Supplement D

to

Interior solution of azimuthally symmetric case of Laplace equation in orthogonal Similar Oblate Spheroidal coordinates

Pavel Strunz

Nuclear Physics Institute of the CAS, Řež, Czech Republic

e-mail: strunz@ujf.cas.cz

Content:

D.1. Polynomials as solution of the angular part of the Laplace equation in the SOS coordinates

It was shown in the previous main text that $s=f_S/h_R$ function is a basis for the found simple solutions of the angular part of the Laplace equation in the SOS coordinates for K_d=0 and 1. Moreover, f_S/h_R function becomes sin(v) in the limit μ=0. At the same time, it is known that powers of sin(v) form (within the Legendre polynomials) the solution of the Laplace equation in the spherical coordinates. Therefore, we test if the angular part of the Laplace equation in the SOS coordinates for various K_d could be written generally in the form of Legendre-like polynomials, which employ f_S/h_R (i.e. the generalized sine divided by the metric scale factor of R coordinate) instead of the standard sine as an argument, i.e.

$$F_{A\beta_2}(s) = \sum_{j=\beta_1}^{\beta_2} c_j \left(\frac{f_2}{h_R}\right)^j = \sum_{j=\beta_1}^{\beta_2} c_j s^j \quad . \tag{D1}$$

Here c_j are unknown constants, which ratio we are looking for, and β_1, β_2 ($\beta_2 \geq \beta_1$) are some integer numbers determining the number and power of the solution terms.

The angular part of the Laplace equation (see (68) in the main text) can be – with this model – written in the form

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\frac{d^2F_{A\beta_2}(s)}{ds^2} + s[-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d+\mu(3-2K_d)s^2]\frac{dF_{A\beta_2}(s)}{ds} + K_d[(K_d-2)\mu s^2+(1+\mu)K_d+(1+\mu)^2]F_{A\beta_2}(s) = 0 \quad , \tag{D2}$$

Calculation of the first derivative of the model function leads to

$$\frac{dF_{A\beta_2}(s)}{ds} = \frac{d}{ds}\left[\sum_{j=\beta_1}^{\beta_2} c_j s^j\right] = \sum_{j=\beta_1}^{\beta_2} c_j j s^{j-1} \quad . \qquad \text{(D3)}$$

The second derivative can be then written as

$$\frac{d^2F_{A\beta_2}(s)}{ds^{\,2}} = \frac{d}{ds}\left[\frac{dF_{A\beta_2}(s)}{ds}\right] = \frac{d}{ds}\left[\sum_{j=\beta_1}^{\beta_2} c_j j s^{j-1}\right] = \sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} \quad .\text{(D4)}$$

Then, we insert $F_{A\beta_2}(s)$ and its derivatives to (D2) with the following result:

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} + s[-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d + \mu(3-2K_d)s^2]\sum_{j=\beta_1}^{\beta_2} c_j j s^{j-1} + K_d[(K_d-2)\mu s^2+(1+\mu)K_d+(1+\mu)^2]\sum_{j=\beta_1}^{\beta_2} c_j s^j = 0 \quad , \qquad \text{(D5)}$$

By expanding, we get

$$[(1+\mu)^3-(1+2\mu)(1+\mu)s^2+\mu s^4]\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} + [\{-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d\}s+\mu(3-2K_d)s^3]\sum_{j=\beta_1}^{\beta_2} c_j j s^{j-1} + [K_d(K_d-2)\mu s^2+K_d\{(1+\mu)K_d+(1+\mu)^2\}]\sum_{j=\beta_1}^{\beta_2} c_j s^j = 0 \quad . \qquad \text{(D6)}$$

In a series of subsequent steps of expansion and simplification, we receive

$$\left[(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} - (1+2\mu)(1+\mu)\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2}\,s^2 + \mu\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2}\,s^4\right] + \left[\{-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d\}\sum_{j=\beta_1}^{\beta_2} c_j j s^{j-1}\,s + \mu(3-2K_d)\sum_{j=\beta_1}^{\beta_2} c_j j s^{j-1}\,s^3\right] + \left[K_d(K_d-2)\mu\sum_{j=\beta_1}^{\beta_2} c_j s^j\,s^2 + K_d\{(1+\mu)K_d+(1+\mu)^2\}\sum_{j=\beta_1}^{\beta_2} c_j s^j\right] = 0 \quad , \qquad \text{(D7)}$$

$$\left[(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} - (1+2\mu)(1+\mu)\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j} + \mu\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j+2}\right] + \left[\{-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d\}\sum_{j=\beta_1}^{\beta_2} c_j j s^{j} + \mu(3-2K_d)\sum_{j=\beta_1}^{\beta_2} c_j j s^{j+2}\right] + \left[K_d(K_d-2)\mu\sum_{j=\beta_1}^{\beta_2} c_j s^{j+2} + K_d\{(1+\mu)K_d+(1+\mu)^2\}\sum_{j=\beta_1}^{\beta_2} c_j s^j\right] = 0 \quad , \qquad \text{(D8)}$$

$$\left[(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2}\right] + \left[-(1+2\mu)(1+\mu)\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j} + \{-(3\mu+2)(1+\mu) + 2\mu(1+\mu)K_d\}\sum_{j=\beta_1}^{\beta_2} c_j j s^{j} + K_d\{(1+\mu)K_d+(1+\mu)^2\}\sum_{j=\beta_1}^{\beta_2} c_j s^j\right] + \left[\mu\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j+2} + \mu(3-2K_d)\sum_{j=\beta_1}^{\beta_2} c_j j s^{j+2} + K_d(K_d-2)\mu\sum_{j=\beta_1}^{\beta_2} c_j s^{j+2}\right] = 0 \quad , \text{(D9)}$$

$$\left[(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2}\right] + (1+\mu)\left[-(1+2\mu)\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j} + \{-(3\mu+2) + 2\mu K_d\}\sum_{j=\beta_1}^{\beta_2} c_j j s^{j} + K_d\{K_d+(1+\mu)\}\sum_{j=\beta_1}^{\beta_2} c_j s^j\right] + \mu\left[\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j+2} + (3-2K_d)\sum_{j=\beta_1}^{\beta_2} c_j j s^{j+2} + K_d(K_d-2)\sum_{j=\beta_1}^{\beta_2} c_j s^{j+2}\right] = 0 \quad , \text{(D10)}$$

$$\left[(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2}\right] + (1+\mu)\left[-\sum_{j=\beta_1}^{\beta_2}(1+2\mu)c_j j(j-1)s^{j} + \sum_{j=\beta_1}^{\beta_2}\{-(3\mu+2) + 2\mu K_d\}c_j j s^{j} + \sum_{j=\beta_1}^{\beta_2} K_d\{K_d+(1+\mu)\}c_j s^j\right] + \mu\left[\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j+2} + \sum_{j=\beta_1}^{\beta_2}(3-2K_d)c_j j s^{j+2} + \sum_{j=\beta_1}^{\beta_2} K_d(K_d-2)c_j s^{j+2}\right] = 0 \quad , \qquad \text{(D11)}$$

$$\left[(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2}\right] + (1+\mu)\left[\sum_{j=\beta_1}^{\beta_2}\left[-(1+2\mu)c_j j(j-1) + \{-(3\mu+2)+2\mu K_d\}c_j j + K_d\{K_d+(1+\mu)\}c_j\right]s^j\right] + \mu\left[\sum_{j=\beta_1}^{\beta_2}\left[c_j j(j-1) + (3-2K_d)c_j j + K_d(K_d-2)c_j\right]s^{j+2}\right] = 0 \quad , \qquad \text{(D12)}$$

$$(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} + (1+\mu)\sum_{j=\beta_1}^{\beta_2} c_j[(-(1+2\mu)j^2+(1+2\mu)j)+\{-(3\mu+2)+2\mu K_d\}j+K_d\{K_d+(1+\mu)\}]s^j + \mu\sum_{j=\beta_1}^{\beta_2} c_j[(j^2-j)+(3-2K_d)j+K_d(K_d-2)]s^{j+2}=0 \quad ,\text{(D13)}$$

$$(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} + (1+\mu)\sum_{j=\beta_1}^{\beta_2} c_j[-(1+2\mu)j^2+\{(1+2\mu)-(3\mu+2)+2\mu K_d\}j+K_d\{K_d+(1+\mu)\}]s^j + \mu\sum_{j=\beta_1}^{\beta_2} c_j[j^2+(-1+3-2K_d)j+K_d(K_d-2)]s^{j+2}=0 \quad , \quad \text{(D14)}$$

$$(1+\mu)^3\sum_{j=\beta_1}^{\beta_2} c_j j(j-1)s^{j-2} + (1+\mu)\sum_{j=\beta_1}^{\beta_2} c_j[-(1+2\mu)j^2+\{2\mu K_d-(1+\mu)\}j+K_d\{K_d+(1+\mu)\}]s^j + \mu\sum_{j=\beta_1}^{\beta_2} c_j[j^2+2(1-K_d)j+K_d(K_d-2)]s^{j+2}=0 \quad , \quad \text{(D15)}$$

$$\sum_{j=\beta_1}^{\beta_2} c_j(1+\mu)^3 j(j-1)s^{j-2} + \sum_{j=\beta_1}^{\beta_2} c_j(1+\mu)[-(1+2\mu)j^2+\{2\mu K_d-(1+\mu)\}j+K_d\{K_d+(1+\mu)\}]s^2s^{j-2} + \sum_{j=\beta_1}^{\beta_2} c_j\mu[j^2+2(1-K_d)j+K_d(K_d-2)]s^4s^{j-2}=0 \quad , \quad \text{(D16)}$$

$$\sum_{j=\beta_1}^{\beta_2} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2+\{2\mu K_d-(1+\mu)\}j+K_d[K_d+(1+\mu)]]s^2 + \mu[j^2+2(1-K_d)j+K_d(K_d-2)]s^4\}s^{j-2}=0 \quad . \quad \text{(D17)}$$

This is the angular part of the Laplace equation in SOS coordinates when (D1) polynomial is used as a solution model.

First, test the case when μ is equal to zero (μ=0), i.e. the pure spherical case. Then the last term in (D17) equals to zero and we have the reduced equation:

$$\sum_{j=\beta_1}^{\beta_2} c_j\{j(j-1)+[-j^2-j+K_d[K_d+1]]s^2\}s^{j-2}=0 \quad , \quad \text{(D18)}$$

If we select $\beta_2 \geq \beta_1$ (it could be done equivalently in an opposite way, only the counting direction would change), it can be seen that the term with the pre-factor $j(j-1)$ is always the term with the lowest power when j is equal to β_1, and – moreover – it cannot be compensated by any other term as no other has that low power of s. Therefore, this term has to be zero for j equal to β_1. This restricts the bottom limit of the sum (i.e. β_1) to either 0 or 1. We conservatively select β_1=0.

Further, it can be seen that the term with the second power in s will always give in the sum the term with the largest power in s when j is equal to β_2. Moreover, it cannot be compensated by any other term as no other has that large power. Therefore, the pre-factor has to be zero itself (in order the particular $c_{\beta 2}$ could be nonzero). Then

$$-{\beta_2}^2-\beta_2+K_d[K_d+1]=0 \quad . \quad \text{(D19)}$$

It can be easily shown that two possible values of the upper limit β_2 exist in case μ=0:

$$\beta_2=K_d \quad \text{or} \quad \beta_2=-(K_d+1) \quad . \quad \text{(D20)}$$

According to our condition $\beta_2 \geq \beta_1$, and as we found that β_1 is zero, therefore K_d has to be a positive integer or zero in the first case, while it has to be a negative integer in the second case. The first case equation is thus

$$\sum_{j=0}^{K_d} c_j\{j(j-1)+[-j^2-j+K_d[K_d+1]]s^2\}s^{j-2}=0 \quad , \quad \text{(D21)}$$

while the second case equation is

$$\sum_{j=0}^{-K_d-1} c_j\{j(j-1)+[-j^2-j+K_d[K_d+1]]s^2\}s^{j-2}=0 \quad . \quad \text{(D22)}$$

When we substitute K'_d=–(K_d+1) to the second case, i.e. K'_d is positive integer or zero, we obtain

$$\sum_{j=0}^{K'_d} c_j\{j(j-1)+[-j^2-j+[K'_d+1]K'_d]s^2\}s^{j-2}=0 \ , \qquad \text{(D23)}$$

which is formally the same relation as the first case (D21), only the transformation between the variables K'_d and K_d has to be taken into account when solving the the full Laplace equation (i.e. also the radial part). This findings lead to the well-known result for the spherical case which allows both the positive and the negative integer exponent for the radial coordinate in the solution of the radial part of the separated Laplace equation, and thus both solution for the interior region and for the exterior region up to infinity.

Now, we come back to the general case, i.e. to nonzero value of the parameter μ ($\mu \neq 0$), and thus to the solution in SOS coordinates. Situation is here significantly different than for the spherical case, as the term with s^4 in (D17) is now non-zero.

It can be seen that the term with the pre-factor $j(j-1)(1+\mu)^3$ in (D17) will be always the term with the lowest power when j is equal to β_1, and – moreover – it cannot be compensated by any other term in the sum as no other has that low power. Therefore, it has to be zero for j equal to β_1. Similarly, this term cannot be compensated by any other term (as no other has that power of s) also for j equal to β_1+1. This restricts the bottom limit of the sum β_1 to 0. In such case ($\beta_1=0$), both abovementioned terms are zero thanks to the pre factor $j(j-1)$. Thus, we select $\beta_1=0$ in what follows.

Further, it can be seen that the term with the fourth power in s will always give in the sum the term with the largest power in $s=f_S/h_R$ when j is equal to β_2. Moreover, it cannot be compensated by any other term in the sum as no other has that large power. Therefore, the pre-factor has to be zero itself (in order the particular c_{β_2} could be nonzero). It can be easily shown that in order to fulfill

$$\mu[\beta_2^2+2(1-K_d)\beta_2+K_d(K_d-2)]=0 \ , \qquad \text{(D24)}$$

the roots has to be $\beta_2=K_d$ or $\beta_2=K_d-2$. Moreover, it can be seen that the term with the fourth power in s in (D17) will always give in the sum the term with the second largest power in s when j is equal to β_2-1. It cannot be compensated by any other term as no other has that power. It can be easily shown that, in order to fulfill

$$\mu[(\beta_2-1)^2+2(1-K_d)(\beta_2-1)+K_d(K_d-2)]=0 \ , \qquad \text{(D25)}$$

then the roots has to be $\beta_2=K_d\pm1$, or c_{β_2-1} has to be zero. As the roots for the abovementioned cases $j=\beta_2$ and $j=\beta_2-1$ are mutually incompatible, it is clear that c_{β_2-1} has to be zero, and the upper limit of the sum β_2 is either K_d or K_d-2.

This restricts the upper limit of the sum β_2 to either K_d or K_d-2. We conservatively select the larger one, i.e. K_d. Moreover, as the upper limit of the sum β_2 has to be an integer number, then K_d has to be integer, too. Further, it has to be positive or zero (i.e. $K_d\geq0$), according to our condition $\beta_2\geq\beta_1$. Thus, unlike in the spherical case, K_d cannot be negative. This is a significant result, as it is different from the spherical case where negative K_d is also allowed. It means that (D1) model polynomial cannot lead to a solution for any negative separation constant K_d, and further search for solutions with negative exponents K_d in the radial part (i.e. in fact for the exterior points) has still later to be done for the spheroidal case.

We further employ the above derived limits in the sum of (D17), and we then have the basic equation for (D1) model in the form

$$\sum_{j=0}^{K_d} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{2\mu K_d - (1+\mu)\}j + K_d[K_d + (1+\mu)]]s^2 + \mu[j^2 + 2(1-K_d)j + K_d(K_d-2)]s^4\}s^{j-2} = 0 \quad , \tag{D26}$$

D.2. Special cases: K_d=2, K_d=3, K_d=4, K_d=5 and K_d=6

We already know the solution in the form of polynomial for K_d=0 and 1 (see (71) and (67) in the main text). Now, we try to find, using (D26) the solutions for higher values of the positive integer separation constant K_d.

First, let's search for a solution for K_d=2. For this value of the separation constant, (D26) becomes

$$\sum_{j=0}^{2} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{2\mu\cdot 2 - (1+\mu)\}j + 2[2 + (1+\mu)]]s^2 + \mu[j^2 + 2(1-2)j + 2(2-2)]s^4\}s^{j-2} = 0 \quad , \tag{D27}$$

and subsequent algebraic manipulations lead to

$$\sum_{j=0}^{2} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{4\mu - (1+\mu)\}j + 2(3+\mu)]s^2 + \mu[j^2 - 2j]s^4\}s^{j-2} = 0\,, \tag{D28}$$

$$\sum_{j=0}^{2} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{3\mu - 1\}j + (6+2\mu)]s^2 + \mu[j^2 - 2j]s^4\}s^{j-2} = 0, \tag{D29}$$

$$c_0\{(1+\mu)^3 0(0-1) + (1+\mu)[-(1+2\mu)0^2 + \{3\mu-1\}0 + (6+2\mu)]s^2 + \mu[0^2 - 2\cdot 0]s^4\}s^{0-2} + c_1\{(1+\mu)^3(1-1) + (1+\mu)[-(1+2\mu)1^2 + \{3\mu-1\} + (6+2\mu)]s^2 + \mu[1^2-2]s^4\}s^{1-2} + c_2\{(1+\mu)^3 2(2-1) + (1+\mu)[-(1+2\mu)2^2 + \{3\mu-1\}2 + (6+2\mu)]s^2 + \mu[2^2 - 2\cdot 2]s^4\}s^{2-2} = 0, \tag{D30}$$

$$c_0\{(1+\mu)(6+2\mu)s^2\}s^{-2} + c_1\{(1+\mu)^3 0 + (1+\mu)[-(1+2\mu) + \{3\mu-1\} + (6+2\mu)]s^2 - \mu s^4\}s^{-1} + c_2\{2(1+\mu)^3 + (1+\mu)[-4(1+2\mu) + 2\{3\mu-1\} + (6+2\mu)]s^2 + \mu[0]s^4\}s^0 = 0 \quad , \tag{D31}$$

$$c_0(1+\mu)(6+2\mu) + c_1\{(1+\mu)[4+3\mu]s^2 - \mu s^4\}s^{-1} + c_2\{2(1+\mu)^3 + (1+\mu)[(-4-8\mu) + \{6\mu-2\} + (6+2\mu)]s^2\} = 0 \quad , \tag{D32}$$

$$c_0(1+\mu)(6+2\mu) + \{c_1(1+\mu)[4+3\mu]s - c_1\mu s^3\} + c_2\{2(1+\mu)^3 + (1+\mu)[0]s^2\} = 0 \quad , \tag{D33}$$

$$c_0(1+\mu)(6+2\mu) + c_1(1+\mu)[4+3\mu]s - c_1\mu s^3 + 2c_2(1+\mu)^3 = 0 \quad , \tag{D34}$$

$$c_0(1+\mu)(6+2\mu) + c_1(1+\mu)[4+3\mu]s - c_1\mu s^3 + 2c_2(1+\mu)^3 = 0 \quad , \tag{D35}$$

The pre-factors by the first and the third power of s have to be clearly zero. Therefore, c_1=0 and

$$c_0(1+\mu)(6+2\mu) + 2c_2(1+\mu)^3 = 0 \quad . \tag{D36}$$

Then we have the equation for the coefficients in the form

$$c_0(3+\mu) + c_2(1+\mu)^2 = 0 \quad , \tag{D37}$$

which results in the ratio of the coefficients

$$\frac{c_2}{c_0} = -\frac{\mu+3}{(1+\mu)^2} \quad . \tag{D38}$$

Then, for K_d=2 and with an initial normalization which is for μ=0 the same as for the spherical case (i.e. the polynomial has value 1 when the variable is 1),

$$c_0 = -\frac{1}{2}(1+\mu)^2, \quad c_1 = 0, \quad c_2 = \frac{1}{2}(\mu+3) \,. \tag{D39}$$

Then, the polynomial solution is

$$F_{A2}(s) = \frac{1}{2}[(\mu+3)s^2 - (1+\mu)^2] \quad . \tag{D40}$$

Secondly, we perform the same procedure for K_d=3. By inserting this value to (D26), we obtain

$$\sum_{j=0}^{3} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)\left[-(1+2\mu)j^2 + \{2\mu\cdot 3 - (1+\mu)\}j + 3[3+(1+\mu)]\right]s^2 + \mu[j^2 + 2(1-3)j + 3(3-2)]s^4\}s^{j-2} = 0 \ , \tag{D41}$$

$$\sum_{j=0}^{3} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{5\mu-1\}j + 3(4+\mu)]s^2 + \mu(j^2 - 4j + 3)s^4\}s^{j-2} = 0 \ , \tag{D42}$$

$$c_0\{(1+\mu)^3 0(0-1) + (1+\mu)[-(1+2\mu)0^2 + \{5\mu-1\}0 + 3(4+\mu)]s^2 + \mu(0^2 - 4\cdot 0 + 3)s^4\}s^{0-2} + c_1\{(1+\mu)^3(1-1) + (1+\mu)[-(1+2\mu)1^2 + \{5\mu-1\} + 3(4+\mu)]s^2 + \mu(1^2 - 4 + 3)s^4\}s^{1-2} + c_2\{(1+\mu)^3 2(2-1) + (1+\mu)[-(1+2\mu)2^2 + \{5\mu-1\}2 + 3(4+\mu)]s^2 + \mu(2^2 - 4\cdot 2 + 3)s^4\}s^{2-2} + c_3\{(1+\mu)^3 3(3-1) + (1+\mu)[-(1+2\mu)3^2 + \{5\mu-1\}3 + 3(4+\mu)]s^2 + \mu(3^2 - 4\cdot 3 + 3)s^4\}s^{3-2} = 0 \ , \tag{D43}$$

$$c_0\{(1+\mu)[3(4+\mu)]s^2 + 3\mu s^4\}s^{-2} + c_1\{(1+\mu)[-1 - 2\mu + 5\mu - 1 + 12 + 3\mu]s^2 + \mu(0)s^4\}s^{-1} + c_2\{2(1+\mu)^3 + (1+\mu)[-(4+8\mu) + \{10\mu-2\} + (12+3\mu)]s^2 - \mu s^4\}s^0 + c_3\{6(1+\mu)^3 + (1+\mu)[-9 - 18\mu + 15\mu - 3 + 12 + 3\mu]s^2 + \mu(0)s^4\}s^1 = 0 \ , \tag{D44}$$

$$c_0\{3(1+\mu)(4+\mu)s^2 + 3\mu s^4\}s^{-2} + c_1\{(1+\mu)(6\mu+10)s^2\}s^{-1} + c_2\{2(1+\mu)^3 + (1+\mu)[-4 - 8\mu + 10\mu - 2 + 12 + 3\mu]s^2 - \mu s^4\} + c_3\{6(1+\mu)^3 + (1+\mu)[0]s^2\}s = 0 \ , \tag{D45}$$

$$\{3c_0(1+\mu)(4+\mu) + 3c_0\mu s^2\} + c_1(1+\mu)(6\mu+10)s^1 + c_2\{2(1+\mu)^3 + (1+\mu)[5\mu+6]s^2 - \mu s^4\} + c_3 6(1+\mu)^3 s = 0 \ , \tag{D46}$$

$$3c_0(1+\mu)(4+\mu) + 3c_0\mu s^2 + c_1(1+\mu)(6\mu+10)s + 2c_2(1+\mu)^3 + c_2(1+\mu)(5\mu+6)s^2 - c_2\mu s^4 + 6c_3(1+\mu)^3 s = 0 \ , \tag{D47}$$

The pre-factors by the fourth power of s has to be clearly zero, as the term with that power is only one, and cannot thus re-combine with another one. Therefore, c_2=0. We also put together the same power terms of s and have

$$3c_0(1+\mu)(4+\mu) + 3c_0\mu s^2 + c_1(1+\mu)(6\mu+10)s + 6c_3(1+\mu)^3 s = 0 \ , \tag{D48}$$

It is further clear that c_0=0, as the second power term cannot be zero otherwise. Then the remaining relation is

$$c_1(1+\mu)(6\mu+10)s + 6c_3(1+\mu)^3 s = 0 \ , \tag{D49}$$

and thus

$$\frac{c_3}{c_1} = -\frac{3\mu+5}{3(1+\mu)^2} \quad . \tag{D50}$$

Then, for K_d=3 and with normalization selected in such a way that it is for μ=0 the same as for the spherical case (i.e. the polynomial has value 1 when the variable is 1), we have the coefficients

$$c_0 = 0, \quad c_1 = -\frac{1}{2}3(1+\mu)^2, \quad c_2 = 0, \quad c_3 = \frac{1}{2}(3\mu+5) \ , \tag{D51}$$

Then, the polynomial solution is

$$F_{A3}(s) = \frac{1}{2}[(3\mu+5)s^3 - 3(1+\mu)^2 s] \quad . \tag{D52}$$

Further, let's search for a solution for K_d=4. For this value of the separation constant, (D26) becomes

$$\sum_{j=0}^{4} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)\big[-(1+2\mu)j^2 + \{2\mu\cdot 4 - (1+\mu)\}j + 4\cdot[4+(1+\mu)]\big]s^2 + \mu[j^2 + 2(1-4)j + 4\cdot(4-2)]s^4\}s^{j-2} = 0 \quad , \tag{D53}$$

$$\sum_{j=0}^{K_4} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)\big[-(1+2\mu)j^2 + \{8\mu - (1+\mu)\}j + 4\cdot[5+\mu]\big]s^2 + \mu[j^2 + 2(-3)j + 4\cdot(2)]s^4\}s^{j-2} = 0 \quad , \tag{D54}$$

$$\sum_{j=0}^{4} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{7\mu-1\}j + 20 + 4\mu]s^2 + \mu[j^2 - 6j + 8]s^4\}s^{j-2} = 0 \quad , \tag{D55}$$

$$\begin{aligned}
&c_0\{(1+\mu)^3 0\cdot(0-1) + (1+\mu)[-(1+2\mu)\cdot 0^2 + \{7\mu-1\}\cdot 0 + 20 + 4\mu]s^2 + \mu[0^2 - 6\cdot 0 + 8]s^4\}s^{0-2} + \\
&c_1\{(1+\mu)^3 1\cdot(1-1) + (1+\mu)[-(1+2\mu)\cdot 1^2 + \{7\mu-1\}\cdot 1 + 20 + 4\mu]s^2 + \mu[1^2 - 6\cdot 1 + 8]s^4\}s^{1-2} + \\
&c_2\{(1+\mu)^3 2\cdot(2-1) + (1+\mu)[-(1+2\mu)\cdot 2^2 + \{7\mu-1\}\cdot 2 + 20 + 4\mu]s^2 + \mu[2^2 - 6\cdot 2 + 8]s^4\}s^{2-2} + \\
&c_3\{(1+\mu)^3 3\cdot(3-1) + (1+\mu)[-(1+2\mu)\cdot 3^2 + \{7\mu-1\}\cdot 3 + 20 + 4\mu]s^2 + \mu[3^2 - 6\cdot 3 + 8]s^4\}s^{3-2} + \\
&c_4\{(1+\mu)^3 4\cdot(4-1) + (1+\mu)[-(1+2\mu)\cdot 4^2 + \{7\mu-1\}\cdot 4 + 20 + 4\mu]s^2 + \mu[4^2 - 6\cdot 4 + 8]s^4\}s^{4-2} = 0 \quad ,
\end{aligned} \tag{D56}$$

$$c_0\{(1+\mu)^3 0\cdot(-1) + (1+\mu)[-(1+2\mu)\cdot 0 + 0 + 20 + 4\mu]s^2 + \mu[0 - 0 + 8]s^4\}s^{-2} + c_1\{(1+\mu)^3 1\cdot(0) + (1+\mu)[-(1+2\mu)\cdot 1 + \{7\mu-1\} + 20 + 4\mu]s^2 + \mu[1 - 6 + 8]s^4\}s^{-1} + c_2\{(1+\mu)^3 2\cdot(1) + (1+\mu)[-(1+2\mu)\cdot 4 + \{14\mu-2\} + 20 + 4\mu]s^2 + \mu[4 - 12 + 8]s^4\}s^0 + c_3\{(1+\mu)^3 3\cdot(2) + (1+\mu)[-(1+2\mu)\cdot 9 + \{21\mu-3\} + 20 + 4\mu]s^2 + \mu[9 - 18 + 8]s^4\}s^1 + c_4\{(1+\mu)^3 4\cdot(3) + (1+\mu)[-(1+2\mu)\cdot 16 + \{28\mu-4\} + 20 + 4\mu]s^2 + \mu[16 - 24 + 8]s^4\}s^2 = 0 \quad , \tag{D57}$$

$$c_0\{0 + (1+\mu)[0 + 0 + 20 + 4\mu]s^2 + \mu[8]s^4\}s^{-2} + c_1\{0 + (1+\mu)[-(1+2\mu) + \{7\mu-1\} + 20 + 4\mu]s^2 + \mu[1 - 6 + 8]s^4\}s^{-1} + c_2\{2(1+\mu)^3 + (1+\mu)[-(4+8\mu) + \{14\mu-2\} + 20 + 4\mu]s^2 + \mu[4 - 12 + 8]s^4\}s^0 + c_3\{6(1+\mu)^3 + (1+\mu)[-(9+18\mu) + \{21\mu-3\} + 20 + 4\mu]s^2 + \mu[9 - 18 + 8]s^4\}s^1 + c_4\{12(1+\mu)^3 + (1+\mu)[-(16+32\mu) + \{28\mu-4\} + 20 + 4\mu]s^2 + \mu[16 - 24 + 8]s^4\}s^2 = 0 \quad , \tag{D58}$$

$$c_0\{(1+\mu)[20+4\mu]s^2 + 8\mu s^4\}s^{-2} + c_1\{(1+\mu)[18+9\mu]s^2 + \mu[3]s^4\}s^{-1} + c_2\{2(1+\mu)^3 + (1+\mu)[14 + 10\mu]s^2 + \mu[0]s^4\}s^0 + c_3\{6(1+\mu)^3 + (1+\mu)[8+7\mu]s^2 + \mu[-1]s^4\}s^1 + c_4\{12(1+\mu)^3 + (1+\mu)[0]s^2 + \mu[0]s^4\}s^2 = 0 \quad , \tag{D59}$$

$$c_0\{(1+\mu)[20+4\mu]s^2 s^{-2} + 8\mu s^4 s^{-2}\} + c_1\{(1+\mu)[18+9\mu]s^2 s^{-1} + 3\mu s^4 s^{-1}\} + c_2\{2(1+\mu)^3 s^0 + (1+\mu)[14+10\mu]s^2 s^0\} + c_3\{6(1+\mu)^3 s^1 + (1+\mu)[8+7\mu]s^2 s^1 - \mu s^4 s^1\} + c_4\{12(1+\mu)^3 s^2\} = 0 \quad , \tag{D60}$$

$$\{(1+\mu)[20+4\mu]c_0 + 8\mu s^2 c_0\} + \{(1+\mu)[18+9\mu]c_1 s^1 + 3\mu c_1 s^3\} + \{2(1+\mu)^3 c_2 + (1+\mu)[14 + 10\mu]c_2 s^2\} + \{6(1+\mu)^3 c_3 s^1 + (1+\mu)[8+7\mu]c_3 s^3 - \mu c_3 s^5\} + \{12(1+\mu)^3 c_4 s^2\} = 0 \quad , \tag{D61}$$

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} + \{(1+\mu)[18+9\mu]c_1 s^1 + 6(1+\mu)^3 c_3 s^1\} + \{8\mu c_0 s^2 + (1+\mu)[14 + 10\mu]c_2 s^2 + 12(1+\mu)^3 c_4 s^2\} + \{+3\mu c_1 s^3 + (1+\mu)[8+7\mu]c_3 s^3\} - \mu c_3 s^5 = 0 \quad , \tag{D62}$$

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} + \{(1+\mu)[18+9\mu]c_1 + 6(1+\mu)^3 c_3\}s^1 + \{8\mu c_0 + (1+\mu)[14+10\mu]c_2 + 12(1+\mu)^3 c_4\}s^2 + \{+3\mu c_1 + (1+\mu)[8+7\mu]c_3\}s^3 - \mu c_3 s^5 = 0 \quad , \tag{D63}$$

The pre-factors by the last term has to be clearly zero, as the term with that power is only one, and cannot thus re-combine with another one. Therefore,

$$-\mu c_3 s^5 = 0 \qquad \Rightarrow \qquad c_3 = 0 \qquad , \tag{D64}$$

Then

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} + \{(1+\mu)[18+9\mu]c_1 + 6(1+\mu)^3 \cdot 0\}s^1 + \{8\mu c_0 + (1+\mu)[14+10\mu]c_2 + 12(1+\mu)^3 c_4\}s^2 + \{+3\mu c_1 + (1+\mu)[8+7\mu]\cdot 0\}s^3 - \mu \cdot 0 s^5 = 0 \ , \tag{D65}$$

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} + \{(1+\mu)[18+9\mu]c_1 + 0\}s^1 + \{8\mu c_0 + (1+\mu)[14+10\mu]c_2 + 12(1+\mu)^3 c_4\}s^2 + \{+3\mu c_1 + 0\}s^3 - 0 = 0 \ , \tag{D66}$$

and thus

$$\{+3\mu c_1 + 0\} = 0 \qquad \Rightarrow \qquad c_1 = 0 \quad , \tag{D67}$$

Further simplification of the equation leads to

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} + \{(1+\mu)[18+9\mu]\cdot 0 + 0\}s^1 + \{8\mu c_0 + (1+\mu)[14+10\mu]c_2 + 12(1+\mu)^3 c_4\}s^2 + \{+3\mu \cdot 0 + 0\}s^3 - 0 = 0 \ , \tag{D68}$$

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} + 0 + \{8\mu c_0 + (1+\mu)[14+10\mu]c_2 + 12(1+\mu)^3 c_4\}s^2 + 0 - 0 = 0 \ , \tag{D69}$$

where clearly the first term has to fulfill

$$\{(1+\mu)[20+4\mu]c_0 + 2(1+\mu)^3 c_2\} = 0 \quad , \tag{D70}$$

and then

$$\frac{c_2}{c_0} = -\frac{2[5+\mu]}{(1+\mu)^2} \qquad . \tag{D71}$$

The power two in s of (D69) has to be zero as well; thus

$$\{8\mu c_0 + (1+\mu)[14+10\mu]c_2 + 12(1+\mu)^3 c_4\}s^2 = 0 \ , \tag{D72}$$

$$8\mu c_0 + (1+\mu)[14+10\mu]\left\{-\frac{2[5+\mu]}{(1+\mu)^2}\right\}c_0 + 12(1+\mu)^3 c_4 = 0 \ , \tag{D73}$$

$$\left\{8\mu - 2[7+5\mu]\frac{2[5+\mu]}{(1+\mu)}\right\}c_0 + 12(1+\mu)^3 c_4 = 0 \ , \tag{D74}$$

$$\left\{\frac{8\mu(1+\mu) - 2[7+5\mu]2[5+\mu]}{(1+\mu)}\right\}c_0 + 12(1+\mu)^3 c_4 = 0 \ , \tag{D75}$$

$$4\frac{(2\mu+2\mu\mu) - 5\mu\mu - 32\mu - 35}{(1+\mu)}c_0 + 12(1+\mu)^3 c_4 = 0 \ , \tag{D76}$$

$$\frac{-3\mu\mu - 30\mu - 35}{(1+\mu)}c_0 + 3(1+\mu)^3 c_4 = 0 \ , \tag{D77}$$

$$\frac{c_4}{c_0} = \frac{1}{3}\frac{(3\mu\mu + 30\mu + 35)}{(1+\mu)^4} \quad , \tag{D78}$$

Then, for K_d=4, and with an initial normalization which is for μ=0 the same as for the spherical case (i.e. the polynomial has value 1 when the variable is 1),

$$c_0 = 3(1+\mu)^4, \quad c_1 = 0, \quad c_2 = -\frac{2[5+\mu]}{(1+\mu)^2}c_0, \quad c_3 = 0, \quad c_4 = \frac{1}{3}\frac{(3\mu\mu+30\mu+35)}{(1+\mu)^4}c_0 \ . \tag{D79}$$

Therefore, one solution is (by inserting to (D63))

$$\left[\frac{1}{3}\frac{(3\mu\mu+30\mu+35)}{(1+\mu)^4}c_0 s^4 - \frac{2(5+\mu)}{(1+\mu)^2}c_0 s^2 + 3(1+\mu)^4\right] \ , \tag{D80}$$

and the general polynomial solution for K_d=4 is

$$F_{A4}(s) = const.[(3\mu\mu + 30\mu + 35)s^4 - 2\cdot 3(\mu+5)(1+\mu)^2 s^2 + 3(1+\mu)^4] \quad . \tag{D82}$$

Let's search also for the solution for K_d=5. For this value of the separation constant, (D26) becomes

$$\sum_{j=0}^{5} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{2\mu \cdot 5 - (1+\mu)\}j + 5\cdot[5 + (1+\mu)]]s^2 + \mu[j^2 + 2(1-5)j + 5\cdot(5-2)]s^4\}s^{j-2} = 0 \ , \tag{D83}$$

$$\sum_{j=0}^{5} c_j\big\{(1+\mu)^3 j(j-1) + (1+\mu)\big[-(1+2\mu)j^2 + \{10\mu-(1+\mu)\}j + 5\cdot[6+\mu]\big]s^2 + \mu[j^2 + 2(-4)j + 5\cdot(3)]s^4\big\}s^{j-2} = 0 \quad , \tag{D84}$$

$$\sum_{j=0}^{5} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{10\mu-(1+\mu)\}j + 30 + 5\mu]s^2 + \mu[j^2 + 2(-4)j + 15]s^4\}s^{j-2} = 0 \quad , \tag{D85}$$

$$\begin{aligned}
&c_0\{(1+\mu)^3 0\cdot(0-1) + (1+\mu)[-(1+2\mu)\cdot 0^2 + \{9\mu-1\}\cdot 0 + 30 + 5\mu]s^2 + \mu[0^2 - 8\cdot 0 + 15]s^4\}s^{0-2} + \\
&c_1\{(1+\mu)^3 1\cdot(1-1) + (1+\mu)[-(1+2\mu)\cdot 1^2 + \{9\mu-1\}\cdot 1 + 30 + 5\mu]s^2 + \mu[1^2 - 8\cdot 1 + 15]s^4\}s^{1-2} + \\
&c_2\{(1+\mu)^3\cdot 2(2-1) + (1+\mu)[-(1+2\mu)\cdot 2^2 + \{9\mu-1\}\cdot 2 + 30 + 5\mu]s^2 + \mu[2^2 - 8\cdot 2 + 15]s^4\}s^{2-2} + \\
&c_3\{(1+\mu)^3\cdot 3(3-1) + (1+\mu)[-(1+2\mu)\cdot 3^2 + \{9\mu-1\}\cdot 3 + 30 + 5\mu]s^2 + \mu[3^2 - 8\cdot 3 + 15]s^4\}s^{3-2} + \\
&c_4\{(1+\mu)^3\cdot 4(4-1) + (1+\mu)[-(1+2\mu)\cdot 4^2 + \{9\mu-1\}\cdot 4 + 30 + 5\mu]s^2 + \mu[4^2 - 8\cdot 4 + 15]s^4\}s^{4-2} + \\
&c_5\{(1+\mu)^3\cdot 5(5-1) + (1+\mu)[-(1+2\mu)\cdot 5^2 + \{10\mu-(1+\mu)\}\cdot 5 + 30 + 5\mu]s^2 + \mu[5^2 - 8\cdot 5 + 15]s^4\}s^{5-2} = 0 \quad ,
\end{aligned} \tag{D86}$$

$$\begin{aligned}
&c_0\{0 + (1+\mu)[0 + 0 + 30 + 5\mu]s^2 + \mu[0 - 0 + 15]s^4\}s^{-2} + c_1\{(1+\mu)^3 1\cdot 0 + (1+\mu)[-(1+2\mu) + \{9\mu-1\} + 30 + 5\mu]s^2 + \\
&\mu[1 - 8 + 15]s^4\}s^{-1} + c_2\{(1+\mu)^3\cdot 2 + (1+\mu)[-(1+2\mu)\cdot 4 + \{9\mu-1\}\cdot 2 + 30 + 5\mu]s^2 + \mu[4 - 16 + 15]s^4\}s^0 + \\
&c_3\{(1+\mu)^3\cdot 3\cdot 2 + (1+\mu)[-(1+2\mu)\cdot 9 + \{9\mu-1\}\cdot 3 + 30 + 5\mu]s^2 + \mu[9 - 24 + 15]s^4\}s^1 + c_4\{(1+\mu)^3\cdot 4\cdot 3 + \\
&(1+\mu)[-(1+2\mu)\cdot 16 + \{9\mu-1\}\cdot 4 + 30 + 5\mu]s^2 + \mu[16 - 32 + 15]s^4\}s^2 + c_5\{(1+\mu)^3\cdot 5\cdot 4 + (1+\mu)[-(1+2\mu)\cdot \\
&25 + \{9\mu-1\}\cdot 5 + 30 + 5\mu]s^2 + \mu[25 - 40 + 15]s^4\}s^3 = 0 \quad ,
\end{aligned} \tag{D87}$$

$$\begin{aligned}
&c_0\{(1+\mu)[30 + 5\mu]s^2 s^{-2} + \mu[15]s^4 s^{-2}\} + c_1\{(1+\mu)[12\mu + 28]s^2 s^{-1} + \mu[8]s^4 s^{-1}\} + c_2\{(1+\mu)^3\cdot 2s^0 + (1+ \\
&\mu)[-(4+8\mu) + \{18\mu - 2\} + 30 + 5\mu]s^2 s^0 + \mu[3]s^4 s^0\} + c_3\{(1+\mu)^3\cdot 6s^1 + (1+\mu)[-(9+18\mu) + \{27\mu - 3\} + 30 + \\
&5\mu]s^2 s^1 + \mu[0]s^4 s^1\} + c_4\{(1+\mu)^3\cdot 12s^2 + (1+\mu)[(-16-32\mu) + \{36\mu - 4\} + 30 + 5\mu]s^2 s^2 + \mu[-1]s^4 s^2\} + \\
&c_5\{(1+\mu)^3\cdot 20s^3 + (1+\mu)[-(1+2\mu)\cdot 25 + \{45\mu - 5\} + 30 + 5\mu]s^2 s^3 + \mu[0]s^4 s^3\} = 0 \quad ,
\end{aligned} \tag{D88}$$

$$\begin{aligned}
&\{(1+\mu)[30 + 5\mu]c_0 + 15\mu c_0 s^2\} + \{(1+\mu)[12\mu + 28]c_1 s^1 + 8\mu c_1 s^3\} + \{(1+\mu)^3\cdot 2c_2 + (1+\mu)[15\mu + 24]c_2 s^2 + \\
&3\mu c_2 s^4\} + \{(1+\mu)^3\cdot 6c_3 s^1 + (1+\mu)[14\mu + 18]c_3 s^3 + \mu[0]c_3 s^5\} + \{(1+\mu)^3\cdot 12c_4 s^2 + (1+\mu)[9\mu + 10]c_4 s^4 - \\
&\mu c_4 s^6\} + \{(1+\mu)^3\cdot 20c_5 s^3 + (1+\mu)[0]c_5 s^5 + \mu[0]c_5 s^7\} = 0 \quad ,
\end{aligned} \tag{D89}$$

$$\begin{aligned}
&\{(1+\mu)[30 + 5\mu]c_0 + (1+\mu)^3\cdot 2c_2\} + \{(1+\mu)[12\mu + 28]c_1 s^1 + (1+\mu)^3\cdot 6c_3 s^1\} + \{+15\mu c_0 s^2 + (1+\mu)[15\mu + \\
&24]c_2 s^2 + (1+\mu)^3\cdot 12c_4 s^2\} + \{+8\mu c_1 s^3 + (1+\mu)[14\mu + 18]c_3 s^3 + (1+\mu)^3\cdot 20c_5 s^3\} + \{+3\mu c_2 s^4 + (1+\mu)[9\mu + \\
&10]c_4 s^4\} + \{-\mu c_4 s^6\} = 0 \quad .
\end{aligned} \tag{D90}$$

The last term with s-power of six has to be zero and then

$$\{-\mu c_4 s^6\} = 0 \qquad \Rightarrow \qquad c_4 = 0 \quad . \tag{D91}$$

The term with with s-power of four in (D88) has to be zero as well:

$$\{+3\mu c_2 s^4 + (1+\mu)[9\mu + 10]c_4 s^4\} = 0 \qquad \Rightarrow \qquad c_2 = 0 \quad . \tag{D92}$$

The term with with s-power of two in (D88) also has to be zero:

$$\{+15\mu c_0 s^2 + (1+\mu)[15\mu + 24]c_2 s^2 + (1+\mu)^3\cdot 12c_4 s^2\} = 0 \qquad \Rightarrow \qquad c_0 = 0 \ , \tag{D93}$$

Thus (D88) simplifies to

$$\{(1+\mu)[12\mu + 28]c_1 s^1 + (1+\mu)^3\cdot 6c_3 s^1\} + \{+8\mu c_1 s^3 + (1+\mu)[14\mu + 18]c_3 s^3 + (1+\mu)^3\cdot 20c_5 s^3\} = 0 \quad , \tag{D94}$$

Then the two equations arise from (D92):

$$\{(1+\mu)[12\mu + 28]c_1 s^1 + (1+\mu)^3\cdot 6c_3 s^1\} = 0 \quad , \tag{D95}$$

$$\{+8\mu c_1 s^3 + (1+\mu)[14\mu + 18]c_3 s^3 + (1+\mu)^3\cdot 20c_5 s^3\} = 0 \quad , \tag{D96}$$

and simplification leads to

$$[6\mu + 14]c_1 + (1+\mu)^2\cdot 3c_3 = 0 \quad , \tag{D97}$$

$$4\mu c_1 + (1+\mu)[7\mu + 9]c_3 + (1+\mu)^3\cdot 10c_5 = 0 \quad . \tag{D98}$$

The ratio of the third and the first coefficient is then

$$\frac{c_3}{c_1} = -\frac{[6\mu+14]}{3(1+\mu)^{2\cdot}} \ , \tag{D99}$$

and (D96) is modified as follows:

$$4\mu + (1+\mu)[7\mu+9]\frac{c_3}{c_1} + (1+\mu)^3 \cdot 10\frac{c_5}{c_1} = 0 \ , \tag{D100}$$

$$10(1+\mu)^3\frac{c_5}{c_1} = -4\mu + (1+\mu)[7\mu+9]\frac{[6\mu+14]}{3(1+\mu)^{2\cdot}} \ , \tag{D101}$$

$$10(1+\mu)^3\frac{c_5}{c_1} = \frac{-4\mu 3(1+\mu)+2[3\mu+7][7\mu+9]}{3(1+\mu)} = 2\frac{-6\mu(1+\mu)+[3\mu+7][7\mu+9]}{3(1+\mu)} = 2\frac{(-6\mu-6\mu\mu)+[21\mu\mu+27\mu+49\mu+63]}{3(1+\mu)} = 2\frac{[15\mu\mu+70\mu+63]}{3(1+\mu)} \ . \tag{D102}$$

Then the ratio of the fifth and the first coefficient is

$$\frac{c_5}{c_1} = \frac{[15\mu\mu+70\mu+63]}{15(1+\mu)^4} \ . \tag{D103}$$

Therefore (by inserting to (D26)) the polynomial

$$\left[\frac{[15\mu\mu+70\mu+63]}{15(1+\mu)^4}s^5 - \frac{[6\mu+14]}{3(1+\mu)^{2\cdot}}s^3 + s^1\right], \tag{D104}$$

represents one solution, and the general polynomial solution for K_d=5 is

$$F_{A5}(s) = const.[(15\mu\mu + 70\mu + 63)s^5 - 10(3\mu+7)(1+\mu)^2 s^3 + 15(1+\mu)^4 s^1] \ . \tag{D105}$$

Further, we try to find, using (D26), the solutions for K_d=6 value of the positive integer separation constant K_d. For this value of the separation constant, (D26) becomes

$$\sum_{j=0}^{6} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{2\mu\cdot 6 - (1+\mu)\}j + 6\cdot[6+(1+\mu)]]s^2 + \mu[j^2 + 2(1-6)j + 6\cdot(6-2)]s^4\}s^{j-2} = 0 \ , \tag{D106}$$

$$\sum_{j=0}^{6} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{12\mu - (1+\mu)\}j + 6\cdot[7+\mu]]s^2 + \mu[j^2 + 2(-5)j + 6\cdot(4)]s^4\}s^{j-2} = 0 \ , \tag{D107}$$

$$\sum_{j=0}^{6} c_j\{(1+\mu)^3 j(j-1) + (1+\mu)[-(1+2\mu)j^2 + \{11\mu - 1\}j + [42+6\mu]]s^2 + \mu[j^2 - 10j + 24]s^4\}s^{j-2} = 0 \ , \tag{D108}$$

$$\begin{aligned}
&c_0\{(1+\mu)^3\cdot 0(0-1) + (1+\mu)[-(1+2\mu)0^2 + \{11\mu-1\}\cdot 0 + [42+6\mu]]s^2 + \mu[0^2 - 10\cdot 0 + 24]s^4\}s^{0-2} +\\
&c_1\{(1+\mu)^3\cdot 1(1-1) + (1+\mu)[-(1+2\mu)1^2 + \{11\mu-1\}\cdot 1 + [42+6\mu]]s^2 + \mu[1^2 - 10\cdot 1 + 24]s^4\}s^{1-2} +\\
&c_2\{(1+\mu)^3\cdot 2(2-1) + (1+\mu)[-(1+2\mu)2^2 + \{11\mu-1\}\cdot 2 + [42+6\mu]]s^2 + \mu[2^2 - 10\cdot 2 + 24]s^4\}s^{2-2} +\\
&c_3\{(1+\mu)^3\cdot 3(3-1) + (1+\mu)[-(1+2\mu)3^2 + \{11\mu-1\}\cdot 3 + [42+6\mu]]s^2 + \mu[3^2 - 10\cdot 3 + 24]s^4\}s^{3-2} +\\
&c_4\{(1+\mu)^3\cdot 4(4-1) + (1+\mu)[-(1+2\mu)4^2 + \{11\mu-1\}\cdot 4 + [42+6\mu]]s^2 + \mu[4^2 - 10\cdot 4 + 24]s^4\}s^{4-2} +\\
&c_5\{(1+\mu)^3\cdot 5(5-1) + (1+\mu)[-(1+2\mu)5^2 + \{11\mu-1\}\cdot 5 + [42+6\mu]]s^2 + \mu[5^2 - 10\cdot 5 + 24]s^4\}s^{5-2} +\\
&c_6\{(1+\mu)^3\cdot 6(6-1) + (1+\mu)[-(1+2\mu)6^2 + \{11\mu-1\}\cdot 6 + [42+6\mu]]s^2 + \mu[6^2 - 10\cdot 6 + 24]s^4\}s^{6-2} = 0 \ ,
\end{aligned} \tag{D109}$$

$$c_0\{(1+\mu)^3\cdot 0(-1) + (1+\mu)[-(1+2\mu)\cdot 0 + 0 + [42+6\mu]]s^2 + \mu[0-0+24]s^4\}s^{-2} + c_1\{(1+\mu)^3\cdot 1(0) + (1+\mu)[-(1+2\mu) + \{11\mu-1\} + [42+6\mu]]s^2 + \mu[1-10+24]s^4\}s^{-1} + c_2\{(1+\mu)^3\cdot 2(1) + (1+\mu)[-(1+2\mu)\cdot 4 + \{22\mu-2\} + [42+6\mu]]s^2 + \mu[4-20+24]s^4\}s^0 + c_3\{(1+\mu)^3\cdot 3(2) + (1+\mu)[-(1+2\mu)\cdot 9 + \{33\mu-3\} + [42+6\mu]]s^2 + \mu[9-30+24]s^4\}s^1 + c_4\{(1+\mu)^3\cdot 4(3) + (1+\mu)[-(1+2\mu)\cdot 16 + \{44\mu-4\} + [42+6\mu]]s^2 + \mu[16-40+24]s^4\}s^2 + c_5\{(1+\mu)^3\cdot 5(4) + (1+\mu)[-(1+2\mu)\cdot 25 + \{55\mu-5\} + [42+6\mu]]s^2 + \mu[25-50+24]s^4\}s^3 + c_6\{(1+\mu)^3\cdot 6(5) + (1+\mu)[-(1+2\mu)\cdot 36 + \{66\mu-6\} + [42+6\mu]]s^2 + \mu[36-60+24]s^4\}s^4 = 0 \ , \tag{D110}$$

$$c_0\{0s^{-2}+(1+\mu)\big[0+0+[42+6\mu]\big]s^2s^{-2}+\mu[24]s^4s^{-2}\}+c_1\{0s^{-1}+(1+\mu)[40+15\mu]s^2s^{-1}+\mu[15]s^4s^{-1}\}+c_2\{(1+\mu)^3\cdot 2s^0+(1+\mu)\big[-(4+8\mu)+[40+28\mu]\big]s^2s^0+\mu[8]s^4s^0\}+c_3\{(1+\mu)^3\cdot 6s^1+(1+\mu)\big[-(9+18\mu)+[39+39\mu]\big]s^2s^1+\mu[3]s^4s^1\}+c_4\{(1+\mu)^3\cdot 12s^2+(1+\mu)\big[-(16+32\mu)+[38+50\mu]\big]s^2s^2+\mu[0]s^4s^2\}+c_5\{(1+\mu)^3\cdot 20s^3+(1+\mu)\big[-(25+50\mu)+[37+61\mu]\big]s^2s^3+\mu[-1]s^4s^3\}+c_6\{(1+\mu)^3\cdot 30s^4+(1+\mu)\big[-(36+72\mu)+[36+72\mu]\big]s^2s^4+\mu[0]s^4s^4\}=0\quad, \tag{D111}$$

$$c_0\{(1+\mu)[42+6\mu]+24\mu s^2\}+c_1\{(1+\mu)[40+15\mu]s^1+15\mu s^3\}+c_2\{(1+\mu)^3\cdot 2+(1+\mu)[36+20\mu]s^2+8\mu s^4\}+c_3\{(1+\mu)^3\cdot 6s^1+(1+\mu)[30+21\mu]s^3+3\mu s^5\}+c_4\{(1+\mu)^3\cdot 12s^2+(1+\mu)[22+18\mu]s^4+0\}+c_5\{(1+\mu)^3\cdot 20s^3+(1+\mu)[12+11\mu]s^5-\mu s^7\}+c_6\{(1+\mu)^3\cdot 30s^4+(1+\mu)[0]s^6+\mu[0]s^8\}=0\quad, \tag{D112}$$

$$\{(1+\mu)[42+6\mu]c_0+24\mu c_0s^2\}+\{(1+\mu)[40+15\mu]c_1s^1+15\mu c_1s^3\}+\{(1+\mu)^3\cdot 2c_2+(1+\mu)[36+20\mu]c_2s^2+8\mu c_2s^4\}+\{(1+\mu)^3\cdot 6c_3s^1+(1+\mu)[30+21\mu]c_3s^3+3\mu c_3s^5\}+\{(1+\mu)^3\cdot 12c_4s^2+(1+\mu)[22+18\mu]c_4s^4\}+\{(1+\mu)^3\cdot 20c_5s^3+(1+\mu)[12+11\mu]c_5s^5-\mu c_5s^7\}+\{(1+\mu)^3\cdot 30c_6s^4\}=0\quad, \tag{D113}$$

$$\{(1+\mu)[42+6\mu]c_0+(1+\mu)^3\cdot 2c_2\}+\{(1+\mu)[40+15\mu]c_1s^1+(1+\mu)^3\cdot 6c_3s^1\}+\{+24\mu c_0s^2+(1+\mu)[36+20\mu]c_2s^2+(1+\mu)^3\cdot 12c_4s^2\}+\{+15\mu c_1s^3+(1+\mu)[30+21\mu]c_3s^3+(1+\mu)^3\cdot 20c_5s^3\}+\{+8\mu c_2s^4+(1+\mu)[22+18\mu]c_4s^4+(1+\mu)^3\cdot 30c_6s^4\}+\{+3\mu c_3s^5+(1+\mu)[12+11\mu]c_5s^5\}+\{-\mu c_5s^7\}=0\quad. \tag{D114}$$

The coefficients by the individual powers of *s* in (D114) has to equal zero; thus

$$\{-\mu c_5s^7\}=0 \qquad\Rightarrow\qquad c_5=0\quad, \tag{D115}$$

$$\{+3\mu c_3s^5+(1+\mu)[12+11\mu]c_5s^5\}=0 \qquad\Rightarrow\qquad c_3=0\quad, \tag{D116}$$

$$\{+15\mu c_1s^3+(1+\mu)[30+21\mu]c_3s^3+(1+\mu)^3\cdot 20c_5s^3\}=0 \qquad\Rightarrow\qquad c_1=0\quad. \tag{D117}$$

Only even-index c_j remain; thus

$$\{(1+\mu)[42+6\mu]c_0+(1+\mu)^3\cdot 2c_2\}+\{24\mu c_0s^2+(1+\mu)[36+20\mu]c_2s^2+(1+\mu)^3\cdot 12c_4s^2\}+\{8\mu c_2s^4+(1+\mu)[22+18\mu]c_4s^4+(1+\mu)^3\cdot 30c_6s^4\}=0\quad, \tag{D118}$$

Therefore

$$\{(1+\mu)[42+6\mu]c_0+(1+\mu)^3\cdot 2c_2\}=0 \qquad\Rightarrow\qquad \frac{c_2}{c_0}=-\frac{[21+3\mu]}{(1+\mu)^2}\quad, \tag{D119}$$

and from the second term of (D118) we get

$$\{24\mu c_0s^2+(1+\mu)[36+20\mu]c_2s^2+(1+\mu)^3\cdot 12c_4s^2\}=0\quad, \tag{D120}$$

$$24\mu+(1+\mu)[36+20\mu]\frac{c_2}{c_0}+(1+\mu)^3\cdot 12\frac{c_4}{c_0}=0\quad, \tag{D121}$$

$$24\mu-(1+\mu)[36+20\mu]\frac{[21+3\mu]}{(1+\mu)^2}+(1+\mu)^3\cdot 12\frac{c_4}{c_0}=0\quad, \tag{D122}$$

$$(1+\mu)^3\cdot 3\frac{c_4}{c_0}=-6\mu+[9+5\mu]3\frac{[7+\mu]}{(1+\mu)}\quad, \tag{D123}$$

$$\frac{c_4}{c_0}=\frac{-2\mu}{(1+\mu)^3}+\frac{(9+5\mu)(7+\mu)}{(1+\mu)^4}\quad, \tag{D124}$$

$$\frac{c_4}{c_0}=\frac{(-2\mu-2\mu\mu)+(63+35\mu+9\mu+5\mu\mu)}{(1+\mu)^4}\quad, \tag{D125}$$

$$\frac{c_4}{c_0}=\frac{(63+42\mu+3\mu\mu)}{(1+\mu)^4}\quad. \tag{D126}$$

From the last term of (D118) we finally get

$$8\mu\frac{c_2}{c_0}+(1+\mu)2[11+9\mu]\frac{c_4}{c_0}+(1+\mu)^3\cdot 30\frac{c_6}{c_0}=0\quad, \tag{D127}$$

$$-4\mu\frac{[21+3\mu]}{(1+\mu)^2}+(1+\mu)[11+9\mu]\frac{(63+42\mu+3\mu\mu)}{(1+\mu)^4}+(1+\mu)^3\cdot 15\frac{c_6}{c_0}=0\quad, \tag{D128}$$

$$\frac{c_6}{c_0} = -\frac{(1+\mu)[-4\cdot 21\mu-4\cdot 3\mu\mu]}{(1+\mu)^6\cdot 15} - \frac{(11\cdot 63+11\cdot 42\mu+11\cdot 3\mu\mu+63\cdot 9\mu+42\cdot 9\mu\mu+3\cdot 9\mu\mu\mu)}{(1+\mu)^6\cdot 15} =$$
$$-\frac{[-4\cdot 21\mu-4\cdot 3\mu\mu-4\cdot 21\mu\mu-4\cdot 3\mu\mu\mu]+(11\cdot 63+11\cdot 42\mu+11\cdot 3\mu\mu+63\cdot 9\mu+42\cdot 9\mu\mu+3\cdot 9\mu\mu\mu)}{(1+\mu)^6\cdot 15} =$$
$$-\frac{[-84\mu-96\mu\mu-12\mu\mu\mu]+(693+1029\mu+411\mu\mu+27\mu\mu\mu)}{15(1+\mu)^6} = -\frac{(693+945\mu+315\mu\mu+15\mu\mu\mu)}{15(1+\mu)^6} \quad , \qquad \text{(D129)}$$
$$\frac{c_6}{c_0} = -\frac{[5\mu^3+105\mu^2+315\mu+231]}{5(1+\mu)^6} \quad . \qquad \text{(D130)}$$
Then, one particular polynomial solution is
$$\left[-\frac{[5\mu^3+105\mu^2+315\mu+231]}{5(1+\mu)^6} s^6 + \frac{(63+42\mu+3\mu\mu)}{(1+\mu)^4} s^4 - \frac{[21+3\mu]}{(1+\mu)^2} s^2 + 1\right] \quad , \qquad \text{(D131)}$$
and the general polynomial solution for K_d=6 is
$$F_{A6}(s) = const.[(5\mu^3 + 105\mu^2 + 315\mu + 231)s^6 - 5(63 + 42\mu + 3\mu^2)(1 + \mu)^2 s^4 + 3\cdot 5(\mu + 7)(1 + \mu)^4 s^2 - 5(1 + \mu)^6]. \qquad \text{(D132)}$$

For the found polynomial solutions with various K_d, normalization factors have to be used which enable the use of a recursion formula which is to be found later.

D.3. Bonnet-like recursion formula application

In this section, the Bonnet-like recursion formula
$$P_{n+1}^{SI}(s) = \frac{(2n+1)}{(n+1)}\frac{1}{1+\mu} s\, P_n^{SI}(s) - \frac{n}{(n+1)}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right) P_{n-1}^{SI}(s) \qquad \text{(D133)}$$
(see (91) in the main text) is applied onto the first two (K_d=0, 1) found generalized Legendre polynomials, and – recursively – the higher degree polynomials (K_d=2, 3, 4, 5 and 6) are determined. This is a test that the formula brings the same result as the already determined solutions of the angular part of the Laplace equation of the first kind (Eqs. (84), (85), (86), (87), (88) in the main text).
For n=1 used in (D133), we obtain
$$P_{1+1}^{SI}(s) = \frac{(2\cdot 1+1)}{(1+1)}\frac{1}{1+\mu} s\, P_1^{SI}(s) - \frac{1}{(1+1)}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right) P_{1-1}^{SI}(s) \quad , \qquad \text{(D134)}$$
$$P_2^{SI}(s) = \frac{3}{2}\frac{1}{1+\mu} s\, P_1^{SI}(s) - \frac{1}{2}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right) P_0^{SI}(s) \quad . \qquad \text{(D135)}$$
When we insert P_0^{SI} and P_1^{SI}, according to (71) of the main text with normalization factor equal to $1/(1+\mu)$, and (67) of the main text with normalization factor equal to 1, we obtain
$$P_2^{SI}(s) = \frac{3}{2}\frac{1}{1+\mu} s \frac{1}{(1+\mu)}\cdot s - \frac{1}{2}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right)\cdot 1 \quad , \qquad \text{(D136)}$$
$$P_2^{SI}(s) = \frac{3}{2}\frac{1}{(1+\mu)^2}s^2 - \frac{1}{2} + \frac{1}{2}\frac{\mu}{(1+\mu)^2}s^2 \quad , \qquad \text{(D137)}$$
$$P_2^{SI}(s) = \frac{1}{2}\frac{1}{(1+\mu)^2}(3 + \mu)s^2 - \frac{1}{2} \quad , \qquad \text{(D138)}$$
and finally
$$P_2^{SI}(s) = \frac{1}{(1+\mu)^2}\cdot\frac{1}{2}[(3 + \mu)s^2 - (1 + \mu)^2] \quad , \qquad \text{(D139)}$$
which is, except the normalization factor, the same as the solution (84) in the main text.

Similarly, we have for n=2
$$P_{2+1}^{SI}(s) = \frac{(2\cdot 2+1)}{(2+1)}\frac{1}{1+\mu} s\, P_2^{SI}(s) - \frac{2}{(2+1)}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right) P_{2-1}^{SI}(s) \quad . \qquad \text{(D140)}$$
$$P_3^{SI}(s) = \frac{5}{3}\frac{1}{1+\mu} s\, P_2^{SI}(s) - \frac{2}{3}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right) P_1^{SI}(s) \quad . \qquad \text{(D141)}$$

$$P_3^{SI}(s) = \frac{5}{3}\frac{1}{1+\mu} s\, P_2^{SI}(s) - \frac{2}{3}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right) P_1^{SI}(s) \quad . \tag{D142}$$

When we insert P_1^{SI} according to formula (67) of the main text with normalization factor equal to $1/(1+\mu)$, and P_2^{SI} according to formula (D139), we obtain

$$P_3^{SI}(s) = \frac{5}{3}\frac{1}{1+\mu} s \frac{1}{2}\frac{1}{(1+\mu)^2}[(3+\mu)s^2 - (1+\mu)^2] - \frac{2}{3}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right)\frac{1}{1+\mu}\cdot s \quad , \tag{D143}$$

$$P_3^{SI}(s) = \frac{5}{3}\frac{1}{2}\frac{1}{(1+\mu)^3}[(3+\mu)s^3 - (1+\mu)^2 s] - \frac{2}{3}\left(\frac{1}{1+\mu}s - \frac{\mu}{(1+\mu)^3}s^2 s\right) \quad , \tag{D144}$$

$$P_3^{SI}(s) = \frac{5}{3}\frac{1}{2}\frac{1}{(1+\mu)^3}(3+\mu)s^3 - \frac{5}{3}\frac{1}{2}\frac{1}{(1+\mu)^3}(1+\mu)^2 s - \frac{2}{3}\frac{1}{1+\mu}s + \frac{2}{3}\frac{\mu}{(1+\mu)^3}s^3 \quad , \tag{D145}$$

$$P_3^{SI}(s) = \left(\frac{5}{3}\frac{1}{2}\frac{1}{(1+\mu)^3}(3+\mu) + \frac{2}{3}\frac{\mu}{(1+\mu)^3}\right)s^3 - \left(\frac{5}{3}\frac{1}{2}\frac{1}{(1+\mu)} + \frac{2}{3}\frac{1}{1+\mu}\right)s \quad , \tag{D146}$$

$$P_3^{SI}(s) = \left(\frac{5(3+\mu)+2\cdot 2\mu}{3\cdot 2\cdot(1+\mu)^3}\right)s^3 - \left(\frac{5+2\cdot 2}{3\cdot 2\cdot(1+\mu)}\right)s \quad , \tag{D147}$$

$$P_3^{SI}(s) = \left(\frac{15+9\mu}{3\cdot 2\cdot(1+\mu)^3}\right)s^3 - \left(\frac{3\cdot 3}{3\cdot 2\cdot(1+\mu)}\right)s \quad , \tag{D148}$$

$$P_3^{SI}(s) = \left(\frac{5+3\mu}{2\cdot(1+\mu)^3}\right)s^3 - \left(\frac{3}{2\cdot(1+\mu)}\right)s \quad , \tag{D149}$$

and finally

$$P_3^{SI}(s) = \frac{1}{(1+\mu)^3}\cdot\frac{1}{2}[(3\mu+5)s^3 - 3(1+\mu)^2 s] \quad , \tag{D150}$$

which is, except the normalization factor, the same as the solution (85) in the main text.

For n=3, we get from (D133)

$$P_{3+1}^{SI}(s) = \frac{(2\cdot 3+1)}{(3+1)}\frac{1}{1+\mu} s\, P_3^{SI}(s) - \frac{3}{(3+1)}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right)P_{3-1}^{SI}(s) \quad , \tag{D151}$$

$$P_4^{SI}(s) = \frac{7}{4}\frac{1}{1+\mu} s\, P_3^{SI}(s) - \frac{3}{4}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right)P_2^{SI}(s) \quad . \tag{D152}$$

When we insert P_2^{SI} and P_3^{SI}, we obtain

$$P_4^{SI}(s) = \frac{7}{4}\frac{1}{1+\mu} s \frac{1}{(1+\mu)^3}\cdot\frac{1}{2}[(3\mu+5)s^3 - 3(1+\mu)^2 s] - \frac{3}{4}\left(1 - \frac{\mu}{(1+\mu)^2}s^2\right)\frac{1}{2}\frac{1}{(1+\mu)^2}[(3+\mu)s^2 - (1+\mu)^2] \quad , \tag{D153}$$

$$P_4^{SI}(s) = \frac{7}{4}\frac{1}{(1+\mu)^4}\cdot\frac{1}{2}[(3\mu+5)s^4 - 3(1+\mu)^2 s^2] - \frac{1}{2}\frac{3}{4}\left(\frac{1}{(1+\mu)^2} - \frac{\mu}{(1+\mu)^4}s^2\right)[(3+\mu)s^2 - (1+\mu)^2] \quad , \tag{D154}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{7}{4}[(3\mu+5)s^4 - 3(1+\mu)^2 s^2] - \frac{3}{4}\left((1+\mu)^2 - \mu s^2\right)[(3+\mu)s^2 - (1+\mu)^2]\right\} \quad , \tag{D155}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{7}{4}(3\mu+5)s^4 - 3\frac{7}{4}(1+\mu)^2 s^2 - \frac{3}{4}[(1+\mu)^2(3+\mu)s^2 - (1+\mu)^2(1+\mu)^2 - \mu s^2(3+\mu)s^2 + \mu s^2(1+\mu)^2]\right\} \quad , \tag{D156}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{7}{4}(3\mu+5)s^4 - 3\frac{7}{4}(1+\mu)^2 s^2 - \frac{3}{4}(1+\mu)^2(3+\mu)s^2 + \frac{3}{4}(1+\mu)^4 + \frac{3}{4}\mu(3+\mu)s^4 - \frac{3}{4}\mu s^2(1+\mu)^2\right\} \quad , \tag{D157}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{7}{4}(3\mu+5)s^4 + \frac{3}{4}\mu(3+\mu)s^4 - 3\frac{7}{4}(1+\mu)^2 s^2 - \frac{3}{4}(1+\mu)^2(3+\mu)s^2 - \frac{3}{4}\mu(1+\mu)^2 s^2 + \frac{3}{4}(1+\mu)^4\right\} \quad , \tag{D158}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{1}{4}[7(3\mu+5) + 3\mu(3+\mu)]s^4 - \frac{3}{4}(1+\mu)^2[7 + (3+\mu) + \mu]s^2 + \frac{3}{4}(1+\mu)^4\right\} \quad , \tag{D159}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{1}{4}[(21\mu+35) + (9\mu+3\mu^2)]s^4 - \frac{3}{4}(1+\mu)^2[10+2\mu]s^2 + \frac{3}{4}(1+\mu)^4\right\}, \tag{D160}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\left\{\frac{1}{4}(3\mu^2+30\mu+35)s^4 - \frac{3}{2}(1+\mu)^2(\mu+5)s^2 + \frac{3}{4}(1+\mu)^4\right\} \quad , \tag{D161}$$

and finally

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2^3}\cdot\{(3\mu^2+30\mu+35)s^4 - 2\cdot 3(\mu+5)(1+\mu)^2 s^2 + 3(1+\mu)^4\} \quad . \tag{D162}$$

which is, except the normalization factor, the same as the solution (86) in the main text.

For *n*=4, we obtain from (D133)

$$P_{4+1}^{SI}(s) = \frac{(2\cdot4+1)}{(4+1)}\frac{1}{1+\mu}s\,P_4^{SI}(s) - \frac{4}{(4+1)}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)P_{4-1}^{SI}(s) \quad , \tag{D163}$$

$$P_5^{SI}(s) = \frac{9}{5}\frac{1}{1+\mu}s\,P_4^{SI}(s) - \frac{4}{5}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)P_3^{SI}(s) \quad . \tag{D164}$$

When we insert P_3^{SI} and P_4^{SI}, we obtain

$$P_5^{SI}(s) = \frac{9}{5}\frac{1}{1+\mu}s\frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\cdot\frac{1}{4}\{(3\mu^2+30\mu+35)s^4 - 2\cdot3(\mu+5)(1+\mu)^2s^2 + 3(1+\mu)^4\} - \frac{4}{5}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)\frac{1}{(1+\mu)^3}\cdot\frac{1}{2}[(3\mu+5)s^3 - 3(1+\mu)^2s] \quad , \tag{D165}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{9}{5}\{(3\mu^2+30\mu+35)s^5 - 2\cdot3(\mu+5)(1+\mu)^2s^3 + 3(1+\mu)^4s\} - \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{4}{5}((1+\mu)^2-\mu s^2)[(3\mu+5)s^3 - 3(1+\mu)^2s] \quad , \tag{D166}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{9}{5}[(3\mu^2+30\mu+35)s^5 - 2\cdot3(\mu+5)(1+\mu)^2s^3 + 3(1+\mu)^4s] - \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{4}{5}[(3\mu+5)(1+\mu)^2s^3 - 3(1+\mu)^4s - \mu s^2(3\mu+5)s^3 + \mu s^2 3(1+\mu)^2s] \quad , \tag{D167}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5}\{9[(3\mu^2+30\mu+35)s^5 - 2\cdot3(\mu+5)(1+\mu)^2s^3 + 3(1+\mu)^4s] - 4\cdot4[(3\mu+5)(1+\mu)^2s^3 - 3(1+\mu)^4s - \mu(3\mu+5)s^5 + \mu3(1+\mu)^2s^3]\} \quad , \tag{D168}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5}\{9(3\mu^2+30\mu+35)s^5 - 2\cdot3\cdot9(\mu+5)(1+\mu)^2s^3 + 3\cdot9(1+\mu)^4s - 4\cdot4(3\mu+5)(1+\mu)^2s^3 + 3\cdot4\cdot4(1+\mu)^4s + 4\cdot4\mu(3\mu+5)s^5 - 3\cdot4\cdot4\mu(1+\mu)^2s^3\} \quad , \tag{D169}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5}\{[9(3\mu^2+30\mu+35) + 4\cdot4\mu(3\mu+5)]s^5 - 2(1+\mu)^2[3\cdot9(\mu+5) + 3\cdot2\cdot4\mu + 2\cdot4(3\mu+5)]s^3 + 3(1+\mu)^4[9+4\cdot4]s\} \quad , \tag{D170}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5}\{[(27\mu^2+270\mu+315) + (48\mu\mu+80\mu)]s^5 - 2(1+\mu)^2[(27\mu+135) + 24\mu + (24\mu+40)]s^3 + 3(1+\mu)^4[25]s\} \quad , \tag{D171}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5}\{(75\mu^2+350\mu+315)s^5 - 2(1+\mu)^2(75\mu+175)s^3 + 3\cdot25(1+\mu)^4s\} \quad . \tag{D172}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{(15\mu^2+70\mu+63)s^5 - 2(1+\mu)^2(15\mu+35)s^3 + 3\cdot5(1+\mu)^4s\} \quad , \tag{D173}$$

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{(15\mu^2+70\mu+63)s^5 - 2\cdot5(1+\mu)^2(3\mu+7)s^3 + 3\cdot5(1+\mu)^4s\} \quad , \tag{D174}$$

and finally

$$P_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2^3}\cdot\{(3\cdot5\mu^2 + 2\cdot5\cdot7\mu + 3\cdot3\cdot7)s^5 - 2\cdot5(3\mu+7)(1+\mu)^2s^3 + 3\cdot5(1+\mu)^4s\} \quad . \tag{D175}$$

which is, except the normalization factor, the same as the solution (87) in the main text.

For *n*=5, we get from (D133)

$$P_{5+1}^{SI}(s) = \frac{(2\cdot5+1)}{(5+1)}\frac{1}{1+\mu}s\,P_5^{SI}(s) - \frac{5}{(5+1)}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)P_{5-1}^{SI}(s) \quad , \tag{D176}$$

$$P_6^{SI}(s) = \frac{11}{6}\frac{1}{1+\mu}s\,P_5^{SI}(s) - \frac{5}{6}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)P_4^{SI}(s) \quad . \tag{D177}$$

When we insert P_4^{SI} and P_5^{SI}, we obtain

$$P_6^{SI}(s) = \frac{11}{6}\frac{1}{1+\mu}s\frac{1}{(1+\mu)^5}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{(3\cdot5\mu^2 + 2\cdot5\cdot7\mu + 3\cdot3\cdot7)s^5 - 2\cdot5(1+\mu)^2(3\mu+7)s^3 + 3\cdot5(1+\mu)^4s\} - \frac{5}{6}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)\frac{1}{(1+\mu)^4}\cdot\frac{1}{2}\cdot\frac{1}{4}\{(3\mu^2+30\mu+35)s^4 - 2\cdot3(\mu+5)(1+\mu)^2s^2 + 3(1+\mu)^4\} \quad , \tag{D178}$$

$$P_6^{SI}(s)=\frac{11}{6}\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot[(3\cdot5\mu^2+2\cdot5\cdot7\mu+3\cdot3\cdot7)s^6-2\cdot5(1+\mu)^2(3\mu+7)s^4+3\cdot5(1+\mu)^4s^2]-\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{5}{6}((1+\mu)^2-\mu s^2)[(3\mu^2+30\mu+35)s^4-2\cdot3(\mu+5)(1+\mu)^2s^2+3(1+\mu)^4] \quad , \quad \text{(D179)}$$

$$P_6^{SI}(s)=\frac{1}{6}\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{11[(3\cdot5\mu^2+2\cdot5\cdot7\mu+3\cdot3\cdot7)s^6-2\cdot5(1+\mu)^2(3\mu+7)s^4+3\cdot5(1+\mu)^4s^2]-5[(1+\mu)^2(3\mu^2+30\mu+35)s^4-2\cdot3(\mu+5)(1+\mu)^4s^2+3(1+\mu)^6-\mu s^2(3\mu^2+30\mu+35)s^4+\mu s^2 2\cdot3(\mu+5)(1+\mu)^2s^2-\mu s^2 3(1+\mu)^4]\} \quad , \quad \text{(D180)}$$

$$P_6^{SI}(s)=\frac{1}{6}\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{[11(3\cdot5\mu^2+2\cdot5\cdot7\mu+3\cdot3\cdot7)s^6-11\cdot2\cdot5(1+\mu)^2(3\mu+7)s^4+11\cdot3\cdot5(1+\mu)^4s^2]-[5(1+\mu)^2(3\mu^2+30\mu+35)s^4-5\cdot2\cdot3(\mu+5)(1+\mu)^4s^2+5\cdot3(1+\mu)^6-5\cdot\mu(3\mu^2+30\mu+35)s^6+5\cdot2\cdot3\mu(\mu+5)(1+\mu)^2s^4-5\cdot\mu3(1+\mu)^4s^2]\} \quad , \quad \text{(D181)}$$

$$P_6^{SI}(s)=\frac{1}{6}\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{[11(3\cdot5\mu^2+2\cdot5\cdot7\mu+3\cdot3\cdot7)+5\cdot\mu(3\mu^2+30\mu+35)]s^6-5(1+\mu)^2[11\cdot2\cdot(3\mu+7)+(3\mu^2+30\mu+35)+2\cdot3\mu(\mu+5)]s^4+5\cdot(1+\mu)^4[11\cdot3+2\cdot3(\mu+5)+3\mu]s^2-5\cdot3(1+\mu)^6\} \quad , \quad \text{(D182)}$$

$$P_6^{SI}(s)=\frac{1}{6}\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{[(165\mu^2+770\mu+693)+(15\mu^3+150\mu^2+175\mu)]s^6-5(1+\mu)^2[(66\mu+154)+(3\mu^2+30\mu+35)+(6\mu^2+30\mu)]s^4+5\cdot(1+\mu)^4[33+(6\mu+30)+3\mu]s^2-5\cdot3(1+\mu)^6\} \quad , \quad \text{(D183)}$$

$$P_6^{SI}(s)=\frac{1}{6}\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\{(15\mu^3+315\mu^2+945\mu+693)s^6-5(1+\mu)^2(9\mu^2+126\mu+189)s^4+5\cdot(1+\mu)^4(9\mu+63)s^2-5\cdot3(1+\mu)^6\} \quad , \quad \text{(D184)}$$

$$P_6^{SI}(s)=\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}\{(3\cdot5\mu^3+3\cdot3\cdot5\cdot7\mu^2+3\cdot3\cdot3\cdot5\cdot7\cdot\mu+3\cdot3\cdot7\cdot11)s^6-5(1+\mu)^2(3\cdot3\mu^2+2\cdot3\cdot3\cdot7\mu+3\cdot3\cdot3\cdot7)s^4+5\cdot(1+\mu)^4(3\cdot3\mu+3\cdot3\cdot7)s^2-5\cdot3(1+\mu)^6\} \quad , \quad \text{(D185)}$$

$$P_6^{SI}(s)=\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}\{3(5\mu^3+3\cdot5\cdot7\mu^2+3\cdot3\cdot5\cdot7\cdot\mu+3\cdot7\cdot11)s^6-3\cdot5(1+\mu)^2(3\mu^2+2\cdot3\cdot7\mu+3\cdot3\cdot7)s^4+3\cdot5\cdot(1+\mu)^4(3\mu+3\cdot7)s^2-5\cdot3(1+\mu)^6\} \quad , \quad \text{(D186)}$$

$$P_6^{SI}(s)=\frac{1}{(1+\mu)^6}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}3\{(5\mu^3+3\cdot5\cdot7\mu^2+3\cdot3\cdot5\cdot7\cdot\mu+3\cdot7\cdot11)s^6-5(1+\mu)^2(3\mu^2+2\cdot3\cdot7\mu+3\cdot3\cdot7)s^4+5(1+\mu)^4(3\mu+3\cdot7)s^2-5(1+\mu)^6\} \quad , \quad \text{(D187)}$$

$$P_6^{SI}(s)=\frac{1}{(1+\mu)^6}\cdot\frac{1}{2^4}\{(5\mu^3+3\cdot5\cdot7\mu^2+3\cdot3\cdot5\cdot7\cdot\mu+3\cdot7\cdot11)s^6-5(1+\mu)^2(3\mu^2+2\cdot3\cdot7\mu+3\cdot3\cdot7)s^4+5(1+\mu)^4(3\mu+3\cdot7)s^2-5(1+\mu)^6\} \quad , \quad \text{(D188)}$$

and finally

$$P_6^{SI}(s)=\frac{1}{(1+\mu)^6}\cdot\frac{1}{2^4}\{(5\mu^3+105\mu^2+315\mu+231)s^6-5(1+\mu)^2(3\mu^2+42\mu+63)s^4+3\cdot5(1+\mu)^4(\mu+7)s^2-5(1+\mu)^6\} \quad . \quad \text{(D189)}$$

which is, except the normalization factor, the same as the solution (88) in the main text.