Single Photon Quantum Ranging: When Sequential Decoding Meets High Dimensional Entanglement

Armanpreet Pannu[1], Han Liu[1], Amr S. Helmy[1], and Hesham El Gamal[2]

[1]The Edward S. Rogers Department of Electrical and Computer Engineering, University of Toronto
[2]School of Electrical and Information Engineering, Faculty of Engineering, The University of Sydney

1 Problem Statement and Modeling

Suppose we have a target at unknown distance in discrete bins $x = 0, ..., D - 1$. The goal is to determine the value of x by probing the target with a beam of light in presence of a bright background noise. We will be using states which lives in $M + D$ possible time-bin modes. This Hilbert space is spanned by the states $|n_1, ...n_{M+D}\rangle$ which is a state with n_i photons in the i-th time-bin.

We assume that noise is low and therefore no more than one photon per detection event. In this case the thermal background can be approximated as:

$$\rho_B = (1 - NM) |vac\rangle \langle vac| + N \sum_{i=1}^{M} |\mathbf{e}_i\rangle \langle \mathbf{e}_i| \tag{1}$$

In the case of a signal photon reflected off the target, we assume that it doesn't interact with the noise since the noise is sparse enough to almost never overlap with a signal photon.

2 Time-Bin Quantum Scheme

2.1 Selecting the Entangled state

The proposed scheme uses the starting state:

$$|\psi_0\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |\mathbf{e}_i\rangle_I \otimes |\mathbf{e}_i\rangle_S \tag{2}$$

$$= \frac{1}{\sqrt{M}} |1, 0, ..., 0\rangle |1, 0, ..., 0\rangle + \frac{1}{\sqrt{M}} |0, 1, ..., 0\rangle |0, 1, ..., 0\rangle + \frac{1}{\sqrt{M}} |0, ..., 0, 1, 0, ..., 0\rangle |0, 0, ..., 0, 1, 0, ..., 0\rangle \tag{3}$$

Here $\mathbf{e}_i = (0, ..., 0, 1, 0, ..., 0)$ is the standard basis vector for $\mathbb{R}^M$ with 1 as the i-th element and 0 elsewhere. The state $|\psi_0\rangle$ is, therefore, a superposition of states with one photon in the signal and one in the idler mode entangled such that both photons can only be measured in the **same** i-th mode (for their respective spatial mode) with uniform phase across the bins as captured by equation (2). Furthermore, we only entangle the starting state across the first M time-bin modes to ensure no information is lost regardless of the delay from the target (as long as it's in the range $0 \leq x \leq D$)

The idler modes are stored locally while the signal mode is sent to the target. In the case the target is present, the back reflected light is collected along with any environment noise in the signal modes and a joint measurement is performed on the idler and signal modes.

2.2 Describing the evolution of the signal modes

The signal beam evolution can be described by two channels acting sequentially: a pure loss channel as a result of the potential absorption of the signal photon and a delay channel which shifts the time-bin modes depending on the distance of the target.

The evolution of the loss channel can be described by a simple beam-splitter with the auxiliary mode traced out which in the case of a single photon input in one port and vaccum in the other results in an output which is a mixture of the signal and the vaccum state.

The latter target delay channel for a delay of x can be represented using the operator:

$$\hat{D}_x = \sum_{\mathbf{n}} |0, 0, ..., n_1, ...n_{M+D-x}\rangle \langle n_1, ..., n_{M+D}| \tag{4}$$

In this case, if the target is at a distance $0 \leq x \leq D$, a signal state $|n_1, ..., n_{M+D}\rangle$ is returned as $= |0, 0, ..., n_1, ...n_{M+D-x}\rangle$. In the case that the target falls outside the maximum range of the $M + D$ time bins (i.e $x \geq M + D$), we simply get the vacuum state back. Intuitively this is because our detection only occurs in the $M + D$ time-bins so if the delay is too long, the signal light will not return in time before the detector has completed the measurement.

In total the state that is collected is:

$$\begin{aligned}
\rho_x =& \eta \hat{D}_x |\psi_0\rangle \langle\psi_0| \hat{D}_x^\dagger + (1-\eta)\Big(\text{Tr}_S[|\psi_0\rangle \langle\psi_0|]\Big) \otimes \rho_B & (5)\\
=& \eta \hat{D}_x |\psi_0\rangle \langle\psi_0| \hat{D}_x^\dagger + \frac{1}{M}(1-\eta)N \sum_{i,j=1}^{M,M+D} |\mathbf{e}_i, \mathbf{e}_j\rangle \langle \mathbf{e}_i, \mathbf{e}_j| + \frac{1}{M}(1-\eta)(1-N(M+D)) \sum_{i=1}^{M+D} |\mathbf{e}_i, vac\rangle \langle \mathbf{e}_i, vac| & (6)
\end{aligned}$$

where $|\psi_0\rangle$ is our starting state. The three terms represent: returning signal photon, returning noise photon and returning vacuum.

2.3 Designing the Measurement

In the case of hypothesis x (Target located at a delay of x), we have that the signal and idler state after collection (assuming the signal photon is reflected) is:

$$\hat{D}_x |\psi_0\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |\mathbf{e}_i, \mathbf{e}_{i+x}\rangle \tag{7}$$

$$\Big[\hat{D}_1 |\psi_0\rangle\Big] \quad \Big[\hat{D}_2 |\psi_0\rangle\Big] \cdots \Big[\hat{D}_D |\psi_0\rangle\Big] \quad \Big[N/A\Big]$$

Note that for different x, these states are orthogonal:

$$\begin{aligned}
\langle\psi_0| \hat{D}_{x'}^\dagger \hat{D}_x |\psi_0\rangle &= \frac{1}{M} \sum_{i,j=1}^{M} \big\langle \mathbf{e}_j, \mathbf{e}_{j+x'} \big| \mathbf{e}_i, \mathbf{e}_{i+x} \big\rangle & (8)\\
&= \frac{1}{M} \sum_{i,j=1}^{M} \langle \mathbf{e}_j | \mathbf{e}_i \rangle_I \big\langle \mathbf{e}_{j+x'} \big| \mathbf{e}_{i+x} \big\rangle_S & (9)\\
&= \frac{1}{M} \sum_{i}^{M} \langle \mathbf{e}_{i+x'} | \mathbf{e}_{i+x} \rangle_S & (10)\\
&= \delta_{x,x'} & (11)
\end{aligned}$$

Since we want to distinguish between the states corresponding to different x, its natural to consider the POVM: $\{M_1, M_2, ..., M_D, id - \sum_x \frac{1}{D} M_x\}$ where each measurement is a projection onto the the **expected** returning state:

$$M_x = \hat{D}_x |\psi_0\rangle \langle\psi_0| \hat{D}_x^\dagger \tag{12}$$

$$= \frac{1}{M} \sum_{i,j=1}^{M} |\mathbf{e}_i, \mathbf{e}_{i+x}\rangle \langle\mathbf{e}_j, \mathbf{e}_{j+x}| \tag{13}$$

2.4 Error Rate

We fist calculate the probabilities **for a single illumination event**:

$$P[x|x'] = Tr[M_x \rho_{x'}] \tag{14}$$

$$= Tr\Bigg[M_x \Bigg(\eta \hat{D}_x |\psi_0\rangle \langle\psi_0| \hat{D}_x^\dagger + \frac{1}{M}(1-\eta)N \sum_{i,j=1}^{M,M+D} |\mathbf{e}_i, \mathbf{e}_j\rangle \langle\mathbf{e}_i, \mathbf{e}_j| \tag{15}$$

$$+ \frac{1}{M}(1-\eta)(1-N(M+D)) \sum_{i=1}^{M+D} |\mathbf{e}_i, vac\rangle \langle\mathbf{e}_i, vac| \Bigg)\Bigg]$$

$$= \eta\delta_{x,x'} + (1-\eta)NTr\left[\frac{1}{M^2} \sum_{i,h=1}^{M} |\mathbf{e}_i, \mathbf{e}_{i+x}\rangle \langle\mathbf{e}_j, \mathbf{e}_{j+x}| \sum_{k,l=1}^{M,M+D} |\mathbf{e}_k, \mathbf{e}_l\rangle \langle\mathbf{e}_k, \mathbf{e}_l|\right] + 0 \tag{16}$$

$$= \eta\delta_{x,x'} + (1-\eta)NTr\left[\frac{1}{M^2} \sum_{i,j,k,l=1}^{M,M,M,M+D} |\mathbf{e}_i, \mathbf{e}_{i+x}\rangle \langle\mathbf{e}_j, \mathbf{e}_{j+x}|\mathbf{e}_k, \mathbf{e}_l\rangle \langle\mathbf{e}_k, \mathbf{e}_l|\right] \tag{17}$$

$$= \eta\delta_{x,x'} + (1-\eta)N\frac{1}{M^2} \sum_{i,j,k,l=1}^{M,M,M,M+D} \langle\mathbf{e}_j, \mathbf{e}_{j+x}|\mathbf{e}_k, \mathbf{e}_l\rangle \langle\mathbf{e}_k, \mathbf{e}_l|\mathbf{e}_i, \mathbf{e}_{i+x}\rangle \tag{18}$$

$$= \eta\delta_{x,x'} + (1-\eta)N\frac{1}{M^2} \sum_{k,l=1}^{M,M+D} \langle\mathbf{e}_k, \mathbf{e}_{k+x}|\mathbf{e}_k, \mathbf{e}_l\rangle \langle\mathbf{e}_k, \mathbf{e}_l|\mathbf{e}_k, \mathbf{e}_{k+x}\rangle \tag{19}$$

$$= \eta\delta_{x,x'} + (1-\eta)N\frac{1}{M^2} \sum_{k,l=1}^{M,M+D} \langle\mathbf{e}_{k+x}|\mathbf{e}_l\rangle \langle\mathbf{e}_l|\mathbf{e}_{k+x}\rangle \tag{20}$$

$$= \eta\delta_{x,x'} + (1-\eta)\frac{N}{M} \tag{21}$$

Note this analysis assumes $M > D > x$ or else some of the delays may mean that we return a vacuum in the first M modes. In this case a vacuum can trick the projector with a probability 1.

Thus we have:

$$P[x|x'] = \begin{cases} \eta + (1-\eta)\frac{N}{M} & \text{if } x = x' \\ (1-\eta)\frac{N}{M} & \text{else} \end{cases} \tag{22}$$

3 Time-Bin Sequential Strategy

3.1 Single Click Error

We have to repeat the process of illumination and detection since the signal photon is absorbed with a non-zero probability. After each shot if the POVM returns measurement x (anything other than the last outcome) then we declare that the target is at position x and stop transmission. If the result is inconclusive, then we proceed to the next transmission until a **conclusive** measurement is made.

If the target is at a position x, the probability of a conclusive outcome (correct or incorrect is given by):

$$p_c = \sum_{x'=1}^{D} P[x|x'] \tag{23}$$

$$= \eta + (1-\eta)\frac{DN}{M} \tag{24}$$

Thus the probability of an incorrect result in a single click is given by:

$$P_e^{SC} = \sum_{n=1}^{\infty} (1-p_c)^{n-1} Pr[\text{incorrect measurement}] \tag{25}$$

$$= \frac{1}{p_c} Pr[\text{incorrect measurement}] \tag{26}$$

$$= (D-1)\frac{(1-\eta)\frac{N}{M}}{\eta + (1-\eta)\frac{DN}{M}} \tag{27}$$

The derivative of this expression is

$$\frac{\partial P_e^{SC}}{\partial M} = -\frac{(D-1)(1-\eta)\eta N_B}{(N_B D(1-\eta) + \eta M)^2} \tag{28}$$

which is negative for $2 \leq D$. Thus we can conclude that the probability of error decreases with increasing M. The average energy use to get a single click is:

$$N_{avg} = \sum_{n=1}^{\infty} nP[\text{conclusive after n}] \tag{29}$$

$$= \sum_{n=1}^{\infty} n(1-p_c)^{n-1} p_c \tag{30}$$

$$= \frac{1}{p_c} \tag{31}$$

$$= \frac{1}{\eta + (1-\eta)\frac{DN}{M}} \tag{32}$$

$$N_{avg} \leq \frac{1}{\eta} \tag{33}$$

The last inequality is important since its the average photons required to get one back reflected photon ignoring the possible clicks from noise photons.

3.2 Sequential Strategy

We now repeat the above strategy but instead of waiting for a single photon and making a decision, we wait for the detector to measure a photon in the same bin R times. Then the average count of signal photons required is:

$$N_{avg} \leq \sum_{n=R}^{\infty} n\binom{n-1}{R-1}\eta^R(1-\eta)^{n-R} \tag{34}$$

$$= \Big(\frac{\eta}{1-\eta}\Big)^R \sum_{n=R}^{\infty} n \binom{n-1}{R-1} (1-\eta)^n \tag{35}$$

$$= \frac{R}{\eta} \tag{36}$$

To simplify the expression, we assumed that a conclusive measurement can only occur from a signal and not a noise photon hence why we obtain an upper bound.

The probability of an incorrect measurement is the probability of a **particular** incorrect measurement multiplied by the number of incorrect measurement which is $(D-1)$. In the expression below, we show the computation assuming $x=0$ is correct and we are computing the probability of measuring $x = D-1$:

$$P_e^{TBS} = (D-1) \sum_{i_0, i_1, \ldots, i_{D-2}=0}^{R-1} (\text{multiplicity}) P[\text{measure } x = D-1,\ R \text{ times}] \prod_{j=0}^{D-2} P[\text{measure } x = j,\ i_j \text{ times}] \tag{37}$$

$$= (D-1) \sum_{i_0, i_1, \ldots, i_{D-2}=0}^{R-1} \binom{R + i_0 + \ldots + i_{D-2} - 1}{(R-1), i_0, \ldots, i_{D-2}} \Bigg(\frac{\eta}{\eta + \frac{(1-\eta)DN_B}{M}}\Bigg)^{i_0} \Bigg(\frac{(1-\eta)N_B}{\eta + \frac{(1-\eta)DN_B}{M}}\Bigg)^{R+i_1+\ldots+i_{D-2}} \tag{38}$$

This expression is difficult to analyze analytically and compute numerically for large D since the expression (38) has number of terms that scale exponentially with D.

We can make an approximation by assuming that η is extremely small so N_{avg} is large. The law of large numbers dictates that N_{avg} illumination events will take place before R clicks in the correct bin. The probability of an incorrect conclusion can be approximated by the probability of getting R clicks in an incorrect time-bin before N_{avg} transmissions multiplied by the number of incorrect time-bins. This can be approximated as:

$$P_e^{TBS} \approx \sum_{r=R}^{N_{avg}} (D-1) \binom{N_{avg}}{r} \Big(\frac{N_B}{M}\Big)^r \Big(1 - \frac{N_B}{M}\Big)^{N_{avg}-r} \tag{39}$$

$$\approx (D-1) \binom{N_{avg}}{R} \Big(\frac{N_B}{M}\Big)^R \Big(1 - \frac{N_B}{M}\Big)^{N_{avg}-R} \qquad \text{First term is dominant} \tag{40}$$

$$\approx (D-1) \frac{(N_{avg})^R}{R!} \Big(\frac{N_B}{M}\Big)^R \Big(1 - \frac{N_B}{M}\Big)^{N_{avg}-R} \qquad \text{Sterling's approximation: } \eta << 1 \Rightarrow R << N_{avg} \tag{41}$$

$$= (D-1) \frac{1}{R!} \Big(\frac{RN_B}{\eta M}\Big)^R \Big(1 - \frac{N_B}{M}\Big)^{\frac{R}{\eta}-R} \tag{42}$$

$$= (D-1) \frac{1}{R!} \Big(\frac{RN_B}{\eta M}\Big)^R \Big(\frac{M - N_B}{M}\Big)^{\frac{R}{\eta}-R} \tag{43}$$

$$= (D-1) \frac{1}{R!} \Big(\frac{RN_B}{\eta M}\Big)^R \Big(\frac{M - N_B}{M}\Big)^{-R} \Big(\frac{M - N_B}{M}\Big)^{\frac{R}{\eta}} \tag{44}$$

$$\approx (D-1) \frac{1}{R!} \Big(\frac{RN_B}{\eta M}\Big)^R \Big(1 - \frac{N_B}{M}\Big)^{\frac{R}{\eta}-R} \tag{45}$$

$$\approx (D-1) \exp\Bigg[-\log(R!) + R \log\Big(\frac{N_B R}{\eta M}\Big) - \Big(\frac{R}{\eta} - R\Big)\frac{N_B}{M} \Bigg] \tag{46}$$

$$\approx (D-1) \exp\Bigg[-R\log(R) + R + R \log\Big(\frac{N_B R}{\eta M}\Big) - \Big(\frac{R}{\eta} - R\Big)\frac{N_B}{M} \Bigg] \qquad \text{Sterling's approximation} \tag{47}$$

$$= (D-1) \exp\Bigg[R + R \log\Big(\frac{N_B}{\eta M}\Big) - \Big(\frac{R}{\eta} - R\Big)\frac{N_B}{M} \Bigg] \tag{48}$$

$$= (D-1) \exp\Bigg[R\Bigg(1 + \log\Big(\frac{N_B}{\eta M}\Big) - \frac{N_B}{M}\Big(\frac{1}{\eta} - 1\Big)\Bigg) \Bigg] \tag{49}$$

This last expression is valid under the assumption $\eta << 1$ and $R >> 1$ (large N_{avg}) and allows analytic comparison between different scheme. This last expression also allows us to numerically plot the result for large D. The accuracy of this approximation is demonstrated with a plot in the main paper.

4 Results and Comparison

The benchmarks used for comparison are: classical sequential, two-mode squeezed vaccum and block classical detection. All of these are outlined in the main paper and have the following probability of error when normalized for energy:

$$P_e^{TMSV} = (D-1)\exp\left[-\frac{R}{D(1+N_B)}\right] \tag{50}$$

$$P_e^{CB} = \frac{D-1}{2D}\exp\left[-\frac{2R}{1+2N_B}\right] \tag{51}$$

Let $T = (M+D)N_B$ The regime in which the time-bin sequential scheme outperforms the TMSV scheme can be approximated as:

$$P_e^{TBS} \leq P_e^{TMSV} \tag{52}$$

$$(D-1)\exp\left[R\left(1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right)\right)\right] \leq (D-1)\exp\left[-\frac{R}{D(1+N_B)}\right] \tag{53}$$

$$\exp\left[R\left(1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right)\right)\right] \leq \exp\left[-\frac{R}{D(1+N_B)}\right] \tag{54}$$

$$R\left(1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right)\right) \leq -\frac{R}{D(1+N_B)} \tag{55}$$

$$1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right) \leq -\frac{1}{D(1+N_B)} \tag{56}$$

$$1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right) \leq -\frac{1}{D} \qquad \text{Assume small } N_B \tag{57}$$

$$1+\log\left(\frac{N_B^2}{\eta(T-DN_B)}\right)-\frac{N_B^2}{T-DN_B}\left(\frac{1}{\eta}-1\right) \leq -\frac{1}{D} \qquad \text{Plug in } M \text{ in terms of } T \tag{58}$$

$$\log\left(\frac{N_B^2}{\eta(T-DN_B)}\right) \leq -\frac{1}{D} \tag{59}$$

$$\frac{N_B^2}{\eta(T-DN_B)} \leq \exp\left[\frac{1}{D}\right] \tag{60}$$

$$N_B^2 \leq \eta(T-DN_B)\exp\left[\frac{1}{D}\right] \tag{61}$$

$$N_B^2 \leq \eta \qquad \text{Large } D \tag{62}$$

The regime in which the time-bin sequential scheme outperforms the block classical:

$$P_e^{TBS} \leq P_e^{CB} \tag{63}$$

$$(D-1)\exp\left[R\left(1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right)\right)\right] \leq \frac{D-1}{2D}\exp\left[-\frac{2R}{1+2N_B}\right] \tag{64}$$

$$\exp\left[R\left(1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right)\right)\right] \leq \exp\left[-\frac{2R}{1+2N_B}-\log(2D)\right] \tag{65}$$

$$R\left(1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right)\right) \leq -\frac{2R}{1+2N_B}-\log(2D) \tag{66}$$

$$1+\log\left(\frac{N_B}{\eta M}\right)-\frac{N_B}{M}\left(\frac{1}{\eta}-1\right) \leq -\frac{2}{1+2N_B}-\frac{\log(2D)}{R} \tag{67}$$

$$1+\log\left(\frac{N_B^2}{\eta(T-DN_B)}\right)-\frac{N_B^2}{T-DN_B}\left(\frac{1}{\eta}-1\right) \leq -\frac{2}{1+2N_B}-\frac{\log(2D)}{R} \tag{68}$$

$$\log\left(\frac{N_B^2}{\eta(T-DN_B)}\right)-\frac{N_B^2}{\eta(T-DN_B)} \leq -2-\frac{\log(2D)}{R} \tag{69}$$

$$\log\left(\frac{N_B^2}{\eta(T-DN_B)}\right) \leq -2-\frac{\log(2D)}{R} \tag{70}$$

$$N_B^2 \leq \eta(T-DN_B)e^{-2-\frac{\log(2D)}{R}} \tag{71}$$

$$N_B^2 \leq \eta(T-DN_B)e^{-2}e^{-\frac{\log(2D)}{R}} \tag{72}$$

$$N_B^2 \leq \eta(T-DN_B)e^{-2}(2D)^{-\frac{1}{R}} \tag{73}$$

$$N_B^2 \leq \eta(2D)^{-\frac{1}{R}} \tag{74}$$

$$N_B^2 \leq \eta(2D)^{-\frac{1}{\eta N_{avg}}} \tag{75}$$

$$\tag{76}$$