Supplement C
to
Interior solution of azimuthally symmetric case of Laplace equation in orthogonal Similar Oblate Spheroidal coordinates

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C.1. The second kind and the first kind solutions of the angular part of Laplace equation in f_S/h_R terms

The found second-kind solution (58), i.e.

$$F_B(W) = c_{B1}\frac{f_S}{h_R}\left[\ln\left(\frac{f_S}{(1+\mu)f_C} + \sqrt{1+\frac{f_S^2}{(1+\mu)^2 f_C^2}}\right) - \sqrt{1+\frac{(1+\mu)^2 f_C^2}{f_S^2}}\right] \quad , \tag{C1}$$

contains relations with f_S/f_C. Nevertheless, it can be written in terms of f_S/h_R as well, as (see (A24) in **Appendix A**)

$$\frac{f_S^2}{f_C^2} = \frac{\frac{f_S^2}{h_R^2}}{1-\frac{1}{1+\mu}\frac{f_S^2}{h_R^2}} \ , \tag{C2}$$

Let's define a new variable

$$s \equiv \frac{f_S}{h_R}. \tag{C3}$$

Then we can express the ratios of the squared generalized sine and cosine as

$$\frac{f_S^2}{f_C^2} = \frac{s^2}{1-\frac{1}{1+\mu}s^2} \qquad \text{and} \qquad \frac{f_C^2}{f_S^2} = \frac{(1+\mu)-s^2}{(1+\mu)s^2} \quad . \tag{C4}$$

By inserting these ratios to (C1) we get

$$F_B(W) = c_{B1} s \left[\ln\left(\frac{1}{(1+\mu)} \sqrt{\frac{s^2}{1-\frac{1}{1+\mu}s^2}} + \sqrt{1 + \frac{1}{(1+\mu)^2} \frac{s^2}{1-\frac{1}{1+\mu}s^2}} \right) - \sqrt{1 + (1+\mu)^2 \frac{(1+\mu)-s^2}{(1+\mu)s^2}} \right] =$$

$$c_{B1} s \left[\ln\left(\sqrt{\frac{1}{(1+\mu)^2} \frac{s^2}{1-\frac{1}{1+\mu}s^2}} + \sqrt{1 + \frac{1}{(1+\mu)} \frac{s^2}{(1+\mu)- s^2}} \right) - \sqrt{1 + (1+\mu) \frac{(1+\mu)-s^2}{s^2}} \right] =$$

$$c_{B1} s \left[\ln\left(\sqrt{\frac{1}{(1+\mu)} \frac{s^2}{(1+\mu)- s^2}} + \sqrt{1 + \frac{1}{(1+\mu)} \frac{s^2}{(1+\mu)- s^2}} \right) - \sqrt{\frac{s^2+(1+\mu)^2-(1+\mu)s^2}{s^2}} \right] = c_{B1} s \left[\ln\left(\frac{1}{\sqrt{1+\mu}} \sqrt{\frac{s^2}{(1+\mu)- s^2}} + \right.\right.$$

$$\left.\left. \frac{1}{\sqrt{1+\mu}} \sqrt{(1+\mu) + \frac{s^2}{(1+\mu)- s^2}} \right) - \frac{\sqrt{(1+\mu)^2-\mu s^2}}{s} \right] =$$

$$c_{B1} \left[s \ln\left(\frac{1}{\sqrt{1+\mu}} \sqrt{\frac{s^2}{(1+\mu)- s^2}} + \frac{1}{\sqrt{1+\mu}} \sqrt{\frac{(1+\mu)[(1+\mu)- s^2]+ s^2}{(1+\mu)- s^2}} \right) - \sqrt{(1+\mu)^2 - \mu s^2} \right] =$$

$$c_{B1} \left[s \ln\left(\frac{1}{\sqrt{1+\mu}} \sqrt{\frac{s^2}{(1+\mu)- s^2}} + \frac{1}{\sqrt{1+\mu}} \sqrt{\frac{[(1+\mu)^2-(1+\mu)\, s^2]+ s^2}{(1+\mu)- s^2}} \right) - \sqrt{(1+\mu)^2 - \mu s^2} \right] =$$

$$c_{B1} \left[s \ln\left(\frac{1}{\sqrt{1+\mu}} \frac{s}{\sqrt{(1+\mu)-s^2}} + \frac{1}{\sqrt{1+\mu}} \sqrt{\frac{(1+\mu)^2-\mu\, s^2}{(1+\mu)- s^2}} \right) - \sqrt{(1+\mu)^2 - \mu s^2} \right] = c_{B1} \left[s \ln\left(\frac{1}{\sqrt{1+\mu}} \frac{s+\sqrt{(1+\mu)^2-\mu\, s^2}}{\sqrt{(1+\mu)-s^2}} \right) - \right.$$

$$\left. \sqrt{(1+\mu)^2 - \mu s^2} \right] = c_{B1} \left[s \ln\left(\left\{ \left[\frac{s+\sqrt{(1+\mu)^2-\mu s^2}}{\sqrt{1+\mu}\sqrt{(1+\mu)-s^2}} \right]^2 \right\}^{\frac{1}{2}} \right) - \sqrt{(1+\mu)^2 - \mu s^2} \right] \quad , \qquad \text{(C5)}$$

and thus

$$F_B(W) = F_B(s) = c_{B1} \left[s \frac{1}{2} \ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]} \right) - \sqrt{(1+\mu)^2 - \mu s^2} \right] \quad \text{(C6)}$$

For μ=0, this simplifies to

$$F_B(s)|_{\mu=0} = c_{B1} \left[s \frac{1}{2} \ln\left(\frac{1+s}{1-s} \right) - 1 \right] \qquad \text{(C7)}$$

which is the Legendre function of the second kind of the first degree.

Similarly, the first determined solution of the angular part of the Laplace equation is written in the terms of s as

$$F_A(W) = F_A(s) = c_{A1} s \qquad . \qquad \text{(C8)}$$

which is the Legendre function of the first kind (or Legendre polynomial) of the first degree.

C.2. Rewrite the angular part of the Laplace equation to f_2/h_R terms

The further step is to rewrite the angular part of the Laplace equation in SOS coordinates ((37) in the main text), i.e.

$$\frac{d^2F(W)}{dW^{\,2}} + \frac{1}{\mu} \frac{\{[(\mu\mu-\mu-7)+2K_d(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]h_R^4+(-2\mu-4)(1+\mu)h_R^6+[2-\mu-2K_d]\}}{\{(2+\mu)h_R^2-1\}W} \frac{dF(W)}{dW} +$$
$$\frac{K_d}{\mu^2} \frac{\{[-3\mu-7+(2+\mu)K_d]h_R^2+[(7\mu+8+\mu\mu)-(1+\mu)K_d]h_R^4-(1+\mu)(\mu+3)h_R^6-[K_d-2]\}}{\{(2+\mu)h_R^2-1\}W^2} F(W) = 0 \quad , \text{(C9)}$$

to the form containing only variable $s=f_S/h_R$.

When (A19) from **Appendix A** is used to express h_R^2, we obtain (see (A20) in **Appendix A**) for the denominator term

$$(2+\mu)h_R^2-1=\frac{(1+\mu)-\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2}{\left[\frac{1}{h_R^2}\right]} \quad . \qquad \text{(C10)}$$

The equation (C9) is then multiplied by the denominator term $(2+\mu)h_R^2-1$ and further by $\left[\frac{1}{h_R^2}\right]^3$:

$$\left[\frac{1}{h_R^2}\right]^3\frac{(1+\mu)-\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2}{\left[\frac{1}{h_R^2}\right]}\frac{d^2F(W)}{dW^2}+\frac{1}{\mu W}\left[\frac{1}{h_R^2}\right]^3\{[(\mu\mu-\mu-7)+2K_d(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]h_R^4+(-2\mu-4)(1+\mu)h_R^6+[2-\mu-2K_d]\}\frac{dF(W)}{dW}+\frac{K_d}{\mu^2W^2}\left[\frac{1}{h_R^2}\right]^3\{[-3\mu-7+(2+\mu)K_d]h_R^2+[(7\mu+8+\mu\mu)-(1+\mu)K_d]h_R^4-(1+\mu)(\mu+3)h_R^6-[K_d-2]\}F(W)=0 \quad . \qquad \text{(C11)}$$

By expanding and simplifying, we get

$$\left[\frac{1}{h_R^2}\right]^2\left[(1+\mu)-\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2\right]\frac{d^2F(W)}{dW^2}+\frac{1}{\mu W}\left\{[2-\mu-2K_d]\left[\frac{1}{h_R^2}\right]^3+[(\mu\mu-\mu-7)+2K_d(2+\mu)]\left[\frac{1}{h_R^2}\right]^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]\left[\frac{1}{h_R^2}\right]^1+(-2\mu-4)(1+\mu)\right\}\frac{dF(W)}{dW}+\frac{K_d}{\mu^2W^2}\left\{-[K_d-2]\left[\frac{1}{h_R^2}\right]^3+[-3\mu-7+(2+\mu)K_d]\left[\frac{1}{h_R^2}\right]^2+[(7\mu+8+\mu\mu)-(1+\mu)K_d]\left[\frac{1}{h_R^2}\right]^1-(1+\mu)(\mu+3)\right\}F(W)=0 \quad . \qquad \text{(C12)}$$

Subsequently insert $\frac{1}{h_R^2}$ according to (A19) from **Appendix A** with the result

$$\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^2\left[(1+\mu)-\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2\right]\frac{d^2F(W)}{dW^2}+\frac{1}{\mu W}\left\{[2-\mu-2K_d]\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^3+[(\mu\mu-\mu-7)+2K_d(2+\mu)]\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^1+(-2\mu-4)(1+\mu)\right\}\frac{dF(W)}{dW}+\frac{K_d}{\mu^2W^2}\left\{-[K_d-2]\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^3+[-3\mu-7+(2+\mu)K_d]\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^2+[(7\mu+8+\mu\mu)-(1+\mu)K_d]\left[\frac{\mu}{1+\mu}\left(\frac{f_S}{h_R}\right)^2+1\right]^1-(1+\mu)(\mu+3)\right\}F(W)=0 \quad . \qquad \text{(C13)}$$

Now, the term f_S/h_R is present at many occurrences. For simplification, we use substitution (61) from now onwards to replace it by the variable s. Further we expand the terms and get:

$$\left[\frac{\mu^2}{(1+\mu)^2}s^4+2\frac{\mu}{1+\mu}s^2+1\right]\left[(1+\mu)-\frac{\mu}{1+\mu}s^2\right]\frac{d^2F(W)}{dW^2}+\frac{1}{\mu W}\left\{[2-\mu-2K_d]\left[\frac{\mu^3}{(1+\mu)^3}s^6+3\frac{\mu^2}{(1+\mu)^2}s^4+3\frac{\mu}{1+\mu}s^2+1\right]+[(\mu\mu-\mu-7)+2K_d(2+\mu)]\left[\frac{\mu^2}{(1+\mu)^2}s^4+2\frac{\mu}{1+\mu}s^2+1\right]+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]\left[\frac{\mu}{1+\mu}s^2+1\right]+(-2\mu-4)(1+\mu)\right\}\frac{dF(W)}{dW}+\frac{K_d}{\mu^2W^2}\left\{-[K_d-2]\left[\frac{\mu^3}{(1+\mu)^3}s^6+3\frac{\mu^2}{(1+\mu)^2}s^4+3\frac{\mu}{1+\mu}s^2+1\right]+[-3\mu-7+(2+\mu)K_d]\left[\frac{\mu^2}{(1+\mu)^2}s^4+2\frac{\mu}{1+\mu}s^2+1\right]+[(7\mu+8+\mu\mu)-(1+\mu)K_d]\left[\frac{\mu}{1+\mu}s^2+1\right]-(1+\mu)(\mu+3)\right\}F(W)=0 \quad . \qquad \text{(C14)}$$

Further expanding the terms leads to

$$\Big[(1+\mu)\frac{\mu^2}{(1+\mu)^2}s^4+2(1+\mu)\frac{\mu}{1+\mu}s^2+(1+\mu)-\frac{\mu^2}{(1+\mu)^2}\frac{\mu}{1+\mu}s^2s^4-2\frac{\mu}{1+\mu}\frac{\mu}{1+\mu}s^2s^2-\frac{\mu}{1+\mu}s^2\Big]\frac{d^2F(W)}{dW^2}+\frac{1}{\mu W}\Big\{\Big[[2-\mu-2K_d]\frac{\mu^3}{(1+\mu)^3}s^6+3[2-\mu-2K_d]\frac{\mu^2}{(1+\mu)^2}s^4+3[2-\mu-2K_d]\frac{\mu}{1+\mu}s^2+[2-\mu-2K_d]\Big]+\Big[[(\mu\mu-\mu-7)+2K_d(2+\mu)]\frac{\mu^2}{(1+\mu)^2}s^4+2[(\mu\mu-\mu-7)+2K_d(2+\mu)]\frac{\mu}{1+\mu}s^2+[(\mu\mu-\mu-7)+$$

$$2K_d(2+\mu)\Big]\Bigg] + \Bigg[\Big[(\mu\mu+8\mu+9) - 2K_d(1+\mu)\Big]\frac{\mu}{1+\mu}s^2 + \Big[(\mu\mu+8\mu+9) - 2K_d(1+\mu)\Big]\Bigg] + (-2\mu-4)(1+\mu)\Bigg\}\frac{dF(W)}{dW} + \frac{K_d}{\mu^2W^2}\Bigg\{\Bigg[-[K_d-2]\frac{\mu^3}{(1+\mu)^3}s^6 - 3[K_d-2]\frac{\mu^2}{(1+\mu)^2}s^4 - 3[K_d-2]\frac{\mu}{1+\mu}s^2 - [K_d-2]\Bigg] + \Bigg[[-3\mu-7+(2+\mu)K_d]\frac{\mu^2}{(1+\mu)^2}s^4 + 2[-3\mu-7+(2+\mu)K_d]\frac{\mu}{1+\mu}s^2 + [-3\mu-7+(2+\mu)K_d]\Bigg] + \Bigg[[(7\mu+8+\mu\mu) - (1+\mu)K_d]\frac{\mu}{1+\mu}s^2 + [(7\mu+8+\mu\mu) - (1+\mu)K_d]\Bigg] - (1+\mu)(\mu+3)\Bigg\}F(W) = 0\ . \tag{C15}$$

We now put together the same power of s terms:

$$\Bigg[-\frac{\mu^3}{(1+\mu)^3}s^6 + \frac{\mu^2}{(1+\mu)}s^4 - 2\frac{\mu^2}{(1+\mu)^2}s^4 + 2\mu s^2 - \frac{\mu}{1+\mu}s^2 + (1+\mu)\Bigg]\frac{d^2F(W)}{dW^2} + \frac{1}{\mu W}\Bigg\{[2-\mu-2K_d]\frac{\mu^3}{(1+\mu)^3}s^6 + \Bigg[+3[2-\mu-2K_d]\frac{\mu^2}{(1+\mu)^2}s^4 + [(\mu\mu-\mu-7) + 2K_d(2+\mu)]\frac{\mu^2}{(1+\mu)^2}s^4\Bigg] + \Bigg[+3[2-\mu-2K_d]\frac{\mu}{1+\mu}s^2 + 2[(\mu\mu-\mu-7) + 2K_d(2+\mu)]\frac{\mu}{1+\mu}s^2 + [(\mu\mu+8\mu+9) - 2K_d(1+\mu)]\frac{\mu}{1+\mu}s^2\Bigg] + \Big[+[2-\mu-2K_d] + [(\mu\mu-\mu-7) + 2K_d(2+\mu)] + [(\mu\mu+8\mu+9) - 2K_d(1+\mu)] + (-2\mu-4)(1+\mu)\Big]\Bigg\}\frac{dF(W)}{dW} + \frac{K_d}{\mu^2W^2}\Bigg\{-[K_d-2]\frac{\mu^3}{(1+\mu)^3}s^6 + \Bigg[-3[K_d-2]\frac{\mu^2}{(1+\mu)^2}s^4 + [-3\mu-7+(2+\mu)K_d]\frac{\mu^2}{(1+\mu)^2}s^4\Bigg] + \Bigg[-3[K_d-2]\frac{\mu}{1+\mu}s^2 + 2[-3\mu-7+(2+\mu)K_d]\frac{\mu}{1+\mu}s^2 + [(7\mu+8+\mu\mu) - (1+\mu)K_d]\frac{\mu}{1+\mu}s^2\Bigg] + \Big[-[K_d-2] + [-3\mu-7+(2+\mu)K_d] + [(7\mu+8+\mu\mu) - (1+\mu)K_d]\Big] - (1+\mu)(\mu+3)\Bigg\}F(W) = 0 \tag{C16}$$

and combine them

$$\Bigg[-\frac{\mu^3}{(1+\mu)^3}s^6 + \left(\frac{\mu^2}{(1+\mu)} - 2\frac{\mu^2}{(1+\mu)^2}\right)s^4 + \left(2\mu - \frac{\mu}{1+\mu}\right)s^2 + (1+\mu)\Bigg]\frac{d^2F(W)}{dW^2} + \frac{1}{\mu W}\Bigg\{[2-\mu-2K_d]\frac{\mu^3}{(1+\mu)^3}s^6 + \Big[+3[2-\mu-2K_d] + [(\mu\mu-\mu-7) + 2K_d(2+\mu)]\Big]\frac{\mu^2}{(1+\mu)^2}s^4 + \Big[+3[2-\mu-2K_d] + 2[(\mu\mu-\mu-7) + 2K_d(2+\mu)] + [(\mu\mu+8\mu+9) - 2K_d(1+\mu)]\Big]\frac{\mu}{1+\mu}s^2 + \Big[+[2-\mu-2K_d] + [(\mu\mu-\mu-7) + 2K_d(2+\mu)] + [(\mu\mu+8\mu+9) - 2K_d(1+\mu)] + (-2\mu-4)(1+\mu)\Big]\Bigg\}\frac{dF(W)}{dW} + \frac{K_d}{\mu^2W^2}\Bigg\{-[K_d-2]\frac{\mu^3}{(1+\mu)^3}s^6 + \Big[-3[K_d-2] + [-3\mu-7+(2+\mu)K_d]\Big]\frac{\mu^2}{(1+\mu)^2}s^4 + \Big[-3[K_d-2] + 2[-3\mu-7+(2+\mu)K_d] + [(7\mu+8+\mu\mu) - (1+\mu)K_d]\Big]\frac{\mu}{1+\mu}s^2 + \Big[-[K_d-2] + [-3\mu-7+(2+\mu)K_d] + [(7\mu+8+\mu\mu) - (1+\mu)K_d] - (1+\mu)(\mu+3)\Big]\Bigg\}F(W) = 0. \tag{C17}$$

Then we combine the pre-factors of the individual s powers:

$$\Bigg[-\frac{\mu^3}{(1+\mu)^3}s^6 + \mu^2\frac{(\mu-1)}{(1+\mu)^2}s^4 + \mu\frac{1+2\mu}{1+\mu}s^2 + (1+\mu)\Bigg]\frac{d^2F(W)}{dW^2} + \frac{1}{\mu W}\Bigg\{[(2-\mu) - 2K_d]\frac{\mu^3}{(1+\mu)^3}s^6 + [(\mu\mu-4\mu-1) + 2K_d(\mu-1)]\frac{\mu^2}{(1+\mu)^2}s^4 + [(3\mu\mu+3\mu+1) + 2\mu K_d]\frac{\mu}{1+\mu}s^2 + [0]\Bigg\}\frac{dF(W)}{dW} + \frac{K_d}{\mu^2W^2}\Bigg\{-[K_d-2]\frac{\mu^3}{(1+\mu)^3}s^6 + [(-3\mu-1) + (\mu-1)K_d]\frac{\mu^2}{(1+\mu)^2}s^4 + [(1+\mu) + K_d]\mu\frac{\mu}{1+\mu}s^2 + [0]\Bigg\}F(W) = 0\ . \tag{C18}$$

Finally, we simplify and obtain the expression for the angular part of the Laplace equation in terms of s in the form

$$\left[-\frac{\mu^3}{(1+\mu)^3}s^6+(\mu-1)\frac{\mu^2}{(1+\mu)^2}s^4+(1+2\mu)\frac{\mu}{1+\mu}s^2+(1+\mu)\right]\frac{d^2F(W)}{dW^2}+\frac{1}{W}\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^3}s^6+\right.$$
$$\left.[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)^2}s^4+[(3\mu\mu+3\mu+1)+2\mu K_d]\frac{1}{1+\mu}s^2\right\}\frac{dF(W)}{dW}+\frac{K_d}{W^2}\left\{-[K_d-\right.$$
$$\left.2]\frac{\mu}{(1+\mu)^3}s^6+[(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)^2}s^4+[(1+\mu)+K_d]\frac{1}{1+\mu}s^2\right\}F(W)=0 \quad . \tag{C19}$$

It does worth to notice particularly the multiplication by μ in many terms of the equation, which would make the equation (C19) much simpler in case μ=0, i.e. in the spherical case. But before this test, we should still substitute W and the derivatives (now still with respect to W) with terms depending only on s.

As can be seen in **Appendix A**, Eq. (A31), the variable W can be also written in terms of s as follows:

$$W^2=\frac{\frac{1}{1+\mu}\frac{f_2^2}{h_R^2}}{\left(1-\frac{1}{1+\mu}\frac{f_2^2}{h_R^2}\right)^{1+\mu}}=\frac{\frac{1}{1+\mu}s^2}{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}} \quad , \tag{C20}$$

and thus

$$W=\sqrt{\frac{\frac{1}{1+\mu}s^2}{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}}} \quad . \tag{C21}$$

Then, the derivatives of W with respect to s, which will be needed in further derivation, are

$$\frac{dW}{ds}=(1+\mu)^{\frac{\mu}{2}}[(1+\mu)-s^2]^{-\frac{\mu+3}{2}}[(1+\mu)+\mu s^2] \quad , \tag{C22}$$

and

$$\frac{d^2W}{ds^2}=(1+\mu)^{\frac{\mu}{2}+1}s[(1+\mu)-s^2]^{-\frac{\mu+5}{2}}[3(1+\mu)+\mu s^2] \quad . \tag{C23}$$

With these derivatives by hand, we transform the derivatives of the solution F of the angular part of the Laplace equation from “with respect to W” to “with respect to s”. As

$$\frac{dF(s)}{ds}=\frac{dF(W)}{dW}\frac{dW}{ds} \quad . \tag{C24}$$

Then (see (C22))

$$\frac{dF(W)}{dW}=\frac{\frac{dF(s)}{ds}}{\frac{dW}{ds}}=\frac{dF(s)}{ds}\frac{1}{(1+\mu)^{\frac{\mu}{2}}[(1+\mu)-s^2]^{-\frac{\mu+3}{2}}[(1+\mu)+\mu s^2]}=\frac{dF(s)}{ds}\frac{[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu}{2}}[(1+\mu)+\mu s^2]} \quad . \tag{C25}$$

As further

$$\frac{d^2F(s)}{ds^2}=\frac{d}{ds}\left[\frac{dF(s)}{ds}\right]=\frac{d}{ds}\left[\frac{dF(W)}{dW}\frac{dW}{ds}\right]=\frac{d}{ds}\left[\frac{dF(W)}{dW}\right]\frac{dW}{ds}+\frac{dF(W)}{dW}\frac{d}{ds}\left[\frac{dW}{ds}\right]=\frac{d}{dW}\left[\frac{dF(W)}{dW}\right]\frac{dW}{ds}\frac{dW}{ds}+$$
$$\frac{dF(W)}{dW}\left[\frac{d^2W}{ds^2}\right]=\frac{d^2F(W)}{dW^2}\left(\frac{dW}{ds}\right)^2+\frac{\frac{dF(s)}{ds}}{\frac{dW}{ds}}\left[\frac{d^2W}{ds^2}\right] \quad , \tag{C26}$$

then also (see (C22) and (C23))

$$\frac{d^2F(W)}{dW^2}=\frac{d^2F(s)}{ds^2}\frac{1}{\left(\frac{dW}{ds}\right)^2}-\frac{dF(s)}{ds}\frac{\left[\frac{d^2W}{ds^2}\right]}{\left(\frac{dW}{ds}\right)^3}=$$
$$\frac{d^2F(s)}{ds^2}\frac{1}{\left((1+\mu)^{\frac{\mu}{2}}[(1+\mu)-s^2]^{-\frac{\mu+3}{2}}[(1+\mu)+\mu s^2]\right)^2}-\frac{dF(s)}{ds}\frac{(1+\mu)^{\frac{\mu}{2}+1}s[(1+\mu)-s^2]^{-\frac{\mu+5}{2}}[3(1+\mu)+\mu s^2]}{\left((1+\mu)^{\frac{\mu}{2}}[(1+\mu)-s^2]^{-\frac{\mu+3}{2}}[(1+\mu)+\mu s^2]\right)^3}=$$
$$\frac{d^2F(s)}{ds^2}\frac{1}{(1+\mu)^{\mu}[(1+\mu)-s^2]^{-(\mu+3)}[(1+\mu)+\mu s^2]^2}-\frac{dF(s)}{ds}\frac{(1+\mu)^{\frac{\mu}{2}+1}s[(1+\mu)-s^2]^{-\frac{\mu+5}{2}}[3(1+\mu)+\mu s^2]}{(1+\mu)^{\frac{3\mu}{2}}[(1+\mu)-s^2]^{-\frac{3(\mu+3)}{2}}[(1+\mu)+\mu s^2]^3} \quad . \tag{C27}$$

Therefore, the second derivative with respect to W is finally written I terms of the derivatives with respect to s as

$$\frac{d^2F(W)}{dW^{\,2}} = \frac{d^2F(s)}{ds^2}\,\frac{[(1+\mu)-s^2]^{(\mu+3)}}{(1+\mu)^{\mu}[(1+\mu)+\mu s^2]^2} - \frac{dF(s)}{ds}\,s\,\frac{(1+\mu)^{(1-\mu)}[(1+\mu)-s^2]^{(\mu+2)}[3(1+\mu)+\mu s^2]}{[(1+\mu)+\mu s^2]^3}. \qquad \text{(C28)}$$

We now have all necessary means to transform the angular part of the Laplace equation in SOS coordinates to the form depending explicitly only on s variable. The derivation is lengthy, but it involves only standard algebraic manipulations.

The equation (C19) for the angular part of the Laplace equation in terms of s (i.e. in terms of f_S/h_R) can be then rewritten (see (C20) and (C21)) as follows:

$$\left[-\frac{\mu^3}{(1+\mu)^3}s^6 + (\mu-1)\frac{\mu^2}{(1+\mu)^2}s^4 + (1+2\mu)\frac{\mu}{1+\mu}s^2 + (1+\mu)\right]\frac{d^2F(W)}{dW^{\,2}} + \frac{1}{\sqrt{\frac{\frac{1}{1+\mu}s^2}{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}}}}\Big\{[(2-\mu) - 2K_d]\frac{\mu^2}{(1+\mu)^3}s^6 + [(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)^2}s^4 + [(3\mu\mu+3\mu+1)+2\mu K_d]\frac{1}{1+\mu}s^2\Big\}\frac{dF(W)}{dW} + \frac{K_d}{\frac{\frac{1}{1+\mu}s^2}{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}}}\Big\{-[K_d-2]\frac{\mu}{(1+\mu)^3}s^6 + [(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)^2}s^4 + [(1+\mu)+K_d]\frac{1}{1+\mu}s^2\Big\}F(W) = 0\ . \qquad \text{(C29)}$$

By expressing the derivatives in terms of s (see (C25) and (C28)), we get

$$\left[-\frac{\mu^3}{(1+\mu)^3}s^6 + (\mu-1)\frac{\mu^2}{(1+\mu)^2}s^4 + (1+2\mu)\frac{\mu}{1+\mu}s^2 + (1+\mu)\right]\left[\frac{[(1+\mu)-s^2]^{(\mu+3)}}{(1+\mu)^{\mu}[(1+\mu)+\mu s^2]^2}\frac{d^2F(s)}{ds^2} - s\,\frac{(1+\mu)^{(1-\mu)}[(1+\mu)-s^2]^{(\mu+2)}[3(1+\mu)+\mu s^2]}{[(1+\mu)+\mu s^2]^3}\frac{dF(s)}{ds}\right] + \sqrt{\frac{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}}{\frac{1}{1+\mu}s^2}}\Big\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^3}s^6 + [(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)^2}s^4 + [(3\mu\mu+3\mu+1)+2\mu K_d]\frac{1}{1+\mu}s^2\Big\}\frac{[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu}{2}}[(1+\mu)+\mu s^2]}\frac{dF(s)}{ds} + K_d\frac{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}}{\frac{1}{1+\mu}s^2}\Big\{-[K_d-2]\frac{\mu}{(1+\mu)^3}s^6 + [(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)^2}s^4 + [(1+\mu)+K_d]\frac{1}{1+\mu}s^2\Big\}F(s) = 0\ . \qquad \text{(C30)}$$

This equation is now to be simplified. As part of the second term is

$$\sqrt{\frac{\left(1-\frac{1}{1+\mu}s^2\right)^{1+\mu}}{\frac{1}{1+\mu}s^2}}\frac{[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu}{2}}[(1+\mu)+\mu s^2]} = \frac{\left(1-\frac{1}{1+\mu}s^2\right)^{\frac{1+\mu}{2}}}{\frac{1}{(1+\mu)^{\frac{1}{2}}}s}\frac{[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu}{2}}[(1+\mu)+\mu s^2]} = \frac{\left(\frac{(1+\mu)-s^2}{1+\mu}\right)^{\frac{1+\mu}{2}}}{s}\frac{[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\left(\frac{\mu}{2}-\frac{1}{2}\right)}[(1+\mu)+\mu s^2]} = \frac{\left(\frac{1}{1+\mu}\right)^{\frac{1+\mu}{2}}[(1+\mu)-s^2]^{\frac{1+\mu}{2}}}{s}\frac{[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu-1}{2}}[(1+\mu)+\mu s^2]} = \frac{1}{s}\frac{1}{(1+\mu)^{\frac{1+\mu}{2}}}\frac{[(1+\mu)-s^2]^{\frac{1+\mu}{2}}[(1+\mu)-s^2]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu-1}{2}}[(1+\mu)+\mu s^2]} = \frac{1}{s}\frac{1}{(1+\mu)^{\mu}}\frac{[(1+\mu)-s^2]^{\mu+2}}{[(1+\mu)+\mu s^2]}\ , \qquad \text{(C31)}$$

we have

$$\left[-\frac{\mu^3}{(1+\mu)^3}s^6 + (\mu-1)\frac{\mu^2}{(1+\mu)^2}s^4 + (1+2\mu)\frac{\mu}{1+\mu}s^2 + (1+\mu)\right]\left[\frac{[(1+\mu)-s^2]^{(\mu+3)}}{(1+\mu)^{\mu}[(1+\mu)+\mu s^2]^2}\frac{d^2F(s)}{ds^2} - s\,\frac{(1+\mu)[(1+\mu)-s^2]^{(\mu+2)}[3(1+\mu)+\mu s^2]}{(1+\mu)^{\mu}[(1+\mu)+\mu s^2]^3}\frac{dF(s)}{ds}\right] + \frac{1}{s}\frac{1}{(1+\mu)^{\mu}}\frac{[(1+\mu)-s^2]^{\mu+2}}{[(1+\mu)+\mu s^2]}\Big\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^3}s^6 + [(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)^2}s^4 + [(3\mu\mu+3\mu+1)+2\mu K_d]\frac{1}{1+\mu}s^2\Big\}\frac{dF(s)}{ds} + \frac{K_d}{(1+\mu)^{1+\mu}}\,[(1+$$

$$\mu)\ -s^2\ ]^{(1+\mu)}\left\{-[K_d-2]\frac{\mu}{(1+\mu)^2}s^4+[(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)}s^2+[(1+\mu)+K_d]\right\}F(s)=0 \quad . \tag{C32}$$

Multiply the equation by $(1+\mu)^{1+\mu}$, divide by $[(1+\mu)\ -s^2\ ]^{(1+\mu)}$, and get

$$\left[-\frac{\mu^3}{(1+\mu)^3}s^6+(\mu-1)\frac{\mu^2}{(1+\mu)^2}s^4+(1+2\mu)\frac{\mu}{1+\mu}s^2+(1+\mu)\right]\left[(1+\mu)\ \frac{[(1+\mu)-s^2]^2}{[(1+\mu)+\mu s^2]^2}\frac{d^2F(s)}{ds^2}-s\frac{(1+\mu)^2[(1+\mu)-s^2][3(1+\mu)+\mu s^2]}{[(1+\mu)+\mu s^2]^3}\frac{dF(s)}{ds}\right]+\frac{1}{s}(1+\mu)\frac{[(1+\mu)-s^2]}{[(1+\mu)+\mu s^2]}\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^3}s^6+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)^2}s^4+[(3\mu\mu+3\mu+1)+2\mu K_d]\frac{1}{1+\mu}s^2\ \right\}\frac{dF(s)}{ds}+K_d\left\{-[K_d-2]\frac{\mu}{(1+\mu)^2}s^4+[(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)}s^2+[(1+\mu)+K_d]\right\}F(s)=0 \quad . \tag{C33}$$

Then, put together the terms containing the first derivative:

$$\left[-\frac{\mu^3}{(1+\mu)^2}s^6+(\mu-1)\frac{\mu^2}{1+\mu}s^4+(1+2\mu)\mu s^2+(1+\mu)^2\right]\frac{[(1+\mu)-s^2]^2}{[(1+\mu)+\mu s^2]^2}\frac{d^2F(s)}{ds^2}+\left[-\left\{-\frac{\mu^3}{(1+\mu)}s^7+(\mu-1)\mu^2s^5+(1+2\mu)\mu(1+\mu)s^3+(1+\mu)^3s\right\}\frac{[3(1+\mu)+\mu s^2]}{[(1+\mu)+\mu s^2]^3}+\frac{1}{[(1+\mu)+\mu s^2]}\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}s^3+[(3\mu\mu+3\mu+1)+2\mu K_d]s^1\ \right\}\right][(1+\mu)-s^2]\frac{dF(s)}{ds}+K_d\left\{-[K_d-2]\frac{\mu}{(1+\mu)^2}s^4+[(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)}s^2+[(1+\mu)+K_d]\right\}F(s)=0 \quad . \tag{C34}$$

Further, multiply by $[(1+\mu)+\mu s^2]^3$

$$\left[-\frac{\mu^3}{(1+\mu)^2}s^6+(\mu-1)\frac{\mu^2}{1+\mu}s^4+(1+2\mu)\mu s^2+(1+\mu)^2\right][(1+\mu)-s^2]^2\,[(1+\mu)+\mu s^2]\frac{d^2F(s)}{ds^2}+\left[-\left\{-\frac{\mu^3}{(1+\mu)}s^7+(\mu-1)\mu^2s^5+(1+2\mu)\mu(1+\mu)s^3+(1+\mu)^3s\right\}[3(1+\mu)+\mu s^2]+[(1+\mu)+\mu s^2]^2\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}s^3+[(3\mu\mu+3\mu+1)+2\mu K_d]s^1\ \right\}\right][(1+\mu)-s^2]\frac{dF(s)}{ds}+K_d\ \left\{-[K_d-2]\frac{\mu}{(1+\mu)^2}s^4+[(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)}s^2+[(1+\mu)+K_d]\right\}[(1+\mu)+\mu s^2]^3F(s)=0 \quad . \tag{C35}$$

Further, modify the pre-factors by the second derivative and by the first derivative. The pre-factor of the second derivative can be written as

$$\left[-\frac{\mu^3}{(1+\mu)^2}s^6+(\mu-1)\frac{\mu^2}{1+\mu}s^4+(1+2\mu)\mu s^2+(1+\mu)^2\right][(1+\mu)-s^2]^2\ [(1+\mu)+\mu s^2]=\frac{[(1+\mu)-s^2]^2\ [(1+\mu)+\mu s^2]^3\ [(1+\mu)^2-\mu s^2]}{(1+\mu)^2}. \tag{C36}$$

The pre-factor by the first derivative is derived in what follows. As

$$-\left\{-\frac{\mu^3}{(1+\mu)}s^7+(\mu-1)\mu^2s^5+(1+2\mu)\mu(1+\mu)s^3+(1+\mu)^3s\right\}[3(1+\mu)+\mu s^2]=-\left\{-\frac{\mu^3}{(1+\mu)}3(1+\mu)s^7+(\mu-1)\mu^23(1+\mu)s^5+(1+2\mu)\mu(1+\mu)3(1+\mu)s^3+(1+\mu)^33(1+\mu)s-\frac{\mu^3}{(1+\mu)}\mu s^2s^7+(\mu-1)\mu^2\mu s^2s^5+(1+2\mu)\mu(1+\mu)\mu s^2s^3+(1+\mu)^3\mu s^2s\right\}=-\left\{-3\mu^3s^7+3\mu^2(\mu-1)(1+\mu)s^5+3\mu(1+2\mu)(1+\mu)^2s^3+3(1+\mu)^4s-\frac{\mu^4}{(1+\mu)}s^9+\mu^3(\mu-1)s^7+\mu^2(1+2\mu)(1+\mu)s^5+\mu(1+\mu)^3s^3\right\}=-\left\{-\frac{\mu^4}{(1+\mu)}s^9+\mu^3(\mu-1)s^7-3\mu^3s^7+3\mu^2(\mu-1)(1+\mu)s^5+\mu^2(1+2\mu)(1+\mu)s^5+\mu(1+\mu)^3s^3+3\mu(1+2\mu)(1+\mu)^2s^3+3(1+\mu)^4s\right\}=-\left\{-\frac{\mu^4}{(1+\mu)}s^9+[\mu^3(\mu-1)-3\mu^3]s^7+[3\mu^2(\mu-1)+\mu^2(1+2\mu)](1+\mu)s^5+[\mu(1+\mu)+3\mu(1+2\mu)](1+\mu)^2s^3+3(1+\mu)^4s\right\}=\frac{\mu^4}{(1+\mu)}s^9-\mu^3(\mu-4)s^7-\mu^2(5\mu-2)(1+\mu)s^5-\mu(7\mu+4)(1+\mu)^2s^3-3(1+\mu)^4s, \tag{C37}$$

and as

$$[(1+\mu)+\mu s^2]^2\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}s^3+[(3\mu\mu+3\mu+1)+2\mu K_d]s^1\ \right\}=[(1+\mu)^2+2(1+\mu)\mu s^2+\mu^2s^4]\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}s^3+[(3\mu\mu+3\mu+1)+2\mu K_d]s^1\ \right\}=\left\{[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}(1+\mu)^2s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}(1+\mu)^2s^3+[(3\mu\mu+3\mu+1)+2\mu K_d](1+\mu)^2s^1+\right.$$

$$[(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}2(1+\mu)\mu s^2 s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}2(1+\mu)\mu s^2 s^3+[(3\mu\mu+3\mu+1)+2\mu K_d]2(1+\mu)\mu s^2 s^1+ [(2-\mu)-2K_d]\frac{\mu^2}{(1+\mu)^2}\mu^2 s^4 s^5+[(\mu\mu-4\mu-1)+2K_d(\mu-1)]\frac{\mu}{(1+\mu)}\mu^2 s^4 s^3+[(3\mu\mu+3\mu+1)+2\mu K_d]\mu^2 s^4 s^1\Big\} = \Big\{\mu^2[(2-\mu)-2K_d]s^5+ \mu(1+\mu)[(\mu\mu-4\mu-1)+2K_d(\mu-1)]s^3+(1+\mu)^2[(3\mu\mu+3\mu+1)+2\mu K_d]s^1+2\frac{\mu^3}{(1+\mu)}[(2-\mu)-2K_d]s^7+2\mu^2[(\mu\mu-4\mu-1)+ 2K_d(\mu-1)]s^5+2\mu(1+\mu)[(3\mu\mu+3\mu+1)+2\mu K_d]s^3+\frac{\mu^4}{(1+\mu)^2}[(2-\mu)-2K_d]s^9+\frac{\mu^3}{(1+\mu)}[(\mu\mu-4\mu-1)+2K_d(\mu-1)]s^7+\mu^2[(3\mu\mu+ 3\mu+1)+2\mu K_d]s^5\Big\} = \Big\{+\frac{\mu^4}{(1+\mu)^2}[(2-\mu)-2K_d]s^9+2\frac{\mu^3}{(1+\mu)}[(2-\mu)-2K_d]s^7+\frac{\mu^3}{(1+\mu)}[(\mu\mu-4\mu-1)+2K_d(\mu-1)]s^7+\mu^2[(2-\mu)- 2K_d]s^5+2\mu^2[(\mu\mu-4\mu-1)+2K_d(\mu-1)]s^5+\mu^2[(3\mu\mu+3\mu+1)+2\mu K_d]s^5+\mu(1+\mu)[(\mu\mu-4\mu-1)+2K_d(\mu-1)]s^3+2\mu(1+ \mu)[(3\mu\mu+3\mu+1)+2\mu K_d]s^3+(1+\mu)^2[(3\mu\mu+3\mu+1)+2\mu K_d]s^1\Big\} = +\frac{\mu^4}{(1+\mu)^2}[(2-\mu)-2K_d]s^9+\{[(4-2\mu)-4K_d]+[(\mu\mu-4\mu- 1)+2K_d(\mu-1)]\}\frac{\mu^3}{(1+\mu)}s^7+\{[(2-\mu)-2K_d]+[(2\mu\mu-8\mu-2)+4K_d(\mu-1)]+[(3\mu\mu+3\mu+1)+2\mu K_d]\}\mu^2 s^5+\{[(\mu\mu-4\mu-1)+ 2K_d(\mu-1)]+[(6\mu\mu+6\mu+2)+4\mu K_d]\}\mu(1+\mu)s^3+(1+\mu)^2[(3\mu\mu+3\mu+1)+2\mu K_d]s^1 = +\frac{\mu^4}{(1+\mu)^2}[2-\mu-2K_d]s^9+\{\mu\mu-6\mu+3+ 2K_d\mu-6K_d\}\frac{\mu^3}{(1+\mu)}s^7+\{5\mu\mu-6\mu+1+6K_d\mu-6K_d\}\mu^2 s^5+\{7\mu\mu+2\mu+1+6K_d\mu-2K_d\}\mu(1+\mu)s^3+(1+\mu)^2[3\mu\mu+3\mu+1+ 2\mu K_d]s^1\,, \quad \text{(C38)}$$

then the pre-factor by the first derivative of *F* is

$$\Big[\frac{\mu^4}{(1+\mu)}s^9-\mu^3(\mu-4)s^7-\mu^2(5\mu-2)(1+\mu)s^5-\mu(7\mu+4)(1+\mu)^2 s^3-3(1+\mu)^4 s+\frac{\mu^4}{(1+\mu)^2}[2-\mu-2K_d]s^9+\{\mu\mu-6\mu+3+2K_d\mu- 6K_d\}\frac{\mu^3}{(1+\mu)}s^7+\{5\mu\mu-6\mu+1+6K_d\mu-6K_d\}\mu^2 s^5+\{7\mu\mu+2\mu+1+6K_d\mu-2K_d\}\mu(1+\mu)s^3+(1+\mu)^2[3\mu\mu+3\mu+1+2\mu K_d]s^1\Big][(1+ \mu)-s^2] = \Big[\frac{\mu^4}{(1+\mu)}s^9+\frac{\mu^4}{(1+\mu)^2}[2-\mu-2K_d]s^9+\{\mu\mu-6\mu+3+2K_d\mu-6K_d\}\frac{\mu^3}{(1+\mu)}s^7-\mu^3(\mu-4)s^7+\{5\mu\mu-6\mu+1+6K_d\mu- 6K_d\}\mu^2 s^5-\mu^2(5\mu-2)(1+\mu)s^5+\{7\mu\mu+2\mu+1+6K_d\mu-2K_d\}\mu(1+\mu)s^3-\mu(7\mu+4)(1+\mu)^2 s^3-3(1+\mu)^4 s+(1+\mu)^2[3\mu\mu+3\mu+ 1+2\mu K_d]s^1\Big][(1+\mu)-s^2] = \Big[\{(1+\mu)+[2-\mu-2K_d]\}\frac{\mu^4}{(1+\mu)^2}s^9+\big\{\{\mu\mu-6\mu+3+2K_d\mu-6K_d\}-(1+\mu)(\mu-4)\big\}\frac{\mu^3}{(1+\mu)}s^7+ \big\{\{5\mu\mu-6\mu+1+6K_d\mu-6K_d\}-(5\mu-2)(1+\mu)\big\}\mu^2 s^5+\big\{\{7\mu\mu+2\mu+1+6K_d\mu-2K_d\}-(7\mu+4)(1+\mu)\big\}\mu(1+\mu)s^3+\{[3\mu\mu+3\mu+ 1+2\mu K_d]-3(1+\mu)^2\}(1+\mu)^2 s^1\Big][(1+\mu)-s^2] = \Big[\{3-2K_d\}\frac{\mu^4}{(1+\mu)^2}s^9+\{-3\mu+7+2K_d\mu-6K_d\}\frac{\mu^3}{(1+\mu)}s^7+\{-9\mu+3+6K_d\mu- 6K_d\}\mu^2 s^5+\{-9\mu-3+6K_d\mu-2K_d\}\mu(1+\mu)s^3+\{-3\mu-2+2\mu K_d\}(1+\mu)^2 s^1\Big][(1+\mu)-s^2] = \Big[\{3-2K_d\}\frac{\mu^4}{(1+\mu)}s^9+\{-3\mu+7+ 2K_d\mu-6K_d\}\mu^3 s^7+\{-9\mu+3+6K_d\mu-6K_d\}\mu^2(1+\mu)s^5+\{-9\mu-3+6K_d\mu-2K_d\}\mu(1+\mu)^2 s^3+\{-3\mu-2+2\mu K_d\}(1+\mu)^3 s^1- \{3-2K_d\}\frac{\mu^4}{(1+\mu)^2}s^{11}-\{-3\mu+7+2K_d\mu-6K_d\}\frac{\mu^3}{(1+\mu)}s^9-\{-9\mu+3+6K_d\mu-6K_d\}\mu^2 s^7-\{-9\mu-3+6K_d\mu-2K_d\}\mu(1+\mu)s^5- \{-3\mu-2+2\mu K_d\}(1+\mu)^2 s^3\Big] = \Big[-\{3-2K_d\}\frac{\mu^4}{(1+\mu)^2}s^{11}+\{3-2K_d\}\frac{\mu^4}{(1+\mu)}s^9-\{-3\mu+7+2K_d\mu-6K_d\}\frac{\mu^3}{(1+\mu)}s^9+\{-3\mu+7+2K_d\mu- 6K_d\}\mu^3 s^7-\{-9\mu+3+6K_d\mu-6K_d\}\mu^2 s^7+\{-9\mu+3+6K_d\mu-6K_d\}\mu^2(1+\mu)s^5-\{-9\mu-3+6K_d\mu-2K_d\}\mu(1+\mu)s^5+\{-9\mu-3+ 6K_d\mu-2K_d\}\mu(1+\mu)^2 s^3-\{-3\mu-2+2\mu K_d\}(1+\mu)^2 s^3+\{-3\mu-2+2\mu K_d\}(1+\mu)^3 s^1\Big] = -\{3-2K_d\}\frac{\mu^4}{(1+\mu)^2}s^{11}+[\{3-2K_d\}\mu- \{-3\mu+7+2K_d\mu-6K_d\}]\frac{\mu^3}{(1+\mu)}s^9+[\{-3\mu+7+2K_d\mu-6K_d\}\mu-\{-9\mu+3+6K_d\mu-6K_d\}]\mu^2 s^7+[\{-9\mu+3+6K_d\mu-6K_d\}\mu- \{-9\mu-3+6K_d\mu-2K_d\}]\mu(1+\mu)s^5+[\{-9\mu-3+6K_d\mu-2K_d\}\mu-\{-3\mu-2+2\mu K_d\}](1+\mu)^2 s^3+\{-3\mu-2+2\mu K_d\}(1+\mu)^3 s = -\{3-2K_d\}\frac{\mu^4}{(1+\mu)^2}s^{11}+[6\mu-7-4K_d\mu+6K_d]\frac{\mu^3}{(1+\mu)}s^9+[-3\mu\mu+16\mu-3+2K_d\mu\mu-12K_d\mu+6K_d]\mu^2 s^7+[-9\mu\mu+12\mu+3+6K_d\mu\mu- 12K_d\mu+2K_d]\mu(1+\mu)s^5+[-9\mu\mu+2+6K_d\mu\mu-4K_d\mu](1+\mu)^2 s^3+\{-3\mu-2+2\mu K_d\}(1+\mu)^3 s \quad . \quad \text{(C39)}$$

Still, this can be rewritten in the form of a product of four terms

$$\frac{s[(1+\mu)-s^2][(1+\mu)+\mu s^2]^3[2\mu^2K_d-3\mu^2-2\mu K_d s^2+2\mu K_d+3\mu s^2-5\mu-2]}{(1+\mu)^2} = \frac{s[-(3\mu^2+5\mu+2)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2][(1+\mu)-s^2][(1+\mu)+\mu s^2]^3}{(1+\mu)^2}. \quad \text{(C40)}$$

Then, the equation with the simplified pre-factors is

$$\frac{[(1+\mu)-s^2]^2\,[(1+\mu)+\mu s^2]^3\,[(1+\mu)^2-\mu s^2]}{(1+\mu)^2}\frac{d^2F(s)}{ds^2}+ \frac{s[-(3\mu^2+5\mu+2)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2][(1+\mu)-s^2][(1+\mu)+\mu s^2]^3}{(1+\mu)^2}\frac{dF(s)}{ds}+K_d\Big\{-[K_d-2]\frac{\mu}{(1+\mu)^2}s^4+ [(-3\mu-1)+(\mu-1)K_d]\frac{1}{(1+\mu)}s^2+[(1+\mu)+K_d]\Big\}[(1+\mu)+\mu s^2]^3 F(s)=0 \quad . \quad \text{(C41)}$$

Divide this equation by $[(1+\mu)+\mu s^2]^3$ and multiply by $(1+\mu)^2$. This leads to

$$[(1+\mu)-s^2]^2\,[(1+\mu)^2-\mu s^2]\frac{d^2F(s)}{ds^2}+s[-(3\mu^2+5\mu+2)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2][(1+ \mu)-s^2]\frac{dF(s)}{ds}+K_d\{-[K_d-2]\mu s^4+[(-3\mu-1)+(\mu-1)K_d](1+\mu)s^2+[(1+\mu)+K_d](1+ \mu)^2\}F(s)=0 \quad . \text{(C42)}$$

As the part of the last term in curl brackets can be written in the form

$$\{-[K_d-2]\mu s^4+[(-3\mu-1)+(\mu-1)K_d](1+\mu)s^2+[(1+\mu)+K_d](1+\mu)^2\} = [(1+\mu)-s^2][\mu K_d+\mu K_d s^2+K_d+\mu^2+2\mu-2\mu s^2+1] = [(1+\mu)-s^2][(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2] \quad , \qquad \text{(C43)}$$

we have

$$[(1+\mu)-s^2]^2\,[(1+\mu)^2-\mu s^2]\frac{d^2F(s)}{ds^2}+s[-(3\mu^2+5\mu+2)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2][(1+\mu)-s^2]\frac{dF(s)}{ds}+K_d[(1+\mu)-s^2][(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2]F(s)=0 \quad . \qquad \text{(C44)}$$

After division by $[(1+\mu)-s^2]$, the equation is finally

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\frac{d^2F(s)}{ds^2}+s[-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d+\mu(3-2K_d)s^2]\frac{dF(s)}{ds}+K_d[(K_d-2)\mu s^2+(1+\mu)K_d+(1+\mu)^2]F(s)=0 \quad . \qquad \text{(C45)}$$

or– in the expanded form –

$$[(1+\mu)^3-(1+3\mu+\mu^2)s^2+\mu s^4]\frac{d^2F(s)}{ds^2}+[(2\mu K_d-3\mu-2)(1+\mu)s+\mu(3-2K_d)s^3]\frac{dF(s)}{ds}+[K_d(K_d-2)\mu s^2+K_d(K_d+\mu+1)(1+\mu)]F(s)=0 \quad . \qquad \text{(C46)}$$

Note that the equation simplifies for μ=0 to

$$[1-s^2]\frac{d^2F(s)}{ds^2}-2s\frac{dF(s)}{ds}+K_d[K_d+1]F(s)=0 \quad . \qquad \text{(C47)}$$

which is Legendre equation. Therefore, we will denote the equation (C45) “generalized Legendre equation” in what follows.