on-site unitary transformation). The spin model for the R-matrix in Ex. 3 has been presented in the main text. For the R-matrix in Ex. 4, the local Hilbert space $\mathfrak{V}$ is $m+2$-dimensional, with basis states $|0\rangle, \{|1,b\rangle\}_{b=1}^m, |2\rangle$; the non-zero matrix elements of $\hat{y}_a^\pm$ are $\hat{y}_a^+|0\rangle = |1,a\rangle$, $\hat{y}_a^-|1,b\rangle = \delta_{ab}|0\rangle$, $\hat{y}_a^+|1,b\rangle = c_{ab}|2\rangle$, and $\hat{y}_a^-|2\rangle = \sum_b \lambda_{ab}|1,b\rangle$; and the non-zero matrix elements of $\hat{x}_a^\pm$ are $\hat{x}_a^+|0\rangle = |1,a\rangle$, $\hat{x}_a^-|1,b\rangle = \delta_{ab}|0\rangle$, $\hat{x}_a^+|1,b\rangle = c_{ba}|2\rangle$, and $\hat{x}_a^-|2\rangle = \sum_b \lambda_{ba}|1,b\rangle$. The action of the operators $x_i^\pm, \hat{y}_i^\pm$ and n on the orthonormal basis states are shown in Fig. S1.

The spin model Hamiltonian in Eq. (17) is not Hermitian for the R-matrix in Ex. 4 for $m \geq 3$ [7], since $\hat{x}_a^+ \neq (\hat{x}_a^-)^\dagger, \hat{y}_a^+ \neq (\hat{y}_a^-)^\dagger$. However, $\hat{H}$ is parity-time symmetric [8, 9], and therefore has real eigenvalues and generates unitary time evolution. To be precise, let P be the parity operator that generates the chain reflection symmetry, and let T be the time-reversal symmetry, which, in our spin model, is simply complex conjugation. Using the explicit representation of the matrices λ_{ab}, c_{ab} in Tab. I and footnote [10], we see that $\lambda_{ab}^* = \lambda_{ba}, c_{ab}^* = c_{ba}$, and therefore, the time-reversal operation T simply swaps the operators $\hat{x}_a^\pm \leftrightarrow \hat{y}_a^\pm$ in the Hamiltonian $\hat{H}$, which can subsequently be undone by P. Thus, $\hat{H}$ is invariant under the combined operation PT, and all eigenvalues of $\hat{H}$ are real (as already seen from the exact solution of the spectrum).

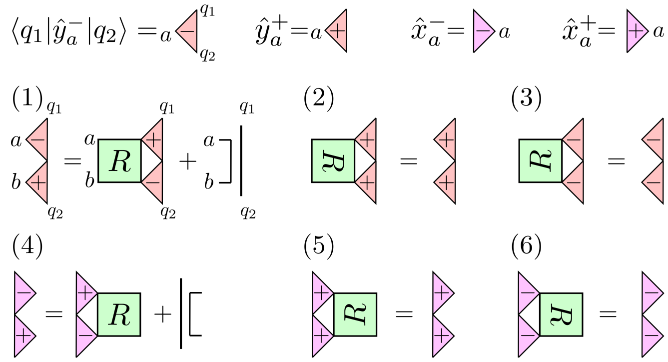

FIG. S2. Graphical representation of the CRs between the local spin operators $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ in Eq. (S12). The matrix elements of each operator $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ is a tensor (represented by the triangles) with two quantum indices (e.g. the indices q_1 and q_2 shown in figure) and one auxiliary index (e.g. the index a), and the R-matrix (represented by a square) is a tensor with four auxiliary indices. Matrix multiplication goes from top to bottom in the quantum space and from left to right in the auxiliary space.

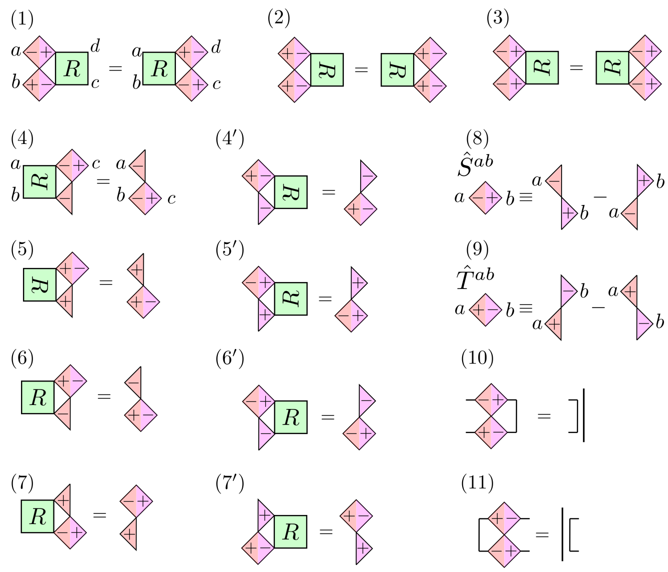

FIG. S3. Graphical representation of the CRs between the local spin operators $\hat{S}^{ab}, \hat{T}^{ab}$ and $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$.

B. Generalized Jordan-Wigner transformations

We now prove that the emergent paraparticles creation and annihilation operators defined by the MPO JWT in Eq. (18) do satisfy the parastatistical CRs in Eq. (6), and the spin Hamiltonian in Eq. (17) is mapped to the free paraparticle Hamiltonian in Eq. (19). An important first step is to prove the algebraic relations between the local spin operators $\hat{S}^{ab}, \hat{T}^{ab}$ and $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ as shown graphically in Fig. S3. For the R-matrices given in Tab. I, all these relations can be explicitly checked by hand, but to prove them for an arbitrary R-matrix, we need to use some techniques of representation the-

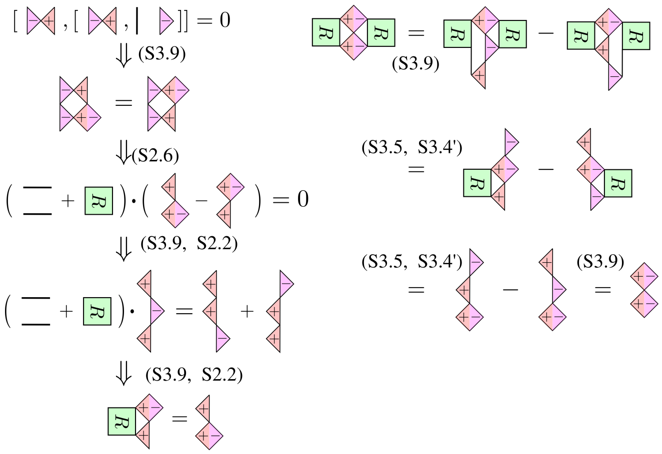

FIG. S4. The graphical proof of relations (S2.2) (right panel) and (S2.5) (left panel) in Fig. S3.

ory. Here we show the proof of relations (2) and (5) as examples, while others can be proven in a similar way. First, it is straightforward to verify that the operators $\hat{S}^+ = \sum_a \hat{x}^+_{1a}\hat{y}^-_{2a}, \hat{S}^- = \sum_a \hat{x}^-_{1a}\hat{y}^+_{2a}, \hat{S}^z = (\hat{n}_1 - \hat{n}_2)/2$ form a representation of the $\mathfrak{sl}_2$ Lie algebra. This can be checked by direct computation using Eqs. (S12,S13). In addition, we have $[\hat{S}^+, \hat{x}^-_{2a}] = 0$ and $[\hat{S}^z, \hat{x}^-_{2a}] = \hat{x}^-_{2a}/2$, so under the adjoint action of this $\mathfrak{sl}_2$, $\hat{x}^-_{2a}$ is a highest weight state with $S^z = 1/2$. The only finite dimensional representation of $\mathfrak{sl}_2$ corresponding to this highest weight state is the spin-1/2 representation, in which we have $[\hat{S}^-, [\hat{S}^-, \hat{x}^-_{2a}]] = 0$. Relation (5) in Fig. S3 can be proven by expanding this equation and using the fifth relation in Eq. (S12), as shown in the left panel of Fig. S4. Then relation (2) can be proven from relation (5) as shown in the right panel of Fig. S4, using tensor graphical manipulations.

With all those relations shown in Fig. S3, Eq. (6) can be proven. For example, in Fig. S5 we show the proof of the first parastatistical commutation relation in Eq. (6) for the $\hat{\psi}^{\pm}_{i,a}$ defined in terms of the spin operators in Eq. (18) and the algebraic relations in Fig. S3. Other relations in Eq. (6) are proven in a similar way. Furthermore, one can insert Eq. (18) into Eq. (19) to reproduce Eq. (17), using the last two relations in Fig. S3 and a similar graphical manipulation as in Fig. S5. This proves the exact mapping from the 1D spin model to free paraparticles.

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[6] Here we are using $\mathcal{X}_R$ (generated by $\{\hat{\psi}^{\pm}_a\}^m_{a=1}$) and $\mathcal{X}^+_R$ (generated by $\{\hat{\psi}^+_a\}^m_{a=1}$) as auxiliary tools to construct the local Hilbert space. One should not confuse the $\hat{\psi}^{\pm}_a$ here (which is defined only on the local Hilbert space) with the paraparticle creation/annihilation operators we obtain from the MPO JWT in Eq. (18), which are non-local operators acting on the global Hilbert space.
[7] The model is still well-defined for $m = 2$, where $\hat{H}$ is Hermitian. But that case is trivial: when $m = 2$, $\hat{H}$ is equal to the sum of two decoupled chains of XY models.
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[10] For example, we can take $\lambda = e^{-M}, c = -e^{M}$, where M is an $m \times m$ antisymmetric matrix, $M^T = -M$, with complex entries satisfying $\mathrm{Tr}[e^{-2M}] = -2$ (one can get an explicit solution using a block-diagonal ansatz for M with maximum block size 2).

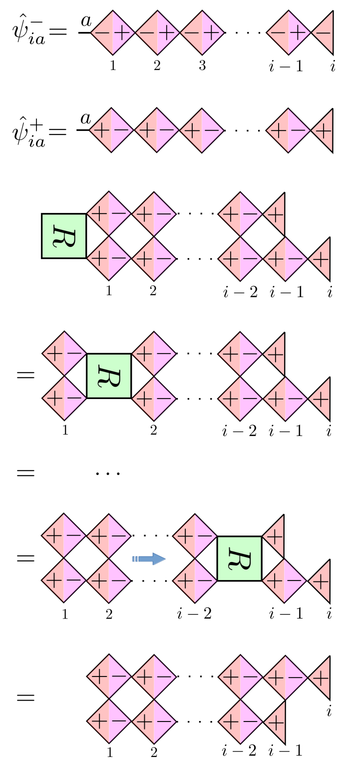

FIG. S5. The graphical proof of the first relation in Eq. (6), using the definition of $\hat{\psi}^{\pm}_{i,a}$ in Eq. (18) and the algebraic relations in Fig. S3.