Supplement E

to

Interior solution of azimuthally symmetric case of Laplace equation in orthogonal Similar Oblate Spheroidal coordinates

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Content:

E.1. The second-degree second-kind solution

Let's simplify the notation of the generalized Legendre equation (68), i.e.,

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\frac{d^2F(s)}{ds^2}+ s[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2]\frac{dF(s)}{ds}+K_d[(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2]F(s)=0 \quad . \tag{E1}$$

It can be rewritten in the following way:

$$H_2(s)\frac{d^2F(s)}{ds^2}+H_1(s)\frac{dF(s)}{ds}+H_0(s)F(s)=0 \quad , \tag{E2}$$

where *H* denote specific polynomials, particularly

$$H_2(s)= [(1+\mu)-s^2][(1+\mu)^2-\mu s^2]=[(1+\mu)^3-(2\mu+1)(1+\mu)s^2+\mu s^4] \tag{E3}$$

$$H_1(s)= s[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2] \tag{E4}$$

$$H_0(s)= K_d[(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2] \tag{E5}$$

Model function

Further, let's use particular shape of the model function. For the second kind solution of higher degree of the generalized Legendre equation (E1), we select the test function on the basis of the already found solutions for K_d=0 and 1, and also based on the experience with the spherical case. The test function in terms of *s* is

$$F_{test2}(s) = (As^{\alpha}+B)\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+(Cs^{\gamma}+D)\sqrt{\frac{(1+\mu)^2}{s^2}-\mu} =$$
$$(As^{\alpha}+B)\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+(Cs^{\gamma}+D)\,\frac{(1+\mu)^2-\mu s^2}{s}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad , \tag{E6}$$

Where A, B, C, D, α, γ are constants to be found. Note that the expression for the model function through the logarithm is

$$F_{test2}(W) = (As^{\alpha}+B)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)+\frac{(Cs^{\gamma}+D)}{s}\sqrt{(1+\mu)^2-\mu s^2} \quad . \tag{E7}$$

The solution of (E6) is extensive, and it is thus split into some subsections. It begins with the calculation of the derivatives of the test function followed by their insertion to the generalized Legendre equation.

Calculation of the derivatives of the test function

The first derivative of the test function is

$$\frac{dF_{test2}(s)}{ds} = \frac{d}{ds}\left[(As^{\alpha}+B)\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+(Cs^{\gamma}+D)\sqrt{\frac{(1+\mu)^2}{s^2}-\mu}\right] = \frac{d}{ds}[(As^{\alpha}+$$
$$B)]\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+(As^{\alpha}+B)\frac{d}{ds}\left[\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)\right]+$$
$$\frac{d}{ds}[(Cs^{\gamma}+D)]\sqrt{\frac{(1+\mu)^2}{s^2}-\mu}+(Cs^{\gamma}+D)\frac{d}{ds}\left[\sqrt{\frac{(1+\mu)^2}{s^2}-\mu}\right] \quad . \tag{E8}$$

To continue, we need some particular derivatives.
Derivative of the $\sinh^{-1}$ part is

$$\frac{d}{ds}\left[\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)\right] = \frac{(1+\mu)}{((1+\mu)-s^2)^{\frac{3}{2}}\sqrt{\mu+\frac{(1+\mu)}{((1+\mu)-s^2)}}} = \frac{(1+\mu)}{((1+\mu)-s^2)^{\frac{3}{2}}\sqrt{\frac{\mu((1+\mu)-s^2)+(1+\mu)}{((1+\mu)-s^2)}}} = \frac{(1+\mu)}{((1+\mu)-s^2)^{\frac{3}{2}}\sqrt{\frac{(1+\mu)^2-\mu s^2}{(1+\mu)-s^2}}} = \frac{(1+\mu)}{((1+\mu)-s^2)\sqrt{(1+\mu)^2-\mu s^2}} =$$
$$\frac{(1+\mu)}{((1+\mu)-s^2)}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad . \tag{E9}$$

The derivative of the square root part is

$$\frac{d}{ds}\left[\sqrt{\frac{(1+\mu)^2}{s^2}-\mu}\right] = \frac{d}{ds}\left[\frac{\sqrt{(1+\mu)^2-\mu s^2}}{s}\right] = \frac{d}{ds}\left[\frac{(1+\mu)^2-\mu s^2}{s}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] = -\frac{(1+\mu)^2}{s^3\sqrt{\frac{(1+\mu)^2}{s^2}-\mu}} = -\frac{(1+\mu)^2}{s^3\sqrt{\frac{(1+\mu)^2-\mu s^2}{s^2}}} = -\frac{(1+\mu)^2}{s^2\sqrt{(1+\mu)^2-\mu s^2}} = -\frac{(1+\mu)^2}{s^2}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad . \tag{E10}$$

The derivative of the reciprocal square root function from the above derivative is

$$\frac{d}{ds}\left[\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] = -\frac{1}{2}\frac{-2\mu s}{[(1+\mu)^2-\mu s^2]\sqrt{(1+\mu)^2-\mu s^2}} = \frac{\mu s}{[(1+\mu)^2-\mu s^2]\sqrt{(1+\mu)^2-\mu s^2}} = \frac{\mu s}{[(1+\mu)^2-\mu s^2]}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} . \tag{E11}$$

The first derivative of the test function is thus

$$\frac{dF_{test2}(s)}{ds} =$$
$$\alpha As^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+(As^{\alpha}+B)\left[\frac{(1+\mu)}{((1+\mu)-s^2)}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right]+\gamma Cs^{\gamma-1}\,\frac{(1+\mu)^2-\mu s^2}{s}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}-(Cs^{\gamma}+D)\left[\frac{(1+\mu)^2}{s^2}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] =$$
$$\alpha As^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+\left\{\frac{(1+\mu)(As^{\alpha}+B)}{((1+\mu)-s^2)}+\gamma Cs^{\gamma-2}\,[(1+\mu)^2-\mu s^2]-\frac{(1+\mu)^2(Cs^{\gamma}+D)}{s^2}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad , \tag{E12}$$

which – in a more compact form – can be written as

$$\frac{dF_{test2}(s)}{ds} = \alpha As^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+\left\{\frac{(1+\mu)(As^{\alpha}+B)}{((1+\mu)-s^2)}+\gamma Cs^{\gamma-2}\,[(1+\mu)^2-\mu s^2]-\right.$$
$$\left.\frac{(1+\mu)^2(Cs^{\gamma}+D)}{s^2}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad . \tag{E13}$$

Further, the second derivative of the test function is:

$$\frac{d^2F_{test2}(s)}{ds^2} = \frac{d}{ds}\left[\frac{dF_{testg}(s)}{ds}\right] = \frac{d}{ds}\bigg[\alpha A s^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \left\{\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)} + \gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\bigg] =$$

$$\frac{d}{ds}\left[\alpha A s^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)\right] + \frac{d}{ds}\left[\left\{\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)} + \gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] =$$

$$\frac{d}{ds}[\alpha A s^{\alpha-1}]\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \alpha A s^{\alpha-1}\frac{d}{ds}\left[\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)\right] +$$

$$\frac{d}{ds}\left[\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)} + \gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\right]\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} + \left\{\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)} + \gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\right\}\frac{d}{ds}\left[\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] =$$

$$[\alpha(\alpha-1)As^{\alpha-2}]\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \alpha A s^{\alpha-1}\left[\frac{(1+\mu)}{((1+\mu)-s^2)}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] + \bigg\{(1+\mu)\frac{d}{ds}\left[\frac{(As^\alpha+B)}{((1+\mu)-s^2)}\right] + \gamma C\frac{d}{ds}\left[s^{\gamma-2}\,[(1+\mu)^2-\mu s^2]\right] -$$

$$(1+\mu)^2\frac{d}{ds}\left[\frac{(Cs^\gamma+D)}{s^2}\right]\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} + \left\{\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)} + \gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\right\}\left[\frac{\mu s}{[(1+\mu)^2-\mu s^2]}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] \quad , \qquad \text{(E14)}$$

where we used the derivative of the reciprocal square root (E11) in the last term. Then, the second derivative is

$$\frac{d^2F_{test2}(s)}{ds^2} = \alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \alpha A s^{\alpha-1}\left[\frac{(1+\mu)}{((1+\mu)-s^2)}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right] + \bigg\{(1+\mu)\left[\frac{\frac{d}{ds}[As^\alpha+B]((1+\mu)-s^2) - (As^\alpha+B)\frac{d}{ds}[(1+\mu)-s^2]}{((1+\mu)-s^2)^2}\right] +$$

$$\gamma C\left[\frac{d}{ds}[s^{\gamma-2}]\,((1+\mu)^2-\mu s^2) + s^{\gamma-2}\frac{d}{ds}[(1+\mu)^2-\mu s^2]\right] - (1+\mu)^2\left[\frac{\frac{d}{ds}[Cs^\gamma+D]s^2 - (Cs^\gamma+D)\frac{d}{ds}[s^2]}{s^4}\right]\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} + \bigg\{\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)}\frac{\mu s}{[(1+\mu)^2-\mu s^2]} +$$

$$\gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2]\frac{\mu s}{[(1+\mu)^2-\mu s^2]} - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\frac{\mu s}{[(1+\mu)^2-\mu s^2]}\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} =$$

$$\alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \bigg\{\alpha A s^{\alpha-1}\frac{(1+\mu)}{((1+\mu)-s^2)} + \bigg[(1+\mu)\frac{\alpha A s^{\alpha-1}((1+\mu)-s^2)+(As^\alpha+B)2s}{((1+\mu)-s^2)^2} + \gamma C[(\gamma-2)s^{\gamma-3}\,((1+\mu)^2-\mu s^2) -$$

$$2\mu s^{\gamma-1}] - (1+\mu)^2\frac{\gamma C s^{\gamma+1}-(Cs^\gamma+D)2s}{s^4}\bigg] +$$

$$\bigg[\frac{(1+\mu)(As^\alpha+B)}{((1+\mu)-s^2)}\frac{\mu s}{[(1+\mu)^2-\mu s^2]} + \gamma C s^{\gamma-2}\,[(1+\mu)^2-\mu s^2]\frac{\mu s}{[(1+\mu)^2-\mu s^2]} - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\frac{\mu s}{[(1+\mu)^2-\mu s^2]}\bigg]\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} =$$

$$\alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \bigg\{\frac{(1+\mu)\alpha A s^{\alpha-1}}{((1+\mu)-s^2)} + \bigg[(1+\mu)\frac{((1+\mu)\alpha A s^{\alpha-1}-\alpha A s^{\alpha+1})+(2As^{\alpha+1}+2Bs)}{((1+\mu)-s^2)^2} + \gamma C[\,((1+\mu)^2(\gamma-2)s^{\gamma-3} -$$

$$\mu(\gamma-2)s^{\gamma-1}) - 2\mu s^{\gamma-1}] - (1+\mu)^2\frac{\gamma C s^{\gamma+1}-(2Cs^{\gamma+1}+2Ds)}{s^4}\bigg] + \bigg[\frac{\mu(1+\mu)(As^\alpha+B)s}{((1+\mu)-s^2)[(1+\mu)^2-\mu s^2]} + \gamma C\mu s^{\gamma-1} - \frac{\mu(1+\mu)^2(Cs^\gamma+D)}{s[(1+\mu)^2-\mu s^2]}\bigg]\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} = \alpha(\alpha -$$

$$1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \bigg\{\frac{(1+\mu)\alpha A s^{\alpha-1}}{((1+\mu)-s^2)}\frac{((1+\mu)-s^2)}{((1+\mu)-s^2)} + \frac{(1+\mu)^2\alpha A s^{\alpha-1}+A(1+\mu)(2-\alpha)s^{\alpha+1}+2(1+\mu)Bs}{((1+\mu)-s^2)^2} + \big[\gamma C\big(-2-(\gamma-2)\big)\mu s^{\gamma-1} +$$

$$\gamma(\gamma-2)C(1+\mu)^2 s^{\gamma-3}\big] - (1+\mu)^2\frac{-2Ds+C(\gamma-2)s^{\gamma+1}}{s^4} + \bigg[\frac{(A\mu(1+\mu)s^{\alpha+1}+B\mu(1+\mu)s)}{((1+\mu)-s^2)[(1+\mu)^2-\mu s^2]} + \gamma C\mu s^{\gamma-1} - \frac{(C\mu(1+\mu)^2 s^\gamma+D\mu(1+\mu)^2)}{s[(1+\mu)^2-\mu s^2]}\bigg]\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad . \qquad \text{(E15)}$$

Therefore

$$\frac{d^2F_{test2}(s)}{ds^2} = \alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right) + \bigg\{\frac{(1+\mu)^2\alpha A s^{\alpha-1}-(1+\mu)\alpha A s^{\alpha+1}}{((1+\mu)-s^2)^2} + \frac{(1+\mu)^2\alpha A s^{\alpha-1}+A(1+\mu)(2-\alpha)s^{\alpha+1}+2(1+\mu)Bs}{((1+\mu)-s^2)^2} - \gamma^2 C\mu s^{\gamma-1} +$$

$$\gamma(\gamma-2)C(1+\mu)^2 s^{\gamma-3} + (1+\mu)^2 2Ds^{-3} - (1+\mu)^2 C(\gamma-2)s^{\gamma-3} +$$

$$\bigg[\frac{(A\mu(1+\mu)s^{\alpha+1}+B\mu(1+\mu)s)}{((1+\mu)-s^2)[(1+\mu)^2-\mu s^2]} + \gamma C\mu s^{\gamma-1} - \frac{(C\mu(1+\mu)^2 s^{\gamma-1}+D\mu(1+\mu)^2 s^{-1})}{[(1+\mu)^2-\mu s^2]}\bigg]\bigg\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}} \quad . \qquad \text{(E16)}$$

The expression in the curl bracket is further expanded individually:

$$\bigg\{\frac{(1+\mu)^2\alpha A s^{\alpha-1}-(1+\mu)\alpha A s^{\alpha+1}+(1+\mu)^2\alpha A s^{\alpha-1}+A(1+\mu)(2-\alpha)s^{\alpha+1}+2(1+\mu)Bs}{((1+\mu)-s^2)^2} - \gamma^2 C\mu s^{\gamma-1} + \gamma C\mu s^{\gamma-1} + \gamma(\gamma-2)C(1+\mu)^2 s^{\gamma-3} - (1+\mu)^2 C(\gamma-2)s^{\gamma-3} +$$

$$(1+\mu)^2 2Ds^{-3} + \bigg[\frac{(A\mu(1+\mu)s^{\alpha+1}+B\mu(1+\mu)s)-((1+\mu)-s^2)(C\mu(1+\mu)^2 s^{\gamma-1}+D\mu(1+\mu)^2 s^{-1})}{((1+\mu)-s^2)[(1+\mu)^2-\mu s^2]}\bigg]\bigg\} = \bigg\{\frac{2(1+\mu)^2\alpha A s^{\alpha-1}+A(1+\mu)(2-2\alpha)s^{\alpha+1}+2(1+\mu)Bs}{((1+\mu)-s^2)^2} + \gamma(1-\gamma)C\mu s^{\gamma-1} +$$

$$C(\gamma-1)(\gamma-2)(1+\mu)^2 s^{\gamma-3} + (1+\mu)^2 2Ds^{-3} + \bigg[\frac{(A\mu(1+\mu)s^{\alpha+1}+B\mu(1+\mu)s)-[C\mu(1+\mu)^3 s^{\gamma-1}+D\mu(1+\mu)^3 s^{-1}-C\mu(1+\mu)^2 s^{\gamma+1}-D\mu(1+\mu)^2 s]}{((1+\mu)-s^2)[(1+\mu)^2-\mu s^2]}\bigg]\bigg\} =$$

$$\bigg\{\frac{2(1+\mu)^2\alpha A s^{\alpha-1}+A(1+\mu)(2-2\alpha)s^{\alpha+1}+2(1+\mu)Bs}{((1+\mu)-s^2)^2} + \frac{[\gamma(1-\gamma)C\mu s^{\gamma-1}+C(\gamma-1)(\gamma-2)(1+\mu)^2 s^{\gamma-3}+(1+\mu)^2 2Ds^{-3}]((1+\mu)^2-2(1+\mu)s^2+s^4)}{((1+\mu)-s^2)^2} +$$

$$\bigg[\frac{A\mu(1+\mu)s^{\alpha+1}+[B+D(1+\mu)]\mu(1+\mu)s-[C\mu(1+\mu)^3 s^{\gamma-1}+D\mu(1+\mu)^3 s^{-1}-C\mu(1+\mu)^2 s^{\gamma+1}]}{((1+\mu)-s^2)[(1+\mu)^2-\mu s^2]}\bigg]\bigg\} =$$

$$\left\{\frac{[2(1+\mu)^2\alpha A s^{\alpha-1}+A(1+\mu)(2-2\alpha)s^{\alpha+1}+2(1+\mu)Bs][(1+\mu)^2-\mu s^2]}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{[\gamma(1-\gamma)C\mu(1+\mu)^2 s^{\gamma-1}+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+(1+\mu)^4 2Ds^{-3}]-[2C\gamma(1-\gamma)\mu(1+\mu)s^{\gamma+1}+2C(\gamma-1)(\gamma-2)(1+\mu)^3 s^{\gamma-1}+4D(1+\mu)^3 s^{-1}]+[\gamma(1-\gamma)C\mu s^{\gamma+3}+C(\gamma-1)(\gamma-2)(1+\mu)^2 s^{\gamma+1}+(1+\mu)^2 2Ds]}{\left((1+\mu)-s^2\right)^2}+$$

$$\left.\left[\frac{[A\mu(1+\mu)s^{\alpha+1}+[B+D(1+\mu)]\mu(1+\mu)s-C\mu(1+\mu)^3 s^{\gamma-1}-D\mu(1+\mu)^3 s^{-1}+C\mu(1+\mu)^2 s^{\gamma+1}]\left((1+\mu)-s^2\right)}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\right\}=$$

$$\left\{\frac{[2(1+\mu)^4\alpha A s^{\alpha-1}+A(1+\mu)^3(2-2\alpha)s^{\alpha+1}+2(1+\mu)^3Bs]-[2(1+\mu)^2\alpha A\mu s^{\alpha+1}+A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}+2(1+\mu)B\mu s^3]}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{[\gamma(1-\gamma)C\mu(1+\mu)^2 s^{\gamma-1}-2C(\gamma-1)(\gamma-2)(1+\mu)^3 s^{\gamma-1}\ ]+[-2C\gamma(1-\gamma)\mu(1+\mu)s^{\gamma+1}+C(\gamma-1)(\gamma-2)(1+\mu)^2 s^{\gamma+1}]+[+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+\gamma(1-\gamma)C\mu s^{\gamma+3}+(1+\mu)^4 2Ds^{-3}-4D(1+\mu)^3 s^{-1}+(1+\mu)^2 2Ds]}{\left((1+\mu)-s^2\right)^2}+$$

$$\left.\left[\frac{[A\mu(1+\mu)^2 s^{\alpha+1}+[B+D(1+\mu)]\mu(1+\mu)^2 s-C\mu(1+\mu)^4 s^{\gamma-1}-D\mu(1+\mu)^4 s^{-1}+C\mu(1+\mu)^3 s^{\gamma+1}]-[A\mu(1+\mu)s^{\alpha+3}+[B+D(1+\mu)]\mu(1+\mu)s^3-C\mu(1+\mu)^3 s^{\gamma+1}-D\mu(1+\mu)^3 s+C\mu(1+\mu)^2 s^{\gamma+3}]}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\right\} \quad \text{(E17)}$$

Simplifications of the particular terms in the expression can be done separately:

$$\begin{aligned}
&[2(1+\mu)^4\alpha A s^{\alpha-1}+A(1+\mu)^3(2-2\alpha)s^{\alpha+1}+2(1+\mu)^3Bs]-[2(1+\mu)^2\alpha A\mu s^{\alpha+1}+A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}+2(1+\mu)B\mu s^3]\\
&\quad=[2(1+\mu)^4\alpha A s^{\alpha-1}+2A[(1+\mu)(1-\alpha)-\alpha\mu](1+\mu)^2 s^{\alpha+1}]-A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}+2(1+\mu)^3Bs-2(1+\mu)B\mu s^3\\
&\quad=2(1+\mu)^4\alpha A s^{\alpha-1}+2A(1+\mu-\alpha-2\alpha\mu)(1+\mu)^2 s^{\alpha+1}-A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}+2(1+\mu)^3Bs-2(1+\mu)B\mu s^3
\end{aligned}$$

$$\begin{aligned}
&[\gamma(1-\gamma)C\mu(1+\mu)^2 s^{\gamma-1}-2C(\gamma-1)(\gamma-2)(1+\mu)^3 s^{\gamma-1}\ ]+[-2C\gamma(1-\gamma)\mu(1+\mu)s^{\gamma+1}+C(\gamma-1)(\gamma-2)(1+\mu)^2 s^{\gamma+1}]\\
&\quad+[+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+\gamma(1-\gamma)C\mu s^{\gamma+3}+(1+\mu)^4 2Ds^{-3}-4D(1+\mu)^3 s^{-1}+(1+\mu)^2 2Ds]\\
&\quad=C[\gamma(1-\gamma)\mu-2(\gamma-1)(\gamma-2)(1+\mu)\ ](1+\mu)^2 s^{\gamma-1}+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)s^{\gamma+1}\\
&\quad+[+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+\gamma(1-\gamma)C\mu s^{\gamma+3}+(1+\mu)^4 2Ds^{-3}-4D(1+\mu)^3 s^{-1}+(1+\mu)^2 2Ds]\\
&\quad=C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^2 s^{\gamma-1}+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)s^{\gamma+1}\\
&\quad+[+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+C\gamma(1-\gamma)\mu s^{\gamma+3}+2D(1+\mu)^4 s^{-3}-4D(1+\mu)^3 s^{-1}+2D(1+\mu)^2 s]
\end{aligned}$$

$$\begin{aligned}
&[A\mu(1+\mu)^2 s^{\alpha+1}+[B+D(1+\mu)]\mu(1+\mu)^2 s-C\mu(1+\mu)^4 s^{\gamma-1}-D\mu(1+\mu)^4 s^{-1}+C\mu(1+\mu)^3 s^{\gamma+1}]\\
&\quad-[A\mu(1+\mu)s^{\alpha+3}+[B+D(1+\mu)]\mu(1+\mu)s^3-C\mu(1+\mu)^3 s^{\gamma+1}-D\mu(1+\mu)^3 s+C\mu(1+\mu)^2 s^{\gamma+3}]\\
&\quad=A\mu(1+\mu)^2 s^{\alpha+1}-A\mu(1+\mu)s^{\alpha+3}-C\mu(1+\mu)^4 s^{\gamma-1}+2C\mu(1+\mu)^3 s^{\gamma+1}-C\mu(1+\mu)^2 s^{\gamma+3}-D\mu(1+\mu)^4 s^{-1}\\
&\quad+[B+2D(1+\mu)]\mu(1+\mu)^2 s-[B+D(1+\mu)]\mu(1+\mu)s^3
\end{aligned}$$

We can then continue with modifications of (E17)

$$\left\{\frac{2(1+\mu)^4\alpha A s^{\alpha-1}+2A(1+\mu-\alpha-2\alpha\mu)(1+\mu)^2 s^{\alpha+1}-A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}+2(1+\mu)^3Bs-2(1+\mu)B\mu s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^2 s^{\gamma-1}+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)s^{\gamma+1}+[+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+C\gamma(1-\gamma)\mu s^{\gamma+3}+2D(1+\mu)^4 s^{-3}-4D(1+\mu)^3 s^{-1}+2D(1+\mu)^2 s]}{\left((1+\mu)-s^2\right)^2}+$$

$$\left.\left[\frac{A\mu(1+\mu)^2 s^{\alpha+1}-A\mu(1+\mu)s^{\alpha+3}-C\mu(1+\mu)^4 s^{\gamma-1}+2C\mu(1+\mu)^3 s^{\gamma+1}-C\mu(1+\mu)^2 s^{\gamma+3}-D\mu(1+\mu)^4 s^{-1}+[B+2D(1+\mu)]\mu(1+\mu)^2 s-[B+D(1+\mu)]\mu(1+\mu)s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\right\} \quad \text{(E18)}$$

The numerator of the second term multiplied by $(1+\mu)^2-\mu s^2$ is:

$$\begin{aligned}
&[(1+\mu)^2-\mu s^2]\{C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^2 s^{\gamma-1}+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)s^{\gamma+1}\\
&\quad+[+C(\gamma-1)(\gamma-2)(1+\mu)^4 s^{\gamma-3}+C\gamma(1-\gamma)\mu s^{\gamma+3}+2D(1+\mu)^4 s^{-3}-4D(1+\mu)^3 s^{-1}+2D(1+\mu)^2 s]\}\\
&=\{C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^4 s^{\gamma-1}+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)^3 s^{\gamma+1}\\
&\quad+[+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}+C\gamma(1-\gamma)\mu(1+\mu)^2 s^{\gamma+3}+2D(1+\mu)^6 s^{-3}-4D(1+\mu)^5 s^{-1}+2D(1+\mu)^4 s]\}\\
&-\{C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^2\mu s^{\gamma+1}+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)\mu s^{\gamma+3}\\
&\quad+[+C(\gamma-1)(\gamma-2)(1+\mu)^4\mu s^{\gamma-1}+C\gamma(1-\gamma)\mu^2 s^{\gamma+5}+2D(1+\mu)^4\mu s^{-1}-4D(1+\mu)^3\mu s+2D(1+\mu)^2\mu s^3]\}\\
&=C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^4 s^{\gamma-1}-C(\gamma-1)(\gamma-2)(1+\mu)^4\mu s^{\gamma-1}\\
&+C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)^3 s^{\gamma+1}-C[(-3\gamma\mu-2\gamma+4+4\mu)\ ](\gamma-1)(1+\mu)^2\mu s^{\gamma+1}\\
&+[+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}+C\gamma(1-\gamma)\mu(1+\mu)^2 s^{\gamma+3}-C[2\gamma\mu+(\gamma-2)(1+\mu)](\gamma-1)(1+\mu)\mu s^{\gamma+3}+2D(1+\mu)^6 s^{-3}\\
&-4D(1+\mu)^5 s^{-1}-2D(1+\mu)^4\mu s^{-1}+2D(1+\mu)^4 s+4D(1+\mu)^3\mu s]+[-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}-2D(1+\mu)^2\mu s^3]\\
&=C\{[(-3\gamma\mu-2\gamma+4+4\mu)\ ]-(\gamma-2)\mu\}(\gamma-1)(1+\mu)^4 s^{\gamma-1}\\
&+C\{[2\gamma\mu+(\gamma-2)(1+\mu)](1+\mu)-[(-3\gamma\mu-2\gamma+4+4\mu)\ ]\mu\}(\gamma-1)(1+\mu)^2 s^{\gamma+1}+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}\\
&+C\{-\gamma(1+\mu)-[2\gamma\mu+(\gamma-2)(1+\mu)]\}(\gamma-1)\mu(1+\mu)s^{\gamma+3}-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}+2D(1+\mu)^6 s^{-3}-4D(1+\mu)^5 s^{-1}\\
&-2D(1+\mu)^4\mu s^{-1}+2D\{(1+\mu)+2\mu\}(1+\mu)^3 s-2D(1+\mu)^2\mu s^3\\
&=C\{-4\gamma\mu-2\gamma+6\mu+4\}(\gamma-1)(1+\mu)^4 s^{\gamma-1}+C\{6\gamma\mu+\gamma-2-8\mu+6\gamma\mu\mu-6\mu\mu\}(\gamma-1)(1+\mu)^2 s^{\gamma+1}\\
&+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}-C\{4\gamma\mu+2\gamma-2-2\mu\}(\gamma-1)\mu(1+\mu)s^{\gamma+3}-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}+2D(1+\mu)^6 s^{-3}\\
&-2D(2+3\mu)(1+\mu)^4 s^{-1}+2D(1+3\mu)(1+\mu)^3 s-2D(1+\mu)^2\mu s^3
\end{aligned}$$

Then, the expression in curl bracket (the pre-factor of the reciprocal square root in the second derivative) is (note: the second term of it has to be split in order the ratio fits one line):

$$\left\{\frac{2(1+\mu)^4\alpha A s^{\alpha-1}+2A(1+\mu-\alpha-2\alpha\mu)(1+\mu)^2 s^{\alpha+1}-A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}+2(1+\mu)^3Bs-2(1+\mu)B\mu s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{C\{-4\gamma\mu-2\gamma+6\mu+4\}(\gamma-1)(1+\mu)^4 s^{\gamma-1}+C\{6\gamma\mu+\gamma-2-8\mu+6\gamma\mu\mu-6\mu\mu\}(\gamma-1)(1+\mu)^2 s^{\gamma+1}+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}-C\{4\gamma\mu+2\gamma-2-2\mu\}(\gamma-1)\mu(1+\mu)s^{\gamma+3}-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+$$

$$\frac{+2D(1+\mu)^6 s^{-3}-2D(2+3\mu)(1+\mu)^4 s^{-1}+2D(1+3\mu)(1+\mu)^3 s-2D(1+\mu)^2\mu s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+$$

$$\left[\frac{A\mu(1+\mu)^2 s^{\alpha+1}-A\mu(1+\mu)s^{\alpha+3}-C\mu(1+\mu)^4 s^{\gamma-1}+2C\mu(1+\mu)^3 s^{\gamma+1}-C\mu(1+\mu)^2 s^{\gamma+3}-D\mu(1+\mu)^4 s^{-1}+[B+2D(1+\mu)]\mu(1+\mu)^2 s-[B+D(1+\mu)]\mu(1+\mu)s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\Bigg\} \quad \text{(E19)}$$

Now, put together the same power in *s* (as there is the same denominator in all four terms, we can move the terms in the numerator from one ratio to the other):

$$\left\{\frac{2(1+\mu)^4\alpha A s^{\alpha-1}+2A(1+\mu-\alpha-2\alpha\mu)(1+\mu)^2 s^{\alpha+1}+A\mu(1+\mu)^2 s^{\alpha+1}-A(1+\mu)(2-2\alpha)\mu s^{\alpha+3}-A\mu(1+\mu)s^{\alpha+3}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{C\{-4\gamma\mu-2\gamma+6\mu+4\}(\gamma-1)(1+\mu)^4 s^{\gamma-1}-C\mu(1+\mu)^4 s^{\gamma-1}+C\{6\gamma\mu+\gamma-2-8\mu+6\gamma\mu\mu-6\mu\mu\}(\gamma-1)(1+\mu)^2 s^{\gamma+1}+2C\mu(1+\mu)^3 s^{\gamma+1}+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}-C\{4\gamma\mu+2\gamma-2-2\mu\}(\gamma-1)\mu(1+\mu)s^{\gamma+3}-C\mu(1+\mu)^2 s^{\gamma+3}-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}$$

$$\left.\left[\frac{+2D(1+\mu)^6 s^{-3}-D\mu(1+\mu)^4 s^{-1}-2D(2+3\mu)(1+\mu)^4 s^{-1}+[B+2D(1+\mu)]\mu(1+\mu)^2 s+2(1+\mu)^3 Bs+2D(1+3\mu)(1+\mu)^3 s-[B+D(1+\mu)]\mu(1+\mu)s^3-2(1+\mu)B\mu s^3-2D(1+\mu)^2\mu s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\right\} \quad \text{(E20)}$$

Join the same power terms

$$\left\{\frac{2(1+\mu)^4\alpha A s^{\alpha-1}+A(2+3\mu-2\alpha-4\alpha\mu)(1+\mu)^2 s^{\alpha+1}-A(3-2\alpha)(1+\mu)\mu s^{\alpha+3}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{C[-4\gamma\gamma\mu-2\gamma\gamma+10\gamma\mu+6\gamma-7\mu-4](1+\mu)^4 s^{\gamma-1}+C[6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu](1+\mu)^2 s^{\gamma+1}+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}+C[-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu](1+\mu)\mu s^{\gamma+3}-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+$$

$$\left.\left[\frac{+2D(1+\mu)^6 s^{-3}-D(4+7\mu)(1+\mu)^4 s^{-1}+\{B(2+3\mu)+2D(1+4\mu)(1+\mu)\}(1+\mu)^2 s-3\{B+D(1+\mu)\}(1+\mu)\mu s^3}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\right\} \quad \text{(E21)}$$

The second derivative is thus

$$\frac{d^2F_{test2}(s)}{ds^2}=$$

$$\alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+$$

$$\left\{\frac{-A(3-2\alpha)(1+\mu)\mu s^{\alpha+3}+A(2+3\mu-2\alpha-4\alpha\mu)(1+\mu)^2 s^{\alpha+1}+2(1+\mu)^4\alpha A s^{\alpha-1}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+\right.$$

$$\frac{-C\gamma(1-\gamma)\mu^2 s^{\gamma+5}+C[-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu](1+\mu)\mu s^{\gamma+3}+C[6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu](1+\mu)^2 s^{\gamma+1}+C[-4\gamma\gamma\mu-2\gamma\gamma+10\gamma\mu+6\gamma-7\mu-4](1+\mu)^4 s^{\gamma-1}+C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-3}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}+$$

$$\left.\left[\frac{-3\{B+D(1+\mu)\}(1+\mu)\mu s^3+\{B(2+3\mu)+2D(1+4\mu)(1+\mu)\}(1+\mu)^2 s-D(4+7\mu)(1+\mu)^4 s^{-1}+2D(1+\mu)^6 s^{-3}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right]\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}, \quad \text{(E22)}$$

which can be rewritten with a help of a set of substitutions

$$\{\eta_{\alpha+3}=-A(3-2\alpha)(1+\mu)\mu,\ \eta_{\alpha+1}=A(2+3\mu-2\alpha-4\alpha\mu)(1+\mu)^2,\ \eta_{\alpha-1}=2(1+\mu)^4\alpha A\},\quad \{\eta_{\gamma+5}=-C\gamma(1-\gamma)\mu^2,\ \eta_{\gamma+3}=C[-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu](1+\mu)\mu,\ \eta_{\gamma+1}=C[6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu](1+\mu)^2,\ \eta_{\gamma-1}=C[-4\gamma\gamma\mu-2\gamma\gamma+10\gamma\mu+6\gamma-7\mu-4](1+\mu)^4,\ \eta_{\gamma-3}=C(\gamma-1)(\gamma-2)(1+\mu)^6\},\quad \{\eta_3=-3\{B+D(1+\mu)\}(1+\mu)\mu,\ \eta_1=\{B(2+3\mu)+2D(1+4\mu)(1+\mu)\}(1+\mu)^2,\ \eta_{-1}=-D(4+7\mu)(1+\mu)^4,\ \eta_{-3}=+2D(1+\mu)^6\} \quad \text{(E23)}$$

to

$$\frac{d^2F_{test2}(s)}{ds^2}=$$

$$\alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+$$

$$\left\{\frac{\eta_{\alpha+3}s^{\alpha+3}+\eta_{\alpha+1}s^{\alpha+1}+\eta_{\alpha-1}s^{\alpha-1}+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3 s^3+\eta_1 s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}. \quad \text{(E24)}$$

Inserting the found derivatives to the generalized Legendre equation

Thus, for the new test function (E6), and its first derivative (E14), and its second derivative (E22), where E23) substitutions are used, the generalized Legendre equation (E2) becomes the form

$$H_2(s)\left[\alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+\left\{\frac{\eta_{\alpha+3}s^{\alpha+3}+\eta_{\alpha+1}s^{\alpha+1}+\eta_{\alpha-1}s^{\alpha-1}+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3 s^3+\eta_1 s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{\left((1+\mu)-s^2\right)^2[(1+\mu)^2-\mu s^2]}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right]+$$

$$H_1(s)\left[\alpha A s^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+\left\{\frac{(1+\mu)(As^{\alpha}+B)}{((1+\mu)-s^2)}+\gamma C s^{\gamma-2}\left[(1+\mu)^2-\mu s^2\right]-\frac{(1+\mu)^2(Cs^{\gamma}+D)}{s^2}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right]+$$
$$H_0(s)\left[(As^{\alpha}+B)\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+(Cs^{\gamma}+D)\frac{(1+\mu)^2-\mu s^2}{s}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}\right]=0 \quad . \qquad \text{(E25)}$$

As $\sinh^{-1}$ and the reciprocal-square root are distinct functions and cannot combine to mutually cancel out, both parts of the equation after substitution of the derivatives and the function itself (the one with $\sinh^{-1}$ and the one with the reciprocal-square root) has to hold separately.

Solving the part of the equation with $\sinh^{-1}$

The part of the equation with $\sinh^{-1}$ (which is the simpler part) is:

$$H_2(s)\,\alpha(\alpha-1)As^{\alpha-2}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+H_1(s)\,\alpha A s^{\alpha-1}\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+H_0(s)\,(As^{\alpha}+B)\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)=0 \quad . \text{(E26)}$$

Divide it by $\sinh^{-1}$ function and we obtain

$$H_2(s)\,\alpha(\alpha-1)As^{\alpha-2}+H_1(s)\,\alpha A s^{\alpha-1}+H_0(s)\,(As^{\alpha}+B)=0 \quad . \qquad \text{(E27)}$$

Then (by inserting (E3), (E4) and (E5)) we get

$$[(1+\mu)^3-(2\mu+1)(1+\mu)s^2+\mu s^4]\alpha(\alpha-1)As^{\alpha-2}+s[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2]\alpha A s^{\alpha-1}+K_d[(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2](As^{\alpha}+B)=0 \quad , \qquad \text{(E28)}$$

$$(1+\mu)^3\alpha(\alpha-1)As^{\alpha-2}-(2\mu+1)(1+\mu)\alpha(\alpha-1)As^{\alpha}+\mu\alpha(\alpha-1)As^{\alpha+2}+[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d]\alpha A s^{\alpha}+\mu(3-2K_d)\alpha A s^{\alpha+2}+K_d(K_d-2)\mu A s^{\alpha+2}+[K_d+(1+\mu)](1+\mu)K_d A s^{\alpha}+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad , \qquad \text{(E29)}$$

$$\mu\alpha(\alpha-1)As^{\alpha+2}+\mu(3-2K_d)\alpha A s^{\alpha+2}+K_d(K_d-2)\mu A s^{\alpha+2}-(2\mu+1)(1+\mu)\alpha(\alpha-1)As^{\alpha}+[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d]\alpha A s^{\alpha}+[K_d+(1+\mu)](1+\mu)K_d A s^{\alpha}+(1+\mu)^3\alpha(\alpha-1)As^{\alpha-2}+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad , \qquad \text{(E30)}$$

$$A\{\alpha(\alpha-1)+(3-2K_d)\alpha+K_d(K_d-2)\}\mu s^{\alpha+2}+A\{-(2\mu+1)\alpha(\alpha-1)+[-(3\mu+2)+2\mu K_d]\alpha+[K_d+(1+\mu)]K_d\}(1+\mu)s^{\alpha}+A(1+\mu)^3\alpha(\alpha-1)s^{\alpha-2}+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad , \qquad \text{(E31)}$$

Now, test several possibilities for α values.

In the case when α=0:

$$AK_d(K_d-2)\mu s^2+A[K_d+(1+\mu)]K_d(1+\mu)+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad . \qquad \text{E32)}$$

Zero-power terms (as well as power-two terms) mutually cancel when B=–A, which means trivial solution regardless of the value of K_d.

Even when K_d=2 or K_d=–(1+μ), the condition B=–A has to be fulfilled.

Second (different) possibility is when K_d=0: then, A and B are arbitrary (but as α=0, they form together only one parameter A+B). This is the already well known solution for K_d=0.

In the case when α=1:

$$A\{(3-2K_d)+K_d(K_d-2)\}\mu s^3+A\{[-(3\mu+2)+2\mu K_d]+[K_d+(1+\mu)]K_d\}(1+\mu)s+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad , \qquad \text{(E33)}$$

$$A[3-4K_d+K_dK_d]\mu s^3+A[K_dK_d+(1+3\mu)K_d-3\mu-2](1+\mu)s+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad . \qquad \text{(E34)}$$

Power-two term is zero when B=0 or when K_d=0 or when K_d=2.
Zero-power term is zero when B=0 or when K_d=0 or when K_d=–(1+μ).
Then either B=0 or K_d=0.
Power term one is zero when A=0 or when K_d=1.
Power term three is zero when A=0 or when K_d=1 or when K_d=3.
All the conditions are fulfilled for K_d=0, A=0, B arbitrary – but this is the same solution as in the previous paragraph for α=0.
All the conditions are fulfilled also for K_d=1, B=0, A arbitrary –this is the already known solution (part of it) for K_d=1 – the one following from the general solution of the Ricatti equation.

In the case when α=2:

$$A\{2+2(3-2K_d)+K_d(K_d-2)\}\mu s^4+A\{-2(2\mu+1)+2[-(3\mu+2)+2\mu K_d]+[K_d+(1+\mu)]K_d\}(1+\mu)s^2+2A(1+\mu)^3+BK_d(K_d-2)\mu s^2+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad , \qquad \text{(E35)}$$

$$A(K_dK_d-6K_d+8)\mu s^4+A\{[K_dK_d+(1+5\mu)K_d-10\mu-6]\}(1+\mu)s^2+BK_d(K_d-2)\mu s^2+2A(1+\mu)^3+B[K_d+(1+\mu)](1+\mu)K_d=0 \quad , \qquad \text{(E36)}$$

$$A(K_dK_d-6K_d+8)\mu s^4+\{[A(1+\mu)+B\mu]K_dK_d+[A(1+5\mu)(1+\mu)-2B\mu]K_d-A(10\mu+6)(1+\mu)\}s^2+[B(1+\mu)K_dK_d+B(1+\mu)^2K_d+2A(1+\mu)^3]=0 \quad . \qquad \text{(E37)}$$

Power term four is zero when A=0 or when K_d=2 or when K_d=4.

A=0: then the equation simplifies to

$$B\mu K_d\{K_d-2\}s^2+B(1+\mu)K_d[K_d+(1+\mu)]=0 \quad , \qquad \text{(E38)}$$

and either K_d=0 or B=0, which both are equivalent to previously discussed cases (as also A=0).

K_d=4: then the equation simplifies to

$$A(0)\mu s^4+\{[A(1+\mu)+B\mu]16+[A(1+5\mu)(1+\mu)-2B\mu]4-A(10\mu+6)(1+\mu)\}s^2+[B(1+\mu)16+B(1+\mu)^2 4+2A(1+\mu)^3]=0 \quad . \qquad \text{(E39)}$$

The power term four is zero.
From the power two term, we have

$$\{[A(1+\mu)+B\mu]16+[A(1+5\mu)(1+\mu)-2B\mu]4-A(10\mu+6)(1+\mu)\}=0 \quad , \qquad \text{(E40)}$$

$$\{[16A(1+\mu)+16B\mu]+[4A(1+5\mu)(1+\mu)-8B\mu]-A(10\mu+6+10\mu\mu+6\mu)\}=0 \quad , \qquad \text{(E41)}$$

$$\{16A(1+\mu)+4A(1+6\mu+5\mu\mu)-A(16\mu+6+10\mu\mu)+16B\mu-8B\mu\}=0 \quad , \qquad \text{(E42)}$$

$$(16A+16A\mu)+(4A+24A\mu+20A\mu\mu)-(16A\mu+6A+10A\mu\mu)+8B\mu=0 \quad , \qquad \text{(E43)}$$

$$2A(7+12\mu+5\mu\mu)+8B\mu=0 \quad , \qquad \text{(E44)}$$

$$\frac{A}{B}=\frac{4\mu}{7+12\mu+5\mu\mu} \quad . \qquad \text{(E45)}$$

From the zero-th power term, we have

$$[B(1+\mu)16+B(1+\mu)^2 4+2A(1+\mu)^3]=0 \quad , \qquad \text{(E46)}$$

$$[8B+2B(1+\mu)+A(1+\mu)^2]=0 \quad , \qquad \text{E47)}$$

$$\frac{A}{B}=\frac{-8+2(1+\mu)}{(1+\mu)^2} \quad , \qquad \text{(E48)}$$

which is different than for the power term two. Then, there is no solution for K_d=4 in this case (i.e. α=2).

K_d=2: then the equation simplifies to

$$A(4-12+8)\mu s^4+\{[A(1+\mu)+B\mu]4+[A(1+5\mu)(1+\mu)-2B\mu]2-A(10\mu+6)(1+\mu)\}s^2+[B(1+\mu)4+2B(1+\mu)^2+2A(1+\mu)^3]=0 \quad , \qquad \text{(E49)}$$

$$A(0)\mu s^4+\{4A(1+\mu)+4B\mu+2A(1+5\mu)(1+\mu)-4B\mu-A(10\mu+6)(1+\mu)\}s^2+[2(3+\mu)B+2A(1+\mu)^2](1+\mu)=0 \quad . \qquad \text{(E50)}$$

The power term four is zero.
From the power term two, we have

$$\{4A(1+\mu)+4B\mu+2A(1+5\mu)(1+\mu)-4B\mu-A(10\mu+6)(1+\mu)\}=0 \quad , \qquad \text{(E51)}$$

$$\{A[4+2(1+5\mu)-(10\mu+6)](1+\mu)+4B\mu-4B\mu\}=0 \quad , \qquad \text{(E52)}$$

$$\{A[6+10\mu-(10\mu+6)](1+\mu)+4B\mu-4B\mu\}=0 \quad , \qquad \text{(E53)}$$

which is always fulfilled.
From the zero-th power term, we have

$$[2(3+\mu)B+2A(1+\mu)^2]=0 \quad \Rightarrow \quad \frac{A}{B}=-\frac{(3+\mu)}{(1+\mu)^2} \quad . \qquad \text{(E54)}$$

Then, we have a partial solution for K_d=2 in the case when α=2, with the given ratio of *A* and *B* constants.
Summary: The tests with α=0 and α=1 led to no solution. In case α=2, a possible solution was found. This solution is to be tested for the second part (square root) of the equation.

Solving the part of the equation with the reciprocal square root

The second part of the equation with the reciprocal square root is:

$$H_2(s)\left\{\frac{\eta_{\alpha+3}s^{\alpha+3}+\eta_{\alpha+1}s^{\alpha+1}+\eta_{\alpha-1}s^{\alpha-1}+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3s^3+\eta_1s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{((1+\mu)-s^2)^2[(1+\mu)^2-\mu s^2]}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}+$$
$$H_1(s)\left\{\frac{(1+\mu)(As^{\alpha}+B)}{((1+\mu)-s^2)}+\gamma Cs^{\gamma-2}\,[(1+\mu)^2-\mu s^2]-\frac{(1+\mu)^2(Cs^{\gamma}+D)}{s^2}\right\}\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}+$$
$$H_0(s)\,(Cs^{\gamma}+D)\,\frac{(1+\mu)^2-\mu s^2}{s}\,\frac{1}{\sqrt{(1+\mu)^2-\mu s^2}}=0 \quad . \qquad \text{(E55)}$$

where

$$\{\eta_{\alpha+3}=-A(3-2\alpha)(1+\mu)\mu,\ \eta_{\alpha+1}=A(2+3\mu-2\alpha-4\alpha\mu)(1+\mu)^2,\ \eta_{\alpha-1}=2(1+\mu)^4\alpha A\},\quad \{\eta_{\gamma+5}=-C\gamma(1-\gamma)\mu^2,\ \eta_{\gamma+3}= C[-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu](1+\mu)\mu,\ \eta_{\gamma+1}=C[6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu](1+\mu)^2,\ \eta_{\gamma-1}= C[-4\gamma\gamma\mu-2\gamma\gamma+10\gamma\mu+6\gamma-7\mu-4](1+\mu)^4,\ \eta_{\gamma-3}=C(\gamma-1)(\gamma-2)(1+\mu)^6\},\quad \{\eta_3=-3\{B+D(1+\mu)\}(1+\mu)\mu,\ \eta_1= \{B(2+3\mu)+2D(1+4\mu)(1+\mu)\}(1+\mu)^2,\ \eta_{-1}=-D(4+7\mu)(1+\mu)^4,\ \eta_{-3}=+2D(1+\mu)^6\} \quad . \qquad \text{(E56)}$$

Multiply (E55) by the reciprocal square root and receive

$$H_2(s)\left\{\frac{\eta_{\alpha+3}s^{\alpha+3}+\eta_{\alpha+1}s^{\alpha+1}+\eta_{\alpha-1}s^{\alpha-1}+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3s^3+\eta_1s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{((1+\mu)-s^2)^2[(1+\mu)^2-\mu s^2]}\right\}+H_1(s)\left\{\frac{(1+\mu)(As^{\alpha}+B)}{((1+\mu)-s^2)}+\right.$$
$$\left.\gamma Cs^{\gamma-2}\,[(1+\mu)^2-\mu s^2]-\frac{(1+\mu)^2(Cs^{\gamma}+D)}{s^2}\right\}+H_0(s)\,(Cs^{\gamma}+D)\,\frac{(1+\mu)^2-\mu s^2}{s}=0 \quad . \qquad \text{(E57)}$$

For the sinh[-1] part, it was found (see (E54)) that a possible solution with α=2 and K_d=2 has to have *A* and *B* ratio

$$\frac{A}{B}=-\frac{(3+\mu)}{(1+\mu)^2} \quad \Rightarrow \quad B=-\frac{(1+\mu)^2}{(3+\mu)}A \quad . \qquad \text{(E58)}$$

Now test this for the second part (square root) equation by inserting the *B* parameter according to (E58) to (E57):

$$H_2(s)\left\{\frac{\eta_{\alpha+3}s^{2+3}+\eta_{\alpha+1}s^{2+1}+\eta_{\alpha-1}s^{2-1}+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3s^3+\eta_1s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{((1+\mu)-s^2)^2[(1+\mu)^2-\mu s^2]}\right\}+$$
$$H_1(s)\left\{\frac{(1+\mu)\left(As^2-\frac{(1+\mu)^2}{(3+\mu)}A\right)}{((1+\mu)-s^2)}+\gamma Cs^{\gamma-2}\,[(1+\mu)^2-\mu s^2]-\frac{(1+\mu)^2(Cs^{\gamma}+D)}{s^2}\right\}+H_0(s)\,(Cs^{\gamma}+D)\,\frac{(1+\mu)^2-\mu s^2}{s}=0 \quad . \qquad \text{(E59)}$$

where the H_i polynomials are given by (E3 (E4) and (E5) and where

$$\{\eta_{\alpha+3}=-A(3-2.2)(1+\mu)\mu,\ \eta_{\alpha+1}=A(2+3\mu-2.2-4.2.\mu)(1+\mu)^2,\ \eta_{\alpha-1}=2(1+\mu)^4.2.A\},\quad \{\eta_{\gamma+5}=-C\gamma(1-\gamma)\mu^2,\ \eta_{\gamma+3}= C[-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu](1+\mu)\mu,\ \eta_{\gamma+1}=C[6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu](1+\mu)^2,\ \eta_{\gamma-1}=$$

$C[-4\gamma\gamma\mu - 2\gamma\gamma + 10\gamma\mu + 6\gamma - 7\mu - 4](1+\mu)^4,\ \ \eta_{\gamma-3} = C(\gamma-1)(\gamma-2)(1+\mu)^6\},\quad \left\{\eta_3 = -3\left\{-\frac{(1+\mu)^2}{(3+\mu)}A + D(1+\mu)\right\}(1+\mu)\mu,\ \eta_1 = \left\{-\frac{(1+\mu)^2}{(3+\mu)}A(2+3\mu) + 2D(1+4\mu)(1+\mu)\right\}(1+\mu)^2,\ \eta_{-1} = -D(4+7\mu)(1+\mu)^4,\ \eta_{-3} = +2D(1+\mu)^6\right\}\quad ,\qquad \text{(E60)}$

which can be further simplified to

$$\{\eta_{\alpha+3} = A(1+\mu)\mu,\ \eta_{\alpha+1} = A(-5\mu-2)(1+\mu)^2,\ \eta_{\alpha-1} = 4A(1+\mu)^4\},\quad \{\eta_{\gamma+5} = -C\gamma(1-\gamma)\mu^2,\ \eta_{\gamma+3} = C[-4\gamma\gamma\mu - 2\gamma\gamma + 6\gamma\mu + 4\gamma - 3 - 3\mu](1+\mu)\mu,\ \eta_{\gamma+1} = C[6\gamma\gamma\mu + \gamma\gamma - 3\gamma - 14\gamma\mu + 6\gamma\gamma\mu\mu - 12\mu\mu\gamma + 2 + 10\mu + 8\mu\mu](1+\mu)^2,\ \eta_{\gamma-1} = C[-4\gamma\gamma\mu - 2\gamma\gamma + 10\gamma\mu + 6\gamma - 7\mu - 4](1+\mu)^4,\ \ \eta_{\gamma-3} = C(\gamma-1)(\gamma-2)(1+\mu)^6\},\quad \left\{\eta_3 = -3\left\{-\frac{(1+\mu)}{(3+\mu)}A + D\right\}(1+\mu)^2\mu,\ \eta_1 = \left\{-\frac{(1+\mu)}{(3+\mu)}A(2+3\mu) + 2D(1+4\mu)\right\}(1+\mu)^3,\ \eta_{-1} = -D(4+7\mu)(1+\mu)^4,\ \eta_{-3} = +2D(1+\mu)^6\right\}\quad . \qquad \text{(E61)}$$

After inserting H polynomials, the equation is

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\left\{\frac{\eta_{\alpha+3}s^5+\eta_{\alpha+1}s^3+\eta_{\alpha-1}s^1+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3s^3+\eta_1s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{((1+\mu)-s^2)^2[(1+\mu)^2-\mu s^2]}\right\} + s[(\mu\mu-\mu-2)-\mu s^2]\left\{\frac{(1+\mu)\left(As^2-\frac{(1+\mu)^2}{(3+\mu)}A\right)}{((1+\mu)-s^2)} + \gamma Cs^{\gamma-2}[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s^2}\right\} + 2(\mu+1)(\mu+3)(Cs^\gamma+D)\frac{(1+\mu)^2-\mu s^2}{s} = 0\quad ,\qquad \text{(E62)}$$

and simplified

$$\left\{\frac{\eta_{\alpha+3}s^5+\eta_{\alpha+1}s^3+\eta_{\alpha-1}s^1+\eta_{\gamma+5}s^{\gamma+5}+\eta_{\gamma+3}s^{\gamma+3}+\eta_{\gamma+1}s^{\gamma+1}+\eta_{\gamma-1}s^{\gamma-1}+\eta_{\gamma-3}s^{\gamma-3}+\eta_3s^3+\eta_1s+\eta_{-1}s^{-1}+\eta_{-3}s^{-3}}{((1+\mu)-s^2)}\right\} + [(\mu\mu-\mu-2)-\mu s^2]\left\{s\frac{(1+\mu)\left(As^2-\frac{(1+\mu)^2}{(3+\mu)}A\right)}{((1+\mu)-s^2)} + \gamma Cs^{\gamma-1}[(1+\mu)^2-\mu s^2] - \frac{(1+\mu)^2(Cs^\gamma+D)}{s}\right\} + 2(\mu+1)(\mu+3)(Cs^\gamma+D)\frac{(1+\mu)^2-\mu s^2}{s} = 0\quad .\qquad \text{(E63)}$$

Multiply it by s with the result

$$\left\{\frac{\eta_{\alpha+3}s^6+\eta_{\alpha+1}s^4+\eta_{\alpha-1}s^2+\eta_{\gamma+5}s^{\gamma+6}+\eta_{\gamma+3}s^{\gamma+4}+\eta_{\gamma+1}s^{\gamma+2}+\eta_{\gamma-1}s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+\eta_3s^4+\eta_1s^2+\eta_{-1}+\eta_{-3}s^{-2}}{((1+\mu)-s^2)}\right\} + [(\mu\mu-\mu-2)-\mu s^2]\left\{\frac{A(1+\mu)\left(s^4-\frac{(1+\mu)^2}{(3+\mu)}s^2\right)}{((1+\mu)-s^2)} + \gamma Cs^{\gamma}[(1+\mu)^2-\mu s^2] - (1+\mu)^2(Cs^\gamma+D)\right\} + 2(\mu+1)(\mu+3)(Cs^\gamma+D)[(1+\mu)^2-\mu s^2] = 0\quad .\qquad \text{(E64)}$$

Expand and simplify the middle term, and expand the last term:

$$\left\{\frac{\eta_{\alpha+3}s^6+\eta_{\alpha+1}s^4+\eta_{\alpha-1}s^2+\eta_{\gamma+5}s^{\gamma+6}+\eta_{\gamma+3}s^{\gamma+4}+\eta_{\gamma+1}s^{\gamma+2}+\eta_{\gamma-1}s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+\eta_3s^4+\eta_1s^2+\eta_{-1}+\eta_{-3}s^{-2}}{((1+\mu)-s^2)}\right\} + [(\mu\mu-\mu-2)-\mu s^2]\left\{\frac{A(1+\mu)\left(s^4-\frac{(1+\mu)^2}{(3+\mu)}s^2\right)}{((1+\mu)-s^2)} + C(\gamma-1)(1+\mu)^2s^\gamma - C\gamma\mu s^{\gamma+2} - (1+\mu)^2D\right\} + 2(\mu+1)(\mu+3)(C(1+\mu)^2s^\gamma + D(1+\mu)^2 - C\mu s^{\gamma+2} - D\mu s^2) = 0\quad .\qquad \text{(E65)}$$

Multiply it by $((1+\mu)-s^2)$, which results in the equation

$$\{\eta_{\alpha+3}s^6 + (\eta_{\alpha+1}+\eta_3)s^4 + (\eta_{\alpha-1}+\eta_1)s^2 + \eta_{\gamma+5}s^{\gamma+6} + \eta_{\gamma+3}s^{\gamma+4} + \eta_{\gamma+1}s^{\gamma+2} + \eta_{\gamma-1}s^\gamma + \eta_{\gamma-3}s^{\gamma-2} + \eta_{-1} + \eta_{-3}s^{-2}\} + [(\mu\mu-\mu-2)-\mu s^2]\left\{\left(A(1+\mu)s^4 - A\frac{(1+\mu)^3}{(3+\mu)}s^2\right) + ((1+\mu)-s^2)C(\gamma-1)(1+\mu)^2s^\gamma - ((1+\mu)-s^2)C\gamma\mu s^{\gamma+2} - ((1+\mu)-s^2)(1+\mu)^2D\right\} + 2(\mu+1)(\mu+3)(C(1+\mu)^2s^\gamma + D(1+\mu)^2 - C\mu s^{\gamma+2} - D\mu s^2)((1+\mu)-s^2) = 0\quad ,\qquad \text{(E66)}$$

where

$$\left\{\eta_{\alpha+3} = A(1+\mu)\mu,\ \eta_{\alpha+1}+\eta_3 = \left[A(-5\mu-2) + 3A\frac{(1+\mu)\mu}{(3+\mu)} - 3D\mu\right](1+\mu)^2,\ \eta_{\alpha-1}+\eta_1 = \left[4A(1+\mu) - A\frac{(1+\mu)(2+3\mu)}{(3+\mu)} + 2D(1+4\mu)\right](1+\mu)^3\right\},\quad \{\eta_{\gamma+5} = -C\gamma(1-\gamma)\mu^2,\ \eta_{\gamma+3} = C[-4\gamma\gamma\mu - 2\gamma\gamma + 6\gamma\mu + 4\gamma - 3 - 3\mu](1+\mu)\mu,\ \eta_{\gamma+1} = C[6\gamma\gamma\mu + \gamma\gamma - 3\gamma - 14\gamma\mu + 6\gamma\gamma\mu\mu - 12\mu\mu\gamma + 2 + 10\mu + 8\mu\mu](1+\mu)^2,\ \eta_{\gamma-1} = C[-4\gamma\gamma\mu - 2\gamma\gamma + 10\gamma\mu + 6\gamma - 7\mu - 4](1+\mu)^4,\ \ \eta_{\gamma-3} = C(\gamma-1)(\gamma-2)(1+\mu)^6\},\quad \{\eta_{-1} = -D(4+7\mu)(1+\mu)^4,\ \eta_{-3} = +2D(1+\mu)^6\}\quad .\qquad \text{(E67)}$$

Expand the terms of the equation:

$$\Big\{\eta_{\alpha+3}s^6+(\eta_{\alpha+1}+\eta_3)s^4+(\eta_{\alpha-1}+\eta_1)s^2+\eta_{\gamma+5}s^{\gamma+6}+\eta_{\gamma+3}s^{\gamma+4}+\eta_{\gamma+1}s^{\gamma+2}+\eta_{\gamma-1}s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+\eta_{-1}+\eta_{-3}s^{-2}\Big\}+[(\mu\mu-\mu-2)-\mu s^2]\Big\{\Big(A(1+\mu)s^4-A\frac{(1+\mu)^3}{(3+\mu)}s^2\Big)+C(\gamma-1)(1+\mu)^3s^{\gamma}-C(\gamma-1)(1+\mu)^2s^{\gamma+2}-(1+\mu)C\gamma\mu s^{\gamma+2}+C\gamma\mu s^{\gamma+4}-(1+\mu)^3D+(1+\mu)^2Ds^2\Big\}+2(\mu+1)(\mu+3)\big(C(1+\mu)^3s^{\gamma}+D(1+\mu)^3-C(1+\mu)\mu s^{\gamma+2}-D(1+\mu)\mu s^2-C(1+\mu)^2s^{\gamma+2}-D(1+\mu)^2s^2+C\mu s^{\gamma+4}+D\mu s^4\big)=0\quad. \tag{E68}$$

Combine the same power terms:

$$\Big\{\eta_{\alpha+3}s^6+(\eta_{\alpha+1}+\eta_3)s^4+(\eta_{\alpha-1}+\eta_1)s^2+\eta_{\gamma+5}s^{\gamma+6}+\eta_{\gamma+3}s^{\gamma+4}+\eta_{\gamma+1}s^{\gamma+2}+\eta_{\gamma-1}s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+\eta_{-1}+\eta_{-3}s^{-2}\Big\}+[(\mu-2)(\mu+1)-\mu s^2]\Big\{+C\gamma\mu s^{\gamma+4}-C[\gamma-1-\mu+2\gamma\mu](1+\mu)s^{\gamma+2}+C(\gamma-1)(1+\mu)^3s^{\gamma}+A(1+\mu)s^4+\Big[-A\frac{(1+\mu)}{(3+\mu)}+D\Big](1+\mu)^2s^2-(1+\mu)^3D\Big\}+2(\mu+1)(\mu+3)\big(C\mu s^{\gamma+4}-C(1+\mu)(1+2\mu)s^{\gamma+2}+C(1+\mu)^3s^{\gamma}+D\mu s^4-D(1+\mu)(1+2\mu)s^2+D(1+\mu)^3\big)=0\quad. \tag{E69}$$

Still expand:

$$\Big\{\eta_{\alpha+3}s^6+(\eta_{\alpha+1}+\eta_3)s^4+(\eta_{\alpha-1}+\eta_1)s^2+\eta_{\gamma+5}s^{\gamma+6}+\eta_{\gamma+3}s^{\gamma+4}+\eta_{\gamma+1}s^{\gamma+2}+\eta_{\gamma-1}s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+\eta_{-1}+\eta_{-3}s^{-2}\Big\}+\Big\{+C\gamma(\mu-2)(\mu+1)\mu s^{\gamma+4}-C[\gamma-1-\mu+2\gamma\mu](\mu-2)(1+\mu)^2s^{\gamma+2}+C(\gamma-1)(\mu-2)(1+\mu)^4s^{\gamma}+A(\mu-2)(1+\mu)^2s^4+\Big[-A\frac{(1+\mu)}{(3+\mu)}+D\Big](\mu-2)(1+\mu)^3s^2-(\mu-2)(1+\mu)^4D\Big\}-\Big\{+C\gamma\mu^2s^{\gamma+6}-C[\gamma-1-\mu+2\gamma\mu](1+\mu)\mu s^{\gamma+4}+C(\gamma-1)(1+\mu)^3\mu s^{\gamma+2}+A(1+\mu)\mu s^6+\Big[-A\frac{(1+\mu)}{(3+\mu)}+D\Big](1+\mu)^2\mu s^4-(1+\mu)^3D\mu s^2\Big\}+[2(\mu+1)(\mu+3)C\mu s^{\gamma+4}-2C(\mu+3)(1+\mu)^2(1+2\mu)s^{\gamma+2}+2C(\mu+3)(1+\mu)^4s^{\gamma}+2D(\mu+1)(\mu+3)\mu s^4-2D(\mu+3)(1+\mu)^2(1+2\mu)s^2+2D(\mu+3)(1+\mu)^4]=0\quad. \tag{E70}$$

Put together the same power terms:

$$\Big\{\eta_{\alpha+3}s^6-A(1+\mu)\mu s^6+(\eta_{\alpha+1}+\eta_3)s^4+A(\mu-2)(1+\mu)^2s^4-\Big[-A\frac{(1+\mu)}{(3+\mu)}+D\Big](1+\mu)^2\mu s^4+2D(\mu+1)(\mu+3)\mu s^4+(\eta_{\alpha-1}+\eta_1)s^2+\Big[-A\frac{(1+\mu)}{(3+\mu)}+D\Big](\mu-2)(1+\mu)^3s^2-2D(\mu+3)(1+\mu)^2(1+2\mu)s^2+(1+\mu)^3D\mu s^2+\eta_{\gamma+5}s^{\gamma+6}-C\gamma\mu^2s^{\gamma+6}+\eta_{\gamma+3}s^{\gamma+4}+C\gamma(\mu-2)(\mu+1)\mu s^{\gamma+4}+C[\gamma-1-\mu+2\gamma\mu](1+\mu)\mu s^{\gamma+4}+2(\mu+1)(\mu+3)C\mu s^{\gamma+4}+\eta_{\gamma+1}s^{\gamma+2}-C[\gamma-1-\mu+2\gamma\mu](\mu-2)(1+\mu)^2s^{\gamma+2}-C(\gamma-1)(1+\mu)^3\mu s^{\gamma+2}-2C(\mu+3)(1+\mu)^2(1+2\mu)s^{\gamma+2}+\eta_{\gamma-1}s^{\gamma}+C(\gamma-1)(\mu-2)(1+\mu)^4s^{\gamma}+2C(\mu+3)(1+\mu)^4s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+\eta_{-1}-(\mu-2)(1+\mu)^4D+2D(\mu+3)(1+\mu)^4+\eta_{-3}s^{-2}\Big\}=0\quad. \tag{E71}$$

Combine the same power terms:

$$[\eta_{\alpha+3}-A(1+\mu)\mu]s^6+\Big[(\eta_{\alpha+1}+\eta_3)+A(\mu-2)(1+\mu)^2+\Big(A\frac{(1+\mu)}{(3+\mu)}-D\Big)(1+\mu)^2\mu+2D(\mu+1)(\mu+3)\mu\Big]s^4+\Big[(\eta_{\alpha-1}+\eta_1)+\Big(-A\frac{(1+\mu)}{(3+\mu)}+D\Big)(\mu-2)(1+\mu)^3-2D(\mu+3)(1+\mu)^2(1+2\mu)+(1+\mu)^3D\mu\Big]s^2+[\eta_{\gamma+5}-C\gamma\mu^2]s^{\gamma+6}+[\eta_{\gamma+3}+C\gamma(\mu-2)(\mu+1)\mu+C(\gamma-1-\mu+2\gamma\mu)(1+\mu)\mu+2(\mu+1)(\mu+3)C\mu]s^{\gamma+4}+[\eta_{\gamma+1}-C(\gamma-1-\mu+2\gamma\mu)(\mu-2)(1+\mu)^2-C(\gamma-1)(1+\mu)^3\mu-2C(\mu+3)(1+\mu)^2(1+2\mu)]s^{\gamma+2}+[\eta_{\gamma-1}+C(\gamma-1)(\mu-2)(1+\mu)^4+2C(\mu+3)(1+\mu)^4]s^{\gamma}+\eta_{\gamma-3}s^{\gamma-2}+[\eta_{-1}-(\mu-2)(1+\mu)^4D+2D(\mu+3)(1+\mu)^4]+\eta_{-3}s^{-2}=0\quad. \tag{E72}$$

Simplify and input the η terms from (E67):

$$[A(1+\mu)\mu-A(1+\mu)\mu]s^6+\Big[\Big[A(-5\mu-2)+3A\frac{(1+\mu)\mu}{(3+\mu)}-3D\mu\Big](1+\mu)^2+A(1+\mu)^2\Big\{2\frac{(\mu+\mu\mu-3)}{(3+\mu)}\Big\}+D\mu(\mu+1)(\mu+5)\Big]s^4+\Big[\Big[4A(1+\mu)-A\frac{(1+\mu)(2+3\mu)}{(3+\mu)}+2D(1+4\mu)\Big](1+\mu)^3-A\frac{(1+\mu)}{(3+\mu)}(\mu-2)(1+\mu)^3+D(1+\mu)^2(-2\mu\mu-14\mu-8)\Big]s^2+[-C\gamma(1-\gamma)\mu^2-C\gamma\mu^2]s^{\gamma+6}+[C[-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu](1+\mu)\mu+C(1+\mu)\mu(3\gamma\mu-\gamma+\mu+5)]s^{\gamma+4}+[C[6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu](1+\mu)^2+C(1+\mu)^2(-3\gamma\mu\mu-2\mu\mu+2\gamma\mu-14\mu+2\gamma-8)]s^{\gamma+2}+[C[-4\gamma\gamma\mu-2\gamma\gamma+10\gamma\mu+6\gamma-7\mu-4](1+\mu)^4+C(1+\mu)^4(\gamma\mu-2\gamma+\mu+8)]s^{\gamma}+C(\gamma-1)(\gamma-2)(1+\mu)^6s^{\gamma-2}+[-D(4+7\mu)(1+\mu)^4+D(1+\mu)^4(\mu+8)]+2D(1+\mu)^6s^{-2}=0\quad. \tag{E73}$$

Then

$$[0]s^6 + \left[A(1+\mu)^2\left[(-5\mu-2)+3\frac{(1+\mu)\mu}{(3+\mu)}+2\frac{(\mu+\mu\mu-3)}{(3+\mu)}\right]+D(\mu+1)\mu[-3(1+\mu)+(\mu+5)]\right]s^4 + \left[A(1+\mu)^3\left[4(1+\mu)-\frac{(1+\mu)(2+3\mu)}{(3+\mu)}-\frac{(1+\mu)}{(3+\mu)}(\mu-2)\right]+D(1+\mu)^2[2(1+\mu)(1+4\mu)+(-2\mu\mu-14\mu-8)]\right]s^2 + [C\gamma\mu^2(\gamma-2)]s^{\gamma+6} + C(1+\mu)\mu[(-4\gamma\gamma\mu-2\gamma\gamma+6\gamma\mu+4\gamma-3-3\mu)+(3\gamma\mu-\gamma+\mu+5)]s^{\gamma+4} + C(1+\mu)^2[(6\gamma\gamma\mu+\gamma\gamma-3\gamma-14\gamma\mu+6\gamma\gamma\mu\mu-12\mu\mu\gamma+2+10\mu+8\mu\mu)+(-3\gamma\mu\mu-2\mu\mu+2\gamma\mu-14\mu+2\gamma-8)]s^{\gamma+2} + C(1+\mu)^4[(-4\gamma\gamma\mu-2\gamma\gamma+10\gamma\mu+6\gamma-7\mu-4)+(\gamma\mu-2\gamma+\mu+8)]s^{\gamma} + C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-2} + D(1+\mu)^4[-(4+7\mu)+(\mu+8)] + 2D(1+\mu)^6 s^{-2} = 0 \quad . \qquad \text{(E74)}$$

Before continuation with the algebraic manipulations of the equation as a whole, a part of the pre-factor by s^2 is simplified separately:

$$\left[4(1+\mu)-\frac{(1+\mu)(2+3\mu)}{(3+\mu)}-\frac{(1+\mu)}{(3+\mu)}(\mu-2)\right] = -(1+\mu)\left[-4+\frac{(2+3\mu)}{(3+\mu)}+\frac{(\mu-2)}{(3+\mu)}\right] = -(1+\mu)\left[\frac{-4(3+\mu)+(2+3\mu)+(\mu-2)}{(3+\mu)}\right] = -(1+\mu)\left[\frac{(-12-4\mu)+(4\mu)}{(3+\mu)}\right] = -(1+\mu)\left[\frac{(-12)}{(3+\mu)}\right] = (1+\mu)\left[\frac{12}{(3+\mu)}\right] \quad . \qquad \text{(E75)}$$

Then

$$\left[A(1+\mu)^2\left(\frac{-12(\mu+1)}{(3+\mu)}\right)+2D(\mu+1)\mu(1-\mu)\right]s^4 + \left[A(1+\mu)^4\left[\frac{12}{(3+\mu)}\right]+D(1+\mu)^2(-6-4\mu+6\mu\mu)\right]s^2 + [C\gamma\mu^2(\gamma-2)]s^{\gamma+6} + C(1+\mu)\mu(-4\gamma\gamma\mu-2\gamma\gamma+9\gamma\mu+3\gamma+2-2\mu)s^{\gamma+4} + C(1+\mu)^2(6\gamma\gamma\mu+\gamma\gamma-\gamma-12\gamma\mu+6\gamma\gamma\mu\mu-15\mu\mu\gamma-6-4\mu+6\mu\mu)s^{\gamma+2} + C(1+\mu)^4(-4\gamma\gamma\mu-2\gamma\gamma+11\gamma\mu+4\gamma-6\mu+4)s^{\gamma} + C(\gamma-1)(\gamma-2)(1+\mu)^6 s^{\gamma-2} + D(1+\mu)^4[-(4+7\mu)+(\mu+8)] + 2D(1+\mu)^6 s^{-2} = 0 \quad . \qquad \text{(E76)}$$

If γ=–2 or lower, then – from the term with power γ–2 (which cannot combine with others) – it follows that *C*=0. Then also *D*=0, as follows from the last term. Then also *A*=0. This is a trivial solution which we are not presently searching for.

If γ=0, then – from the term with power γ+6 (which cannot combine with others) – it follows that *C*=0. Then also *D*=0, as follows from the last two terms. Then also *A*=0. This is again the trivial solution.

If γ=4, then – from the terms with power –2 and 0 (which cannot combine with others) – it follows that *D*=0 and also *C*=0. Then also *A*=0. This is once more a trivial solution.

Similarly for γ>4.

When γ=2, the term with power γ–2 is zero thanks to the pre-factor part (γ–2), and *C* need not to be zero. On the other hand, *D*=0, as follows from the last term (which cannot combine with others). We now try this possibility (i.e. γ=2, *D*=0). The equation simplifies in several steps:

$$\left[A(1+\mu)^2\left(\frac{-12(\mu+1)}{(3+\mu)}\right)\right]s^4 + \left[A(1+\mu)^4\left[\frac{12}{(3+\mu)}\right]\right]s^2 + [C2\mu^2(2-2)]s^{2+6} + C(1+\mu)\mu(-4.2.2\mu-2.2.2+9.2\mu+3.2+2-2\mu)s^{2+4} + C(1+\mu)^2(6.2.2\mu+2.2-2-12.2\mu+6.2.2\mu\mu-15\mu\mu.2-6-4\mu+6\mu\mu)s^{2+2} + C(1+\mu)^4(-4.2.2\mu-2.2.2+11.2\mu+4.2-6\mu+4)s^2 + C(2-1)(2-2)(1+\mu)^6 s^{2-2} = 0 \quad , \qquad \text{(E77)}$$

$$\left[A(1+\mu)^2\left(\frac{-12(\mu+1)}{(3+\mu)}\right)\right]s^4 + \left[A(1+\mu)^4\left[\frac{12}{(3+\mu)}\right]\right]s^2 + [0]s^8 + C(1+\mu)\mu(-16\mu-8+18\mu+6+2-2\mu)s^6 + C(1+\mu)^2(24\mu+4-2-24\mu+24\mu\mu-30\mu\mu-6-4\mu+6\mu\mu)s^4 + C(1+\mu)^4(-16\mu-8+22\mu+8-6\mu+4)s^2 + (0)s^0 = 0 \quad , \qquad \text{(E78)}$$

$$\left[A(1+\mu)^2\left(\frac{-12(\mu+1)}{(3+\mu)}\right)\right]s^4 + \left[A(1+\mu)^4\left[\frac{12}{(3+\mu)}\right]\right]s^2 + (0)s^6 + C(1+\mu)^2(-4-4\mu)s^4 + C(1+\mu)^4(+4)s^2 = 0 \quad , \qquad \text{E79)}$$

$$\left[A\left(\frac{-12(\mu+1)}{(3+\mu)}\right)\right]s^4 + C(-4-4\mu)s^4 + \left[A(1+\mu)^2\left[\frac{12}{(3+\mu)}\right]\right]s^2 + C(1+\mu)^2(+4)s^2 = 0 \quad . \qquad \text{(E80)}$$

Now divide by 4:

$$\left\{A\left(\frac{-3(\mu+1)}{(3+\mu)}\right)-C(1+\mu)\right\}s^4+\left\{A(1+\mu)^2\frac{3}{(3+\mu)}+C(1+\mu)^2\right\}s^2=0\quad . \tag{E81}$$

Then the pre-factors by s^2 and by s^4 both have to be zero. I.e.,

$$A\left(\frac{-3(\mu+1)}{(3+\mu)}\right)-C(1+\mu)=0 \quad \Rightarrow \quad A\left(\frac{-3(\mu+1)}{(3+\mu)}\right)=C(1+\mu) \quad \Rightarrow \quad C=\frac{-3A}{(3+\mu)} \tag{E82}$$

and

$$A(1+\mu)^2\frac{3}{(3+\mu)}+C(1+\mu)^2=0 \quad \Rightarrow \quad C=-\frac{3A}{(3+\mu)}\quad . \tag{E83}$$

Both equations are thus fulfilled for the same C/A ratio. Therefore, the selected model function led to the solution for K_d=2. The found parameters are α=2, γ=2, D=0, $B=-A\frac{(1+\mu)^2}{(3+\mu)}$, $C=-\frac{3A}{(3+\mu)}$. Thus the model function (divided by the constant A) with input the found parameters is

$$F_{test2}(s)/A=\left(s^\alpha+\frac{B}{A}\right)\sinh^{-1}\left(\frac{1}{\sqrt{1+\mu}}\frac{s}{\sqrt{(1+\mu)-s^2}}\right)+\left(\frac{C}{A}s^\gamma+\frac{D}{A}\right)\frac{\sqrt{(1+\mu)^2-\mu s^2}}{s}=\left(s^\alpha+\frac{B}{A}\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)+$$
$$\left(\frac{C}{A}s^\gamma+\frac{D}{A}\right)\frac{\sqrt{(1+\mu)^2-\mu s^2}}{s}=\left(s^2+\frac{B}{A}\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)+\left(\frac{C}{A}s^2+\frac{D}{A}\right)\frac{\sqrt{(1+\mu)^2-\mu s^2}}{s}=$$
$$\left(s^2-\frac{(1+\mu)^2}{(3+\mu)}\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)-\frac{3}{(3+\mu)}s\sqrt{(1+\mu)^2-\mu s^2}\quad . \tag{E84}$$

The second degree Legendre-like function of the second kind (not yet properly normalized, however) is the above solution multiplied by a suitable constant (3+μ)/2:

$$Q_2^S(s)\sim\left(\frac{(3+\mu)}{2}s^2-\frac{(3+\mu)}{2}\frac{(1+\mu)^2}{(3+\mu)}\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)-\frac{(3+\mu)}{2}\frac{3}{(3+\mu)}s\sqrt{(1+\mu)^2-\mu s^2}=$$
$$\frac{(3+\mu)s^2-(1+\mu)^2}{2}\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)-\frac{3}{2}s\sqrt{(1+\mu)^2-\mu s^2}\quad . \tag{E85}$$

Note that, when μ=0, the solution reduces, as expected, to the second kind Legendre function of the second degree, for which C=–3/2=–A=–(3+μ)/2=–(3+0)/2.

Further note, that the first term in the function (E85) is proportional to (see **Table 1** and (92))

$$P_2^S(s)\,Q_0^S(s)=\frac{(3+\mu)s^2-(1+\mu)^2}{2}\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\quad . \tag{E86}$$

E.2. Third degree second kind solution

Model function

In the model function for the second kind third degree solution, we assume that the first part (i.e. $\sinh^{-1}$, or logarithm part) is equal to $P_3^S(s)$ multiplied by $Q_0^S(s)$. Equivalent relations are valid for the lower degree solution functions (see (92), (93) and compare with **Table 1** of the main text, and see also (E86) in the previous section of this Supplement); therefore, we test it also for the third degree solution.

Then, the unknown in the model function remains only the polynomial by the square root. We assume its power to be two, and thus the model function is

$$F_{test3}(s) = P_3^S(s)\,Q_0^S(s) + \frac{1}{(1+\mu)^6}\left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\sqrt{(1+\mu)^2 - \mu s^2} = \frac{1}{(1+\mu)^6}\frac{(\mu+3)}{6}\left((5+3\mu)s^3 - 3(1+\mu)^2 s\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) + \frac{1}{(1+\mu)^6}\left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\sqrt{(1+\mu)^2 - \mu s^2} \quad , \tag{E87}$$

where E and G are constants to be found. The following derivation has an intention to test if the above $F_{test3}(s)$ is a real solution of the angular part of the Laplace equation in the SOS coordinates (68), i.e. if the following equation

$$[(1+\mu) - s^2][(1+\mu)^2 - \mu s^2]\frac{d^2F_{test3}(s)}{ds^2} + s[-(3\mu+2)(\mu+1) + 2\mu(\mu+1)K_d + \mu(3-2K_d)s^2]\frac{dF_{test3}(s)}{ds} + K_d[(K_d-2)\mu s^2 + (\mu+1)K_d + (1+\mu)^2]F_{test3}(s) = 0 \tag{E88}$$

holds, and under which conditions (for which K_{d}, and for which multiplication parameters E and G).

The equation can be rewritten in the following way:

$$H_2(s)\frac{d^2F_{test3}(s)}{ds^2} + H_1(s)\frac{dF_{test3}(s)}{ds} + H_0(s)F_{test3}(s) = 0 \quad , \tag{E89}$$

where H_i denotes specific polynomials (see (E3), (E4), (E5)).

The derivatives of the model function

The first derivative of the model function is

$$\frac{dF_{test3}(s)}{ds} = \frac{d}{ds}\left[\frac{1}{(1+\mu)^6}\frac{(\mu+3)}{6}\left((5+3\mu)s^3 - 3(1+\mu)^2 s\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) + \frac{1}{(1+\mu)^6}\left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\sqrt{(1+\mu)^2 - \mu s^2}\right] =$$
$$\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{6}\frac{d}{ds}\left[\left((5+3\mu)s^3 - 3(1+\mu)^2 s\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right] + \frac{d}{ds}\left[\left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\sqrt{(1+\mu)^2 - \mu s^2}\right]\right\} =$$
$$\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{6}\frac{d}{ds}\left[\left((5+3\mu)s^3 - 3(1+\mu)^2 s\right)\right]\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) +\right.$$
$$\frac{(\mu+3)}{6}\left((5+3\mu)s^3 - 3(1+\mu)^2 s\right)\frac{d}{ds}\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right] + \frac{d}{ds}\left[\left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\right]\sqrt{(1+\mu)^2 - \mu s^2} +$$
$$\left.\left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\frac{d}{ds}\left[\sqrt{(1+\mu)^2 - \mu s^2}\right]\right\} = \frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{6}\left[3(5+3\mu)s^2 - 3(1+\mu)^2\right]\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) +\right.$$
$$\left.\frac{(\mu+3)}{6}\left((5+3\mu)s^3 - 3(1+\mu)^2 s\right)\left(\frac{1+\mu}{[(1+\mu)-s^2]\sqrt{(1+\mu)^2-\mu s^2}}\right) - 5\mathrm{E}s\sqrt{(1+\mu)^2 - \mu s^2} + \left(-\frac{5}{2}\mathrm{E}s^2 + \frac{2}{3}\mathrm{G}\right)\frac{-\mu s}{\sqrt{(1+\mu)^2-\mu s^2}}\right\} \tag{E90}$$

where the derivative of the logarithm function and the derivative of square root in the above relation can be either derived or found using internet portals which calculate derivatives.
Then

$$\frac{dF_{test3}(s)}{ds} = \frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}\left[(5+3\mu)s^2 - (1+\mu)^2\right]\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) + \frac{(\mu+3)(1+\mu)}{6}s\frac{\left((5+3\mu)s^2 - 3(1+\mu)^2\right)}{[(1+\mu)-s^2]\sqrt{(1+\mu)^2-\mu s^2}} - 5\mathrm{E}s\sqrt{(1+\mu)^2 - \mu s^2} + s\frac{\mu\left(\frac{5}{2}\mathrm{E}s^2 - \frac{2}{3}\mathrm{G}\right)}{\sqrt{(1+\mu)^2-\mu s^2}}\right\} \tag{E91}$$

The second part of it (i.e. with square root, not *ln*) is

$$
\frac{(\mu+3)(1+\mu)}{6}s\,\frac{\left((5+3\mu)s^2-3(1+\mu)^2\right)}{[(1+\mu)-s^2]\sqrt{(1+\mu)^2-\mu s^2}} - 5Es\sqrt{(1+\mu)^2-\mu s^2} + s\,\frac{\mu\left(\frac{5}{2}Es^2-\frac{2}{3}G\right)}{\sqrt{(1+\mu)^2-\mu s^2}} =
$$

$$
s\left\{\frac{(\mu+3)(1+\mu)}{6}\,\frac{\left((5+3\mu)s^2-3(1+\mu)^2\right)}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\sqrt{(1+\mu)^2-\mu s^2} - 5E\sqrt{(1+\mu)^2-\mu s^2} + \frac{\mu\left(\frac{15}{6}Es^2-\frac{4}{6}G\right)}{[(1+\mu)^2-\mu s^2]}\sqrt{(1+\mu)^2-\mu s^2}\right\} =
$$

$$
\frac{1}{6}\left\{\frac{(\mu+3)(1+\mu)\left((5+3\mu)s^2-3(1+\mu)^2\right)}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} - 30E + \frac{(15E\mu s^2-4G\mu)}{[(1+\mu)^2-\mu s^2]}\right\}s\sqrt{(1+\mu)^2-\mu s^2} =
$$

$$
\frac{1}{6}\left\{\frac{(\mu\mu+4\mu+3)\left((5+3\mu)s^2-3(1+\mu)^2\right)}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} - \frac{30E[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} + \frac{[(1+\mu)-s^2](15E\mu s^2-4G\mu)}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right\}s\sqrt{(1+\mu)^2-\mu s^2} =
$$

$$
\frac{1}{6}\left\{\frac{\left([\mu\mu(5+3\mu)+4\mu(5+3\mu)+3(5+3\mu)]s^2-3\mu\mu(1+\mu)^2-12\mu(1+\mu)^2-9(1+\mu)^2\right)}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} - \frac{[30E(1+\mu)^3-30E\mu(1+\mu)s^2-30E(1+\mu)^2s^2+30E\mu s^4]}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} + \frac{(15E(1+\mu)\mu s^2-4G(1+\mu)\mu-15E\mu s^4+4G\mu s^2)}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right\}s\sqrt{(1+\mu)^2-\mu s^2} =
$$

$$
\Big\{\left([\mu\mu(5+3\mu)+4\mu(5+3\mu)+3(5+3\mu)]s^2-3\mu\mu(1+\mu)^2-12\mu(1+\mu)^2-9(1+\mu)^2\right) + [-30E(1+\mu)^3+[30E\mu(1+\mu)+30E(1+\mu)^2]s^2-30E\mu s^4] + (-4G(1+\mu)\mu+[15E(1+\mu)\mu+4G\mu]s^2-15E\mu s^4)\Big\}\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} =
$$

$$
\Big\{\left([\mu\mu(5+3\mu)+4\mu(5+3\mu)+3(5+3\mu)]s^2+[30E\mu(1+\mu)+30E(1+\mu)^2]s^2+[15E(1+\mu)\mu+4G\mu]s^2\right) + [-30E\mu s^4-15E\mu s^4] + (-3\mu\mu(1+\mu)^2-12\mu(1+\mu)^2-9(1+\mu)^2-30E(1+\mu)^3-4G(1+\mu)\mu)\Big\}\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} =
$$

$$
\Big\{-45E\mu s^4 + ([(5\mu\mu+3\mu\mu\mu)+(20\mu+12\mu\mu)+(15+9\mu)] + [(30E\mu+30E\mu\mu)+30E(1+2\mu+\mu\mu)] + [(15E\mu+15E\mu\mu)+4G\mu])s^2 + (-3\mu\mu(1+\mu)-12\mu(1+\mu)-9(1+\mu)-30E(1+\mu)^2-4G\mu)(1+\mu)\Big\}\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} \quad . \qquad \text{(E92)}
$$

The pre-factor of s^2 can be simplified as follows:

$$
([(5\mu\mu+3\mu\mu\mu)+(20\mu+12\mu\mu)+(15+9\mu)] + [(30E\mu+30E\mu\mu)+30E(1+2\mu+\mu\mu)] + [(15E\mu+15E\mu\mu)+4G\mu]) = (3\mu\mu\mu + [(5\mu\mu)+(12\mu\mu)+30E\mu\mu+30E\mu\mu+15E\mu\mu] + [20\mu+9\mu+(30E\mu)+(60E\mu)+15E\mu+4G\mu] + [30E+15]) = (3\mu\mu\mu+[17+75E]\mu\mu + [29+105E+4G]\mu + [30E+15]) \quad . \qquad \text{(E93)}
$$

The constant term can be expanded as follows:

$$
(-3\mu\mu(1+\mu)-12\mu(1+\mu)-9(1+\mu)-30E(1+\mu)^2-4G\mu) = \left((-3\mu\mu-3\mu\mu\mu)+(-12\mu-12\mu\mu)+(-9-9\mu)+(-30E-60E\mu-30E\mu\mu)-4G\mu\right) = \left(-3\mu\mu\mu+(-3\mu\mu-12\mu\mu-30E\mu\mu)+(-12\mu-9\mu-60E\mu-4G\mu)+(-9)+(-30E)\right) = \left(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E)\right) \quad . \qquad \text{(E94)}
$$

Thus the second part (i.e. with square root) of the derivative (E92) is

$$
\Big\{-45E\mu s^4 + (3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])s^2 + \left(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E)\right)(1+\mu)\Big\}\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} \quad . \qquad \text{(E95)}
$$

Then the <u>first derivative</u> is

$$
\frac{dF_{test3}(s)}{ds} = \frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}[(5+3\mu)s^2-(1+\mu)^2]\,\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) + \Big[-45E\mu s^4 + (3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])s^2 + \left(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E)\right)(1+\mu)\Big]\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right\} \quad . \qquad \text{(E96)}
$$

The <u>second derivative</u> is

$$\frac{d^2F_{test3}(s)}{ds^2}=\frac{d}{ds}\left[\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}[(5+3\mu)s^2-(1+\mu)^2]\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)+\left[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+\right.\right.\right.$$

$$[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+$$

$$\left.\left.\left.\mu)\right]\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right\}\right]=\frac{1}{(1+\mu)^6}\left\{\frac{d}{ds}\left[\frac{(\mu+3)}{2}[(5+3\mu)s^2-(1+\mu)^2]\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]+\frac{d}{ds}\left[\left[-45\mathrm{E}\mu s^4+\right.\right.\right.$$

$$(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+$$

$$\left.\left.\left.4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+\mu)\right]\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right]\right\}=\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}\left[\frac{d}{ds}[(5+3\mu)s^2-(1+\mu)^2]\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]+\right.\right.$$

$$\left.[(5+3\mu)s^2-(1+\mu)^2]\frac{d}{ds}\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]\right]+\left[\frac{d}{ds}\left[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+\right.\right.$$

$$4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+$$

$$\left.\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+\mu)\right]\left[\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right]+\left[-45\mathrm{E}\mu s^4+\right.$$

$$(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+$$

$$\left.\left.4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+\mu)\right]\frac{d}{ds}\left[\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right]\right]\right\}=\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}\left[2(5+3\mu)s\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]+\right.\right.$$

$$\left.[(5+3\mu)s^2-(1+\mu)^2]\left(\frac{1+\mu}{[(1+\mu)-s^2]\sqrt{(1+\mu)^2-\mu s^2}}\right)\right]+\left[\left[-180\mathrm{E}\mu s^3+2(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+\right.\right.$$

$$\left.[30\mathrm{E}+15])s\right]\left[\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right]+\left[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+\right.$$

$$\left.15])s^2+\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+\mu)\right]\left[\frac{(1+\mu)+s^2}{6[(1+\mu)-s^2]^2\sqrt{(1+\mu)^2-\mu s^2}}+\right.$$

$$\left.\left.\left.\frac{\mu s^2}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]^{\frac{3}{2}}}\right]\right]\right\}=$$

$$\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}\left[2(5+3\mu)s\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]+[(5+3\mu)s^2-(1+\mu)^2]\left(\frac{1+\mu}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\sqrt{(1+\mu)^2-\mu s^2}\right)\right]+\right.$$

$$\left[\left[-180\mathrm{E}\mu s^3+2(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s\right]\left[\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right]+\right.$$

$$\left[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-\right.$$

$$\left.\left.\left.(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+\mu)\right]\left[\frac{(1+\mu)+s^2}{6[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]}+\frac{\mu s^2}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]^2}\right]\sqrt{(1+\mu)^2-\mu s^2}\right]\right\}\quad, \qquad \text{(E97)}$$

where the particular derivatives were found using internet portals which calculate derivatives. The second derivative is thus

$$\frac{d^2F_{test3}(s)}{ds^2}=\frac{1}{(1+\mu)^6}\left\{(\mu+3)(5+3\mu)s\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]+\left[\frac{(\mu+3)}{2}[(5+3\mu)s^2-(1+\mu)^2]\frac{1+\mu}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}+\right.\right.$$

$$\left[-180\mathrm{E}\mu s^3+2(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s\right]\frac{s}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}+\left[-45\mathrm{E}\mu s^4+\right.$$

$$(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+\big(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+$$

$$\left.\left.\left.4\mathrm{G})\mu-(9+30\mathrm{E})\big)(1+\mu)\right]\frac{[(1+\mu)+s^2][(1+\mu)^2-\mu s^2]+\mu s^2[(1+\mu)-s^2]}{6[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\right]\sqrt{(1+\mu)^2-\mu s^2}\right\}=$$

$$\frac{1}{(1+\mu)^6}\left\{(\mu+3)(5+3\mu)s\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]+\left[\frac{(\mu+3)}{2}[(5+3\mu)s^2-(1+\mu)^2]\frac{1+\mu}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}+\left[-180\mathrm{E}\mu s^4+\right.\right.\right.$$

$$2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2] \frac{1}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]} + \Big[-45\mathrm{E}\mu s^4 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)\Big] \frac{(1+\mu)^3+(1+\mu)^2 s^2-2\mu s^4}{6[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big] \sqrt{(1 + \mu)^2 - \mu s^2}\Big\} \quad , \qquad \text{(E98)}$$

where the following identity was used in the last numerator:

$$[(1 + \mu) + s^2][(1 + \mu)^2 - \mu s^2] + \mu s^2[(1 + \mu) - s^2] = [(1 + \mu)^3 - (1 + \mu)\mu s^2 + (1 + \mu)^2 s^2 - \mu s^4] + [(1 + \mu)\mu s^2 - \mu s^4] = [(1 + \mu)^3 + \{-\mu + (1 + \mu)\}(1 + \mu)s^2 + (1 + \mu)\mu s^2 - \mu s^4 - \mu s^4] = [(1 + \mu)^3 + (1 + \mu)s^2 + (1 + \mu)\mu s^2 - 2\mu s^4] = [(1 + \mu)^3 + (1 + \mu)^2 s^2 - 2\mu s^4] \quad . \qquad \text{(E99)}$$

The factor by square root is:

$$\frac{1}{6}\Big[3(\mu + 3)[(5 + 3\mu)s^2 - (1 + \mu)^2] \frac{(1+\mu)[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}{[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2} + [-180\mathrm{E}\mu s^4 + 2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2] \frac{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}{[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2} + \Big[-45\mathrm{E}\mu s^4 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)\Big] \frac{(1+\mu)^3+(1+\mu)^2 s^2-2\mu s^4}{[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big]$$

$$= \frac{1}{6[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[[3(1 + \mu)(\mu + 3)(5 + 3\mu)s^2 - 3(\mu + 3)(1 + \mu)^3][(1 + \mu) - s^2][(1 + \mu)^2 - \mu s^2] + [-180\mathrm{E}\mu s^4 + 2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2][(1 + \mu) - s^2][(1 + \mu)^2 - \mu s^2] + \Big[-45\mathrm{E}\mu s^4 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)\Big][(1 + \mu)^3 + (1 + \mu)^2 s^2 - 2\mu s^4]\Big]$$

$$= \frac{1}{6[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[[3(1 + \mu)^2(\mu + 3)(5 + 3\mu)s^2 - 3(\mu + 3)(1 + \mu)^4 - 3(1 + \mu)(\mu + 3)(5 + 3\mu)s^4 + 3(\mu + 3)(1 + \mu)^3 s^2][(1 + \mu)^2 - \mu s^2] + [-180\mathrm{E}(1 + \mu)\mu s^4 + 2(1 + \mu)(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2 + 180\mathrm{E}\mu s^6 - 2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^4][(1 + \mu)^2 - \mu s^2] + \Big[-45\mathrm{E}(1 + \mu)^3\mu s^4 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])(1 + \mu)^3 s^2 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)^4 - 45\mathrm{E}(1 + \mu)^2\mu s^6 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])(1 + \mu)^2 s^4 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)^3 s^2 + 90\mathrm{E}\mu\mu s^8 - 2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])\mu s^6 - 2\big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)\mu s^4\Big]\Big]$$

$$= \frac{1}{6[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[[3(1 + \mu)^2(\mu + 3)(5 + 3\mu)s^2 + 3(1 + \mu)^2(\mu + 3)(1 + \mu)s^2 - 3(\mu + 3)(1 + \mu)^4 - 3(1 + \mu)(\mu + 3)(5 + 3\mu)s^4][(1 + \mu)^2 - \mu s^2] + [180\mathrm{E}\mu s^6 + (-180\mathrm{E}\mu - 180\mathrm{E}\mu\mu)s^4 - 2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^4 + 2(1 + \mu)(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^2][(1 + \mu)^2 - \mu s^2] + [+90\mathrm{E}\mu\mu s^8 - (45\mathrm{E} + 90\mathrm{E}\mu + 45\mathrm{E}\mu\mu)\mu s^6 - (6\mu\mu\mu + [34 + 150\mathrm{E}]\mu\mu + [58 + 210\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 30])\mu s^6 - (45\mathrm{E} + 90\mathrm{E}\mu + 45\mathrm{E}\mu\mu)\mu(1 + \mu)s^4 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])(1 + \mu)(1 + \mu)s^4 + (6\mu\mu\mu\mu + (30 + 60\mathrm{E})\mu\mu\mu + (42 + 120\mathrm{E} + 8\mathrm{G})\mu\mu + (18 + 60\mathrm{E})\mu)(1 + \mu)s^4 + (3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])(1 + \mu)^3 s^2 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)^3 s^2 + \big(-3\mu\mu\mu - (15 + 30\mathrm{E})\mu\mu - (21 + 60\mathrm{E} + 4\mathrm{G})\mu - (9 + 30\mathrm{E})\big)(1 + \mu)^4\Big]\Big], \qquad \text{(E100)}$$

which is, with the help of the following partial expansions

$$(-180\mathrm{E}\mu - 180\mathrm{E}\mu\mu)s^4 - 2(3\mu\mu\mu + [17 + 75\mathrm{E}]\mu\mu + [29 + 105\mathrm{E} + 4\mathrm{G}]\mu + [30\mathrm{E} + 15])s^4 = -(180\mathrm{E}\mu + 180\mathrm{E}\mu\mu + 6\mu\mu\mu + [34 + 150\mathrm{E}]\mu\mu + [58 + 210\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 30])s^4 = -(+6\mu\mu\mu + [34 + 150\mathrm{E} + 180\mathrm{E}]\mu\mu + [58 + 210\mathrm{E} + 180\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 30])s^4 = -(+6\mu\mu\mu + [34 + 330\mathrm{E}]\mu\mu + [58 + 390\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 30])s^4 \quad , \qquad \text{(E101)}$$

$$-(45\mathrm{E} + 90\mathrm{E}\mu + 45\mathrm{E}\mu\mu)\mu s^6 - (6\mu\mu\mu + [34 + 150\mathrm{E}]\mu\mu + [58 + 210\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 30])\mu s^6 = -(45\mathrm{E} + 90\mathrm{E}\mu + 45\mathrm{E}\mu\mu + 6\mu\mu\mu + [34 + 150\mathrm{E}]\mu\mu + [58 + 210\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 30])\mu s^6 = -(6\mu\mu\mu + [34 + 150\mathrm{E} + 45\mathrm{E}]\mu\mu + [58 + 210\mathrm{E} + 90\mathrm{E} + 8\mathrm{G}]\mu + [60\mathrm{E} + 45\mathrm{E} + 30])\mu s^6 = -(6\mu\mu\mu + [34 + 195\mathrm{E}]\mu\mu + [58 + 300\mathrm{E} + 8\mathrm{G}]\mu + [105\mathrm{E} + 30])\mu s^6 \quad , \qquad \text{(E102)}$$

possible to modify to

$$\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[\Big[-3(1+\mu)(\mu+3)(5+3\mu)s^4+3(1+\mu)^2(\mu+3)(6+4\mu)s^2-3(\mu+3)(1+\mu)^4\Big]\Big[(1+\mu)^2-\mu s^2\Big]+\Big[180\mathbb{E}\mu s^6-(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])s^4+2(1+\mu)(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])s^2\Big]\Big[(1+\mu)^2-\mu s^2\Big]+\Big[+90\mathbb{E}\mu\mu s^8-(6\mu\mu\mu+[34+195\mathbb{E}]\mu\mu+[58+300\mathbb{E}+8\mathbb{G}]\mu+[105\mathbb{E}+30])\mu s^6+\{-(45\mathbb{E}+90\mathbb{E}\mu+45\mathbb{E}\mu\mu)\mu+(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15]+3\mu\mu\mu\mu+[17+75\mathbb{E}]\mu\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu\mu+[30\mathbb{E}+15]\mu)+(6\mu\mu\mu\mu+(30+60\mathbb{E})\mu\mu\mu+(42+120\mathbb{E}+8\mathbb{G})\mu\mu+(18+60\mathbb{E})\mu)\}(1+\mu)s^4+\Big\{(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])+\big(-3\mu\mu\mu-(15+30\mathbb{E})\mu\mu-(21+60\mathbb{E}+4\mathbb{G})\mu-(9+30\mathbb{E})\big)\Big\}(1+\mu)^3s^2+\big(-3\mu\mu\mu-(15+30\mathbb{E})\mu\mu-(21+60\mathbb{E}+4\mathbb{G})\mu-(9+30\mathbb{E})\big)(1+\mu)^4\Big]\Big]$$

$$=\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[\Big[-3(1+\mu)^3(\mu+3)(5+3\mu)s^4+6(1+\mu)^4(\mu+3)(3+2\mu)s^2-3(\mu+3)(1+\mu)^6+3(1+\mu)(\mu+3)(5+3\mu)\mu s^6-6(1+\mu)^2(\mu+3)(3+2\mu)\mu s^4+3(\mu+3)(1+\mu)^4\mu s^2\Big]+\Big[180\mathbb{E}(1+\mu)^2\mu s^6-(1+\mu)^2(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])s^4+2(1+\mu)^3(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])s^2-180\mathbb{E}\mu\mu s^8+(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])\mu s^6-2(1+\mu)(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])\mu s^4\Big]+\Big[+90\mathbb{E}\mu\mu s^8-(6\mu\mu\mu+[34+195\mathbb{E}]\mu\mu+[58+300\mathbb{E}+8\mathbb{G}]\mu+[105\mathbb{E}+30])\mu s^6+\{9\mu\mu\mu\mu+(50+90\mathbb{E})\mu\mu\mu+[88+210\mathbb{E}+12\mathbb{G}]\mu\mu+[62+150\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15]\}(1+\mu)s^4+([2+45\mathbb{E}]\mu\mu+[8+45\mathbb{E}]\mu+6)(1+\mu)^3s^2+\big(-3\mu\mu\mu-(15+30\mathbb{E})\mu\mu-(21+60\mathbb{E}+4\mathbb{G})\mu-(9+30\mathbb{E})\big)(1+\mu)^4\Big]\Big]$$

$$=\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[\Big[+3(1+\mu)(\mu+3)(5+3\mu)\mu s^6-3(5+14\mu+7\mu\mu)(1+\mu)^2(\mu+3)s^4+3(6+5\mu)(\mu+3)(1+\mu)^4s^2-3(\mu+3)(1+\mu)^6\Big]+\Big[-180\mathbb{E}\mu\mu s^8+(+6\mu\mu\mu+[34+510\mathbb{E}]\mu\mu+[58+750\mathbb{E}+8\mathbb{G}]\mu+[240\mathbb{E}+30])\mu s^6+\{-(1+\mu)(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])-2(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])\mu\}(1+\mu)s^4+2(1+\mu)^3(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])s^2\Big]+\Big[+90\mathbb{E}\mu\mu s^8-(6\mu\mu\mu+[34+195\mathbb{E}]\mu\mu+[58+300\mathbb{E}+8\mathbb{G}]\mu+[105\mathbb{E}+30])\mu s^6+\{9\mu\mu\mu\mu+(50+90\mathbb{E})\mu\mu\mu+[88+210\mathbb{E}+12\mathbb{G}]\mu\mu+[62+150\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15]\}(1+\mu)s^4+([2+45\mathbb{E}]\mu\mu+[8+45\mathbb{E}]\mu+6)(1+\mu)^3s^2+\big(-3\mu\mu\mu-(15+30\mathbb{E})\mu\mu-(21+60\mathbb{E}+4\mathbb{G})\mu-(9+30\mathbb{E})\big)(1+\mu)^4\Big]\Big], \tag{E103}$$

where the following partial expansions were used:

$$\{-(45\mathbb{E}+90\mathbb{E}\mu+45\mathbb{E}\mu\mu)\mu+(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15]+3\mu\mu\mu\mu+[17+75\mathbb{E}]\mu\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu\mu+[30\mathbb{E}+15]\mu)+(6\mu\mu\mu\mu+(30+60\mathbb{E})\mu\mu\mu+(42+120\mathbb{E}+8\mathbb{G})\mu\mu+(18+60\mathbb{E})\mu)\}=\{3\mu\mu\mu\mu+6\mu\mu\mu\mu+(3\mu\mu\mu-45\mathbb{E}\mu\mu\mu+(30+60\mathbb{E})\mu\mu\mu+[17+75\mathbb{E}]\mu\mu\mu+[17+75\mathbb{E}-90\mathbb{E}]\mu\mu+(42+120\mathbb{E}+8\mathbb{G})\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu\mu+[29+105\mathbb{E}-45\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15]\mu+(18+60\mathbb{E})\mu+[30\mathbb{E}+15])\}=\{9\mu\mu\mu\mu+(50+90\mathbb{E})\mu\mu\mu+[88+210\mathbb{E}+12\mathbb{G}]\mu\mu+[62+150\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15]\}, \tag{E104}$$

$$\big\{(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])+\big(-3\mu\mu\mu-(15+30\mathbb{E})\mu\mu-(21+60\mathbb{E}+4\mathbb{G})\mu-(9+30\mathbb{E})\big)\big\}(1+\mu)^3s^2=([2+45\mathbb{E}]\mu\mu+[8+45\mathbb{E}]\mu+6)(1+\mu)^3s^2, \tag{E105}$$

$$[-3(1+\mu)^3(\mu+3)(5+3\mu)s^4-6(1+\mu)^2(\mu+3)(3+2\mu)\mu s^4]=-3[(1+\mu)(5+3\mu)+2(3+2\mu)\mu](\mu+3)(1+\mu)^2s^4=-3[(5+3\mu+5\mu+3\mu\mu)+(6\mu+4\mu\mu)](\mu+3)(1+\mu)^2s^4=-3(5+14\mu+7\mu\mu)(\mu+3)(1+\mu)^2s^4, \tag{E106}$$

$$[6(1+\mu)^4(\mu+3)(3+2\mu)s^2+3(\mu+3)(1+\mu)^4\mu s^2]=3[2(3+2\mu)+\mu](\mu+3)(1+\mu)^4s^2=3(6+5\mu)(\mu+3)(1+\mu)^4s^2, \tag{E107}$$

$$[180\mathbb{E}(1+\mu)^2\mu s^6+(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])\mu s^6]=[(180\mathbb{E}+360\mathbb{E}\mu+180\mathbb{E}\mu\mu)+(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])]\mu s^6=[(+6\mu\mu\mu+[34+330\mathbb{E}+180\mathbb{E}]\mu\mu+[58+390\mathbb{E}+360\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+180\mathbb{E}+30])]\mu s^6=[(+6\mu\mu\mu+[34+510\mathbb{E}]\mu\mu+[58+750\mathbb{E}+8\mathbb{G}]\mu+[240\mathbb{E}+30])]\mu s^6, \tag{E108}$$

$$-(1+\mu)^2(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30])s^4-2(1+\mu)(3\mu\mu\mu+[17+75\mathbb{E}]\mu\mu+[29+105\mathbb{E}+4\mathbb{G}]\mu+[30\mathbb{E}+15])\mu s^4=\{-(+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+30]+6\mu\mu\mu\mu+[34+330\mathbb{E}]\mu\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu\mu+[60\mathbb{E}+30]\mu)-(6\mu\mu\mu\mu+[34+150\mathbb{E}]\mu\mu\mu+[58+210\mathbb{E}+8\mathbb{G}]\mu\mu+[60\mathbb{E}+30]\mu)\}(1+\mu)s^4=\{-(+6\mu\mu\mu\mu+6\mu\mu\mu\mu+6\mu\mu\mu+[34+330\mathbb{E}]\mu\mu\mu+[34+150\mathbb{E}]\mu\mu\mu+[34+330\mathbb{E}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu\mu+[58+210\mathbb{E}+8\mathbb{G}]\mu\mu+[58+390\mathbb{E}+8\mathbb{G}]\mu+[60\mathbb{E}+$$

$$30]\mu + [60E + 30]\mu + [60E + 30])\}(1+\mu)s^4 = \{-(+12\mu\mu\mu\mu + [74 + 480E]\mu\mu\mu + [150 + 930E + 16G]\mu\mu + [118 + 510E + 8G]\mu + [60E + 30])\}(1+\mu)s^4 \quad . \qquad (E109)$$

Now, put together the-same-power terms:

$$\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[-180E\mu\mu s^8 + 90E\mu\mu s^8 + [+3(1+\mu)(\mu+3)(5+3\mu)\mu s^6 + (+6\mu\mu\mu + [34 + 510E]\mu\mu + [58 + 750E + 8G]\mu + [240E + 30])\mu s^6 - (6\mu\mu\mu + [34 + 195E]\mu\mu + [58 + 300E + 8G]\mu + [105E + 30])\mu s^6] + [-3(5 + 14\mu + 7\mu\mu)(1+\mu)^2(\mu+3)s^4 + \{-(+12\mu\mu\mu\mu + [74 + 480E]\mu\mu\mu + [150 + 930E + 16G]\mu\mu + [118 + 510E + 8G]\mu + [60E + 30])\}(1+\mu)s^4 + \{9\mu\mu\mu\mu + (50 + 90E)\mu\mu\mu + [88 + 210E + 12G]\mu\mu + [62 + 150E + 4G]\mu + [30E + 15]\}(1+\mu)s^4] + [+2(1+\mu)^3(3\mu\mu\mu + [17 + 75E]\mu\mu + [29 + 105E + 4G]\mu + [30E + 15])s^2 + ([2 + 45E]\mu\mu + [8 + 45E]\mu + 6)\,(1+\mu)^3 s^2 + 3(6 + 5\mu)(\mu+3)(1+\mu)^4 s^2] - 3(\mu+3)(1+\mu)^6 + \big(-3\mu\mu\mu - (15 + 30E)\mu\mu - (21 + 60E + 4G)\mu - (9 + 30E)\big)(1+\mu)^4\Big] . \qquad (E110)$$

When we employ the following expansions

$$[+3(1+\mu)(\mu+3)(5+3\mu)\mu s^6 + (+6\mu\mu\mu + [34 + 510E]\mu\mu + [58 + 750E + 8G]\mu + [240E + 30])\mu s^6 - (6\mu\mu\mu + [34 + 195E]\mu\mu + [58 + 300E + 8G]\mu + [105E + 30])\mu s^6] = [+3(1+\mu)(5\mu + 3\mu\mu + 15 + 9\mu) + (+[+510E]\mu\mu + [+750E + 8G]\mu + [240E]) - (+[+195E]\mu\mu + [+300E + 8G]\mu + [105E])]\mu s^6 = [+(3 + 3\mu)(14\mu + 3\mu\mu + 15) + (+[+315E]\mu\mu + [+450E]\mu + [135E])]\mu s^6 = [(42\mu + 9\mu\mu + 45 + 42\mu\mu + 9\mu\mu\mu + 45\mu) + (+[+315E]\mu\mu + [+450E]\mu + [135E])]\mu s^6 = (9\mu\mu\mu + [51 + 315E]\mu\mu + [87 + 450E]\mu + [45 + 135E])\mu s^6 \quad , \qquad (E111)$$

$$[-3(5 + 14\mu + 7\mu\mu)(1+\mu)^2(\mu+3)s^4 + \{-(+12\mu\mu\mu\mu + [74 + 480E]\mu\mu\mu + [150 + 930E + 16G]\mu\mu + [118 + 510E + 8G]\mu + [60E + 30])\}(1+\mu)s^4 + \{9\mu\mu\mu\mu + (50 + 90E)\mu\mu\mu + [88 + 210E + 12G]\mu\mu + [62 + 150E + 4G]\mu + [30E + 15]\}(1+\mu)s^4] = [-3(5 + 14\mu + 7\mu\mu)(\mu+3)(1+\mu) - (+12\mu\mu\mu\mu + [74 + 480E]\mu\mu\mu + [150 + 930E + 16G]\mu\mu + [118 + 510E + 8G]\mu + [60E + 30]) + \{9\mu\mu\mu\mu + (50 + 90E)\mu\mu\mu + [88 + 210E + 12G]\mu\mu + [62 + 150E + 4G]\mu + [30E + 15]\}](1+\mu)s^4 = [-3(+7\mu\mu\mu + 15 + 47\mu + 35\mu\mu)(1+\mu) - (3\mu\mu\mu\mu + [24 + 390E]\mu\mu\mu + [62 + 720E + 4G]\mu\mu + [56 + 360E + 4G]\mu + [30E + 15])](1+\mu)s^4 == [-(21\mu\mu\mu\mu + 126\mu\mu\mu + 246\mu\mu + 186\mu + 45) - (3\mu\mu\mu\mu + [24 + 390E]\mu\mu\mu + [62 + 720E + 4G]\mu\mu + [56 + 360E + 4G]\mu + [30E + 15])](1+\mu)s^4 = [-(24\mu\mu\mu\mu + [150 + 390E]\mu\mu\mu + [308 + 720E + 4G]\mu\mu + [242 + 360E + 4G]\mu + [30E + 60])](1+\mu)s^4, \qquad (E112)$$

$$[+3(6 + 5\mu)(\mu+3)(1+\mu)^4 s^2 + 2(1+\mu)^3(3\mu\mu\mu + [17 + 75E]\mu\mu + [29 + 105E + 4G]\mu + [30E + 15])s^2 + ([2 + 45E]\mu\mu + [8 + 45E]\mu + 6)\,(1+\mu)^3 s^2] = \{+3(6\mu + 5\mu\mu + 18 + 15\mu)(1+\mu) + (6\mu\mu\mu + [34 + 150E]\mu\mu + [58 + 210E + 8G]\mu + [60E + 30]) + ([2 + 45E]\mu\mu + [8 + 45E]\mu + 6)\}(1+\mu)^3 s^2 = \{+3(5\mu\mu + 21\mu + 18)(1+\mu) + (6\mu\mu\mu + [34 + 150E]\mu\mu + [58 + 210E + 8G]\mu + [60E + 30]) + ([2 + 45E]\mu\mu + [8 + 45E]\mu + 6)\}(1+\mu)^3 s^2 = \{(15\mu\mu + 63\mu + 54)(1+\mu) + (6\mu\mu\mu + [36 + 195E]\mu\mu + [66 + 255E + 8G]\mu + [60E + 36])\}(1+\mu)^3 s^2 = \{(15\mu\mu + 63\mu + 54 + (15\mu\mu\mu + 63\mu\mu + 54\mu)) + (6\mu\mu\mu + [36 + 195E]\mu\mu + [66 + 255E + 8G]\mu + [60E + 36])\}(1+\mu)^3 s^2 = \{(15\mu\mu\mu + 78\mu\mu + 117\mu + 54) + (6\mu\mu\mu + [36 + 195E]\mu\mu + [66 + 255E + 8G]\mu + [60E + 36])\}(1+\mu)^3 s^2 = (21\mu\mu\mu + [114 + 195E]\mu\mu + [183 + 255E + 8G]\mu + [60E + 90])(1+\mu)^3 s^2 \quad , \qquad (E113)$$

$$-3(\mu+3)(1+\mu)^6 + \big(-3\mu\mu\mu - (15 + 30E)\mu\mu - (21 + 60E + 4G)\mu - (9 + 30E)\big)(1+\mu)^4 = [-3(\mu+3)(1+\mu)^2 + \big(-3\mu\mu\mu - (15 + 30E)\mu\mu - (21 + 60E + 4G)\mu - (9 + 30E)\big)](1+\mu)^4 = [-(3\mu + 9)(1 + 2\mu + \mu\mu) + \big(-3\mu\mu\mu - (15 + 30E)\mu\mu - (21 + 60E + 4G)\mu - (9 + 30E)\big)](1+\mu)^4 = [-(3\mu + 6\mu\mu + 3\mu\mu\mu + 9 + 18\mu + 9\mu\mu) + \big(-3\mu\mu\mu - (15 + 30E)\mu\mu - (21 + 60E + 4G)\mu - (9 + 30E)\big)](1+\mu)^4 = \big(-6\mu\mu\mu - (15 + 15 + 30E)\mu\mu - (21 + 21 + 60E + 4G)\mu - (9 + 9 + 30E)\big)(1+\mu)^4 = \big(-6\mu\mu\mu - (30 + 30E)\mu\mu - (42 + 60E + 4G)\mu - (18 + 30E)\big)(1+\mu)^4, \qquad (E114)$$

then we obtain

$$\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[-90E\mu\mu s^8 + (9\mu\mu\mu + [51 + 315E]\mu\mu + [87 + 450E]\mu + [45 + 135E])\mu s^6 - (24\mu\mu\mu\mu + [150 + 390E]\mu\mu\mu + [308 + 720E + 4G]\mu\mu + [242 + 360E + 4G]\mu + [30E + 60])(1+\mu)s^4 + (21\mu\mu\mu + [114 + 195E]\mu\mu + [183 + 255E + 8G]\mu + [90 + 60E])\,(1+\mu)^3 s^2 + \big(-6\mu\mu\mu - (30 + 30E)\mu\mu - (42 + 60E + 4G)\mu - (18 + 30E)\big)(1+\mu)^4\Big] \quad . \qquad (E115)$$

The factor by the square root can be thus filled to the overall second derivative relation, and we have it finally in the form

$$\frac{d^2F_{test3}(s)}{ds^2} = \frac{1}{(1+\mu)^6}\left\{(\mu+3)(5+3\mu)s\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right] + \frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[-90E\mu\mu s^8 + (9\mu\mu\mu + [51 + 315E]\mu\mu + [87 + 450E]\mu + [45 + 135E])\mu s^6 - (24\mu\mu\mu\mu + [150 + 390E]\mu\mu\mu + [308 + 720E + 4G]\mu\mu + [242 + 360E + 4G]\mu + [30E + 60])(1+\mu)s^4 + (21\mu\mu\mu + [114 + 195E]\mu\mu + [183 + 255E + 8G]\mu + [90 + 60E])\,(1+\mu)^3 s^2 + \big(-6\mu\mu\mu - (30 + 30E)\mu\mu - (42 + 60E + 4G)\mu - (18 + 30E)\big)(1+\mu)^4\Big]\sqrt{(1+\mu)^2-\mu s^2}\right\} \qquad (E116)$$

Logarithm part of the angular part of the Laplace equation

Now, let's focus on the *ln* part of the angular Laplace equation for the model function, for which we suppose that it has to be separately equal to zero. When we put together the corresponding logarithm parts of the second (E116) and the first derivatives, and the function itself, we have:

$$H_2(s)\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}\left[2(5+3\mu)s\left[\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right]\right]\right\}+ H_1(s)\frac{1}{(1+\mu)^6}\left\{\frac{(\mu+3)}{2}[(5+3\mu)s^2-(1+\mu)^2]\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)\right\}+H_0(s)\frac{1}{(1+\mu)^6}\frac{(\mu+3)}{6}\left((5+3\mu)s^3-3(1+\mu)^2s\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)=0 \quad . \qquad (E117)$$

Divide it by the common factors of all terms, which results in

$$H_2(s)\left\{\frac{1}{2}[2(5+3\mu)s]\right\}+H_1(s)\left\{\frac{1}{2}[(5+3\mu)s^2-(1+\mu)^2]\right\}+H_0(s)\frac{1}{6}\left((5+3\mu)s^3-3(1+\mu)^2s\right)=0 \quad . \qquad (E118)$$

Then input the H_i polynomials according to (E3), (E4) and (E5):

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2][(5+3\mu)s]+s[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2]\frac{1}{2}[(5+3\mu)s^2-(1+\mu)^2]+K_d[(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2]\frac{1}{6}\left((5+3\mu)s^3-3(1+\mu)^2s\right)=0 \quad . \qquad (E119)$$

Dividing by s leads to

$$[(1+\mu)-s^2][(1+\mu)^2-\mu s^2][(5+3\mu)]+[-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d+\mu(3-2K_d)s^2]\frac{1}{2}[(5+3\mu)s^2-(1+\mu)^2]+K_d[(K_d-2)\mu s^2+(\mu+1)K_d+(1+\mu)^2]\frac{1}{6}\left((5+3\mu)s^2-3(1+\mu)^2\right)=0 \quad . \qquad (E120)$$

Now expand the terms

$$[(1+\mu)^3-(1+\mu)\mu s^2-(1+\mu)^2s^2+\mu s^4][(5+3\mu)]+[\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}+\mu(3-2K_d)s^2]\left[\frac{1}{2}(5+3\mu)s^2-\frac{1}{2}(1+\mu)^2\right]+[K_d(K_d-2)\mu s^2+K_d\{(\mu+1)K_d+(1+\mu)^2\}]\left(\frac{1}{6}(5+3\mu)s^2-\frac{1}{2}(1+\mu)^2\right)=0 \quad , \qquad (E121)$$

$$[(1+\mu)^3-(1+\mu)(1+2\mu)s^2+\mu s^4][(5+3\mu)]+\left[\frac{1}{2}(5+3\mu)\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}s^2+\frac{1}{2}(5+3\mu)(3-2K_d)\mu s^4-\frac{1}{2}(1+\mu)^2\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}-\frac{1}{2}(1+\mu)^2(3-2K_d)\mu s^2\right]+\left[\frac{1}{6}(5+3\mu)K_d(K_d-2)\mu s^4+\frac{1}{6}(5+3\mu)K_d\{(\mu+1)K_d+(1+\mu)^2\}s^2-\frac{1}{2}(1+\mu)^2K_d(K_d-2)\mu s^2-\frac{1}{2}(1+\mu)^2K_d\{(\mu+1)K_d+(1+\mu)^2\}\right]=0 \quad . \qquad (E122)$$

Still expand and start to put together the-same-power s terms:

$$[(1+\mu)^3(5+3\mu)-(1+\mu)(1+2\mu)(5+3\mu)s^2+(5+3\mu)\mu s^4]+\left[+\frac{1}{2}(5+3\mu)(3-2K_d)\mu s^4+\frac{1}{6}(5+3\mu)K_d(K_d-2)\mu s^4\right]+\left[-\frac{1}{2}(1+\mu)^2(3-2K_d)\mu s^2+\frac{1}{2}(5+3\mu)\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}s^2+\frac{1}{6}(5+3\mu)K_d\{(\mu+1)K_d+(1+\mu)^2\}s^2-\frac{1}{2}(1+\mu)^2K_d(K_d-2)\mu s^2\right]-\frac{1}{2}(1+\mu)^2\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}-\frac{1}{2}(1+\mu)^2K_d\{(\mu+1)K_d+(1+\mu)^2\}=0 \quad , \qquad (E123)$$

$$\left[+(5+3\mu)\mu s^4+\frac{1}{2}(5+3\mu)(3-2K_d)\mu s^4+\frac{1}{6}(5+3\mu)K_d(K_d-2)\mu s^4\right]+\left[-(1+\mu)(1+2\mu)(5+3\mu)s^2-\frac{1}{2}(1+\mu)^2(3-2K_d)\mu s^2+\frac{1}{2}(5+3\mu)\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}s^2+\frac{1}{6}(5+3\mu)K_d\{(\mu+1)K_d+\right.$$

$$(1+\mu)^2\}s^2-\frac{1}{2}(1+\mu)^2K_d(K_d-2)\mu s^2\Big]+\Big[(1+\mu)^3(5+3\mu)-\frac{1}{2}(1+\mu)^2\{-(3\mu+2)(\mu+1)+2\mu(\mu+1)K_d\}-\frac{1}{2}(1+\mu)^2K_d\{(\mu+1)K_d+(1+\mu)^2\}\Big]=0 \quad , \qquad \text{(E124)}$$

$$\frac{1}{6}[6(5+3\mu)+3(5+3\mu)(3-2K_d)+(5+3\mu)K_d(K_d-2)]\mu s^4+\frac{1}{6}[-6(1+2\mu)(5+3\mu)-3(1+\mu)(3-2K_d)\mu+3(5+3\mu)\{-(3\mu+2)+2\mu K_d\}+(5+3\mu)K_d\{K_d+(1+\mu)\}-3(1+\mu)K_d(K_d-2)\mu](1+\mu)s^2+\frac{1}{2}[2(5+3\mu)-\{-(3\mu+2)+2\mu K_d\}-K_d\{K_d+(1+\mu)\}](1+\mu)^3=0 \quad , \qquad \text{(E125)}$$

$$\frac{1}{6}[(30+18\mu)+(45-30K_d+27\mu-18\mu K_d)+(5K_dK_d-10K_d+3\mu K_dK_d-6\mu K_d)]\mu s^4+\frac{1}{6}[(-30-18\mu-60\mu-36\mu\mu)+(-9\mu+6\mu K_d-9\mu\mu+6K_d\mu\mu)+\{-(15+9\mu)(3\mu+2)+2\mu K_d(15+9\mu)\}+\{K_d(5K_d+3\mu K_d)+(1+\mu)(5K_d+3\mu K_d)\}+(-3\mu K_dK_d+6\mu K_d-3\mu\mu K_dK_d+6\mu\mu K_d)](1+\mu)s^2+\frac{1}{2}[(10+6\mu)+\{(3\mu+2)-2\mu K_d\}+\{-K_dK_d-(1+\mu)K_d\}](1+\mu)^3=0 \quad , \qquad \text{(E126)}$$

$$\frac{1}{6}[(75+45\mu)+(-40-24\mu)K_d+(5+3\mu)K_dK_d]\mu s^4+\frac{1}{6}[(-30-87\mu-45\mu\mu)+(+6\mu K_d+6K_d\mu\mu)+(-45\mu-27\mu\mu-30-18\mu)+(30\mu K_d+18\mu\mu K_d)+(5K_dK_d+3\mu K_dK_d)+(5K_d+3\mu K_d+5\mu K_d+3\mu\mu K_d)+(-3\mu K_dK_d-3\mu\mu K_dK_d+6\mu K_d+6\mu\mu K_d)](1+\mu)s^2+\frac{1}{2}[(12+9\mu)-2\mu K_d-(1+\mu)K_d-K_dK_d](1+\mu)^3=0 \quad , \qquad \text{(E127)}$$

$$\frac{1}{6}[(75+45\mu)+(-40-24\mu)K_d+(5+3\mu)K_dK_d]\mu s^4+\frac{1}{6}[(-60-150\mu-72\mu\mu)+(50\mu K_d+33K_d\mu\mu)+(5K_dK_d\cancel{+3\mu K_dK_d})+(5K_d)+(\cancel{-3\mu K_dK_d}-3\mu\mu K_dK_d)](1+\mu)s^2+\frac{1}{2}[(12+9\mu)-(1+3\mu)K_d-K_dK_d](1+\mu)^3=0 \quad , \qquad \text{(E128)}$$

$$\frac{1}{6}[(75+45\mu)+(-40-24\mu)K_d+(5+3\mu)K_dK_d]\mu s^4+\frac{1}{6}[(-60-150\mu-72\mu\mu)+(5+50\mu+33\mu\mu)K_d+(5-3\mu\mu)K_dK_d](1+\mu)s^2+\frac{1}{2}[(12+9\mu)-(1+3\mu)K_d-K_dK_d](1+\mu)^3=0 \quad . \qquad \text{(E129)}$$

Then we have (from the term with s with power 4)

$$(75+45\mu)+(-40-24\mu)K_d+(5+3\mu)K_dK_d=0 \quad , \qquad \text{(E130)}$$

and thus

$$K_{d1,2}=\frac{(40+24\mu)\pm\sqrt{(40+24\mu)^2-4(5+3\mu)(75+45\mu)}}{2(5+3\mu)}=\frac{2(20+12\mu)\pm\sqrt{(2(20+12\mu))^2-4(375+225\mu+225\mu+135\mu\mu)}}{2(5+3\mu)}=\frac{2(20+12\mu)\pm2\sqrt{(400+480\mu+144\mu\mu)-(375+450\mu+135\mu\mu)}}{2(5+3\mu)}=\frac{(20+12\mu)\pm\sqrt{(25+30\mu+9\mu\mu)}}{(5+3\mu)}=\frac{4(5+3\mu)\pm\sqrt{(5+3\mu)^2}}{(5+3\mu)}=4\pm1=3 \quad or \quad 5 \quad . \qquad \text{(E131)}$$

We have from the term with s with power 2

$$[(-60-150\mu-72\mu\mu)+(5+50\mu+33\mu\mu)K_d+(5-3\mu\mu)K_dK_d]=0 \quad , \qquad \text{(E132)}$$

and thus

$$K_{d1,2}=\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{(5+50\mu+33\mu\mu)^2-4(5-3\mu\mu)(-60-150\mu-72\mu\mu)}}{2(5-3\mu\mu)}=$$

$$\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{((5+50\mu)^2+66(5+50\mu)\mu\mu+1089\mu\mu\mu\mu)+4(300+750\mu+360\mu\mu-180\mu\mu-450\mu\mu\mu-216\mu\mu\mu\mu)}}{2(5-3\mu\mu)}=$$

$$\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{((25+500\mu+2500\mu\mu)+(330\mu\mu+3300\mu\mu\mu)+1089\mu\mu\mu\mu)+(1200+3000\mu+1440\mu\mu-720\mu\mu-1800\mu\mu\mu-864\mu\mu\mu\mu)}}{2(5-3\mu\mu)}=$$

$$\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{(25+500\mu+2830\mu\mu+3300\mu\mu\mu+1089\mu\mu\mu\mu)+(1200+3000\mu+720\mu\mu-1800\mu\mu\mu-864\mu\mu\mu\mu)}}{2(5-3\mu\mu)}=$$

$$\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{(1225+3500\mu+3550\mu\mu+1500\mu\mu\mu+225\mu\mu\mu\mu)}}{2(5-3\mu\mu)}=\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{25(49+140\mu+142\mu\mu+60\mu\mu\mu+9\mu\mu\mu\mu)}}{2(5-3\mu\mu)}=$$

$$\frac{-(5+50\mu+33\mu\mu)\pm\sqrt{25(7+10\mu+3\mu\mu)^2}}{2(5-3\mu\mu)} = \frac{-(5+50\mu+33\mu\mu)\pm 5(7+10\mu+3\mu\mu)}{2(5-3\mu\mu)} = \frac{-(5+50\mu+33\mu\mu)+(35+50\mu+15\mu\mu)}{2(5-3\mu\mu)} \quad or \quad \frac{-(5+50\mu+33\mu\mu)-(35+50\mu+15\mu\mu)}{2(5-3\mu\mu)} = \frac{(30-18\mu\mu)}{2(5-3\mu\mu)} \quad or \quad \frac{-(40+100\mu+48\mu\mu)}{2(5-3\mu\mu)} = \frac{6(5-3\mu\mu)}{2(5-3\mu\mu)} \quad or \quad -2\frac{(10+25\mu+12\mu\mu)}{(5-3\mu\mu)} = 3 \quad or \quad -2\frac{(10+25\mu+12\mu\mu)}{(5-3\mu\mu)} \ . \tag{E133}$$

And – finally – we have from the term with *s* with power 0

$$[(12+9\mu)-(1+3\mu)K_d-K_dK_d] = 0 \ , \tag{E134}$$

and thus

$$K_{d1,2} = \frac{(1+3\mu)\pm\sqrt{(1+3\mu)^2-4(-1)(12+9\mu)}}{2(-1)} = \frac{(1+3\mu)\pm\sqrt{(1+6\mu+9\mu\mu)+(48+36\mu)}}{-2} = \frac{-(1+3\mu)\mp\sqrt{(49+42\mu+9\mu\mu)}}{2} = \frac{-(1+3\mu)\mp\sqrt{(7+3\mu)^2}}{2} = \frac{-(1+3\mu)\mp(7+3\mu)}{2} = \frac{-(8+6\mu)}{2} \quad or \quad \frac{6}{2} = -(4+3\mu) \quad or \quad 3 \ . \tag{E135}$$

Common for all three power terms is the solution with K_d=3. It confirms that this part of the equation is fulfilled with (and only with) K_d=3.

For the second part of the equation, i.e. the square root part (which is more complicated), K_d can be thus set firmly to K_d=3.

The square root part of the equation

Further, let's focus on the *square root* part of the equation, for which we suppose that it has to be separately equal to zero. When we put together the corresponding *square root* parts of the second and the first derivatives (see (E96) and (E116)), and of the test function itself (see (E87)), we obtain

$$H_2(s)\frac{1}{(1+\mu)^6}\left\{\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[-90\mathrm{E}\mu\mu s^8+(9\mu\mu\mu+[51+315\mathrm{E}]\mu\mu+[87+450\mathrm{E}]\mu+[45+135\mathrm{E}])\mu s^6-(24\mu\mu\mu\mu+[150+390\mathrm{E}]\mu\mu\mu+[308+720\mathrm{E}+4\mathrm{G}]\mu\mu+[242+360\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195\mathrm{E}]\mu\mu+[183+255\mathrm{E}+8\mathrm{G}]\mu+[90+60\mathrm{E}])(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30\mathrm{E})\mu\mu-(42+60\mathrm{E}+4\mathrm{G})\mu-(18+30\mathrm{E}))(1+\mu)^4\Big]\sqrt{(1+\mu)^2-\mu s^2}\right\}+H_1(s)\frac{1}{(1+\mu)^6}\left\{\Big[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E}))(1+\mu)\Big]\frac{s\sqrt{(1+\mu)^2-\mu s^2}}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right\}+H_0(s)\frac{1}{(1+\mu)^6}\left(-\frac{5}{2}\mathrm{E}s^2+\frac{2}{3}\mathrm{G}\right)\sqrt{(1+\mu)^2-\mu s^2}=0 \ . \tag{E136}$$

Divide it by the common factors of all the terms:

$$H_2(s)\left\{\frac{1}{6\,[(1+\mu)-s^2]^2[(1+\mu)^2-\mu s^2]^2}\Big[-90\mathrm{E}\mu\mu s^8+(9\mu\mu\mu+[51+315\mathrm{E}]\mu\mu+[87+450\mathrm{E}]\mu+[45+135\mathrm{E}])\mu s^6-(24\mu\mu\mu\mu+[150+390\mathrm{E}]\mu\mu\mu+[308+720\mathrm{E}+4\mathrm{G}]\mu\mu+[242+360\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195\mathrm{E}]\mu\mu+[183+255\mathrm{E}+8\mathrm{G}]\mu+[90+60\mathrm{E}])(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30\mathrm{E})\mu\mu-(42+60\mathrm{E}+4\mathrm{G})\mu-(18+30\mathrm{E}))(1+\mu)^4\Big]\right\}+H_1(s)\left\{\Big[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E}))(1+\mu)\Big]\frac{s}{6[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\right\}+H_0(s)\left(-\frac{5}{2}\mathrm{E}s^2+\frac{2}{3}\mathrm{G}\right)=0 \ . \tag{E137}$$

Multiply by 6 and by the rest of the denominator of the middle term:

$$H_2(s)\left\{\frac{1}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\Big[-90\mathrm{E}\mu\mu s^8+(9\mu\mu\mu+[51+315\mathrm{E}]\mu\mu+[87+450\mathrm{E}]\mu+[45+135\mathrm{E}])\mu s^6-(24\mu\mu\mu\mu+[150+390\mathrm{E}]\mu\mu\mu+[308+720\mathrm{E}+4\mathrm{G}]\mu\mu+[242+360\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195\mathrm{E}]\mu\mu+[183+255\mathrm{E}+8\mathrm{G}]\mu+[90+60\mathrm{E}])(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30\mathrm{E})\mu\mu-(42+60\mathrm{E}+4\mathrm{G})\mu-(18+30\mathrm{E}))(1+\mu)^4\Big]\right\}+H_1(s)\left\{\Big[-45\mathrm{E}\mu s^4+(3\mu\mu\mu+[17+75\mathrm{E}]\mu\mu+[29+105\mathrm{E}+4\mathrm{G}]\mu+[30\mathrm{E}+15])s^2+(-3\mu\mu\mu-(15+30\mathrm{E})\mu\mu-(21+60\mathrm{E}+4\mathrm{G})\mu-(9+30\mathrm{E}))(1+\mu)\Big]s\right\}+H_0(s)\,(-15\mathrm{E}s^2+4\mathrm{G})[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]=0 \ . \tag{E138}$$

The last-term expansion leads to

$$\begin{aligned}&H_2(s)\left\{\frac{1}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\Big[-90E\mu\mu s^8+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+[45+135E])\mu s^6-\right.\\&(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+[30E+60])(1+\mu)s^4+\\&(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])\,(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30E)\mu\mu-\\&\left.(42+60E+4G)\mu-(18+30E))(1+\mu)^4\Big]\right\}+H_1(s)\{[-45E\mu s^4+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+\\&4G]\mu+[30E+15])s^2+(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)]s\}+\\&H_0(s)\,(-15Es^2+4G)[(1+\mu)^3-(1+2\mu)(1+\mu)s^2+\mu s^4]=0\quad.\end{aligned}\tag{E139}$$

With the help of the expansion in the last term

$$\begin{aligned}&(-15Es^2+4G)[(1+\mu)^3-(1+2\mu)(1+\mu)s^2+\mu s^4]=[-15Es^2(1+\mu)^3+15Es^2(1+2\mu)(1+\mu)s^2-15Es^2\mu s^4+4G(1+\mu)^3-\\&4G(1+2\mu)(1+\mu)s^2+4G\mu s^4]=[-15E\mu s^6+15E(1+2\mu)(1+\mu)s^4+4G\mu s^4-15E(1+\mu)^3s^2-4G(1+2\mu)(1+\mu)s^2+4G(1+\mu)^3]=\\&[-15E\mu s^6+\{15E(1+2\mu)(1+\mu)+4G\mu\}s^4+\{-15E(1+\mu)^2-4G(1+2\mu)\}(1+\mu)s^2+4G(1+\mu)^3]=[-15E\mu s^6+\{15E(1+2\mu+\mu+\\&2\mu\mu)+4G\mu\}s^4+\{-(15E+30E\mu+15E\mu\mu)-(4G+8G\mu)\}(1+\mu)s^2+4G(1+\mu)^3]=[-15E\mu s^6+(15E+[45E+4G]\mu+30E\mu\mu)s^4-\\&([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)s^2+4G(1+\mu)^3],\end{aligned}\tag{E140}$$

we get

$$\begin{aligned}&H_2(s)\left\{\frac{1}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\Big[-90E\mu\mu s^8+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+[45+135E])\mu s^6-\right.\\&(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+[30E+60])(1+\mu)s^4+\\&(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])\,(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30E)\mu\mu-\\&\left.(42+60E+4G)\mu-(18+30E))(1+\mu)^4\Big]\right\}+H_1(s)\{[-45E\mu s^4+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+\\&4G]\mu+[30E+15])s^2+(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)]s\}+\\&H_0(s)\,[-15E\mu s^6+(15E+[45E+4G]\mu+30E\mu\mu)s^4-([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)s^2+4G(1+\mu)^3]=0\quad.\end{aligned}\tag{E141}$$

Then insert the polynomials (E3), (E4) and (E5),

$$\begin{aligned}&[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\left\{\frac{1}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\Big[-90E\mu\mu s^8+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+\right.\\&[45+135E])\mu s^6-(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+\\&[30E+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])\,(1+\mu)^3s^2+\\&\left.(-6\mu\mu\mu-(30+30E)\mu\mu-(42+60E+4G)\mu-(18+30E))(1+\mu)^4\Big]\right\}+s[-(3\mu+2)(\mu+1)+\\&2\mu(\mu+1)K_d+\mu(3-2K_d)s^2]\{[-45E\mu s^4+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])s^2+\\&(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)]s\}+K_d[(K_d-2)\mu s^2+(\mu+1)K_d+\\&(1+\mu)^2][-15E\mu s^6+(15E+[45E+4G]\mu+30E\mu\mu)s^4-([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)s^2+4G(1+\mu)^3]=0\\&,\end{aligned}\tag{E142}$$

to which we substitute K_d=3 (an abbreviation [....] is used for the repeated parts of the equation):

$$\begin{aligned}&[(1+\mu)-s^2][(1+\mu)^2-\mu s^2][\ldots.]+s[-(3\mu+2)(\mu+1)+6\mu(\mu+1)+\mu(-3)s^2][\ldots.]+3[\mu s^2+\\&3(\mu+1)+(1+\mu)^2][\ldots.]=0\quad,\end{aligned}\tag{E143}$$

$$\begin{aligned}&[(1+\mu)-s^2][(1+\mu)^2-\mu s^2][\ldots.]+s[-(3\mu\mu+5\mu+2)+(6\mu\mu+6\mu)-3\mu s^2][\ldots.]+3[\mu s^2+\\&(\mu+1)(\mu+4)][\ldots.]=0\quad,\end{aligned}\tag{E144}$$

and thus

$$\begin{aligned}&[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]\left\{\frac{1}{[(1+\mu)-s^2][(1+\mu)^2-\mu s^2]}\Big[-90E\mu\mu s^8+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+\right.\\&[45+135E])\mu s^6-(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+\\&[30E+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])\,(1+\mu)^3s^2+\\&\left.(-6\mu\mu\mu-(30+30E)\mu\mu-(42+60E+4G)\mu-(18+30E))(1+\mu)^4\Big]\right\}+\\&s[(3\mu\mu+\mu-2)-3\mu s^2]\{[-45E\mu s^4+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])s^2+\\&(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)]s\}+3[\mu s^2+(\mu+1)(\mu+4)][-15E\mu s^6+\\&(15E+[45E+4G]\mu+30E\mu\mu)s^4-([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)s^2+4G(1+\mu)^3]=0\quad.\end{aligned}\tag{E145}$$

In the first term, mutual cancellations in the numerator and the denominator lead to a simplification. Thus

$$\{[-90E\mu\mu s^8+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+[45+135E])\mu s^6-(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+[30E+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30E)\mu\mu-(42+60E+4G)\mu-(18+30E))(1+\mu)^4]\}+[(3\mu\mu+\mu-2)s^2-3\mu s^4]\{[-45E\mu s^4+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])s^2+(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)]\}+[3\mu s^2+3(\mu+1)(\mu+4)][-15E\mu s^6+(15E+[45E+4G]\mu+30E\mu\mu)s^4-([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)s^2+4G(1+\mu)^3]=0\quad . \tag{E146}$$

Further expand the terms:

$$\{[-90E\mu\mu s^8+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+[45+135E])\mu s^6-(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+[30E+60])(1+\mu)s^4+(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])(1+\mu)^3s^2+(-6\mu\mu\mu-(30+30E)\mu\mu-(42+60E+4G)\mu-(18+30E))(1+\mu)^4]\}+\{[-45(3\mu\mu+\mu-2)E\mu s^6+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])(3\mu\mu+\mu-2)s^4+(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)(3\mu\mu+\mu-2)s^2]-[-135E\mu\mu s^8+3(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])\mu s^6+3(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)\mu s^4]\}+[-45E\mu\mu s^8+3(15E+[45E+4G]\mu+30E\mu\mu)\mu s^6-3([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)\mu s^4+12G(1+\mu)^3\mu s^2]+[-45E(\mu+1)(\mu+4)\mu s^6+3(\mu+1)(\mu+4)(15E+[45E+4G]\mu+30E\mu\mu)s^4-3(\mu+4)([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)^2s^2+12G(\mu+4)(1+\mu)^4]=0\quad . \tag{E147}$$

Put together the-same-power terms (also take into the account that $(3\mu\mu+\mu-2)=(3\mu-2)(1+\mu)$):

$$-90E\mu\mu s^8+135E\mu\mu s^8-45E\mu\mu s^8+\{+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+[45+135E])\mu s^6-3(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])\mu s^6-45(3\mu\mu+\mu-2)E\mu s^6+3(15E+[45E+4G]\mu+30E\mu\mu)\mu s^6-45E(\mu+1)(\mu+4)\mu s^6\}+\{-(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+[30E+60])(1+\mu)s^4+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])(3\mu-2)(1+\mu)s^4-3(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)\mu s^4-3([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)\mu s^4+3(\mu+1)(\mu+4)(15E+[45E+4G]\mu+30E\mu\mu)s^4\}+\{(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(1+\mu)(3\mu-2)(1+\mu)s^2+(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])(1+\mu)^3s^2+12G(1+\mu)^3\mu s^2-3(\mu+4)([15E+4G]+[30E+8G]\mu+15E\mu\mu)(1+\mu)^2s^2\}+[+(-6\mu\mu\mu-(30+30E)\mu\mu-(42+60E+4G)\mu-(18+30E))(1+\mu)^4+12G(\mu+4)(1+\mu)^4]=0\quad . \tag{E148}$$

Combine the same power terms:

$$[0]E\mu\mu s^8+\{+(9\mu\mu\mu+[51+315E]\mu\mu+[87+450E]\mu+[45+135E])-3(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])-45(3\mu\mu+\mu-2)E+3(15E+[45E+4G]\mu+30E\mu\mu)-45E(\mu+1)(\mu+4)\}\mu s^6+\{-(24\mu\mu\mu\mu+[150+390E]\mu\mu\mu+[308+720E+4G]\mu\mu+[242+360E+4G]\mu+[30E+60])+(3\mu\mu\mu+[17+75E]\mu\mu+[29+105E+4G]\mu+[30E+15])(3\mu-2)-3(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))\mu-3([15E+4G]+[30E+8G]\mu+15E\mu\mu)\mu+3(\mu+4)(15E+[45E+4G]\mu+30E\mu\mu)\}(1+\mu)s^4+\{(-3\mu\mu\mu-(15+30E)\mu\mu-(21+60E+4G)\mu-(9+30E))(3\mu-2)+(21\mu\mu\mu+[114+195E]\mu\mu+[183+255E+8G]\mu+[90+60E])(1+\mu)+12G(1+\mu)\mu-3(\mu+4)([15E+4G]+[30E+8G]\mu+15E\mu\mu)\}(1+\mu)^2s^2+[+(-6\mu\mu\mu-(30+30E)\mu\mu-(42+60E+4G)\mu-(18+30E))+12G(\mu+4)](1+\mu)^4=0\quad . \tag{E149}$$

In this expression, calculate individually the pre-factors of s power terms (done with a help of algebraic-manipulation-procedure on internet).

The power six part results in 0 for any E and any G.

The power two pre-factor can be expanded using internet-based algebraic machine to $\{[60E\mu\mu\mu+60E\mu\mu-60E\mu-60E-16G\mu\mu-80G\mu-48G+12\mu\mu\mu\mu+96\mu\mu\mu+264\mu\mu+288\mu+108]\}$,

which can be further modified in a series of the following steps:

$$\{[60\mu\mu\mu+60\mu\mu-60\mu-60]E-8[2\mu\mu+10\mu+6]G+12\mu\mu\mu\mu+96\mu\mu\mu+264\mu\mu+288\mu+108\}$$
$$\{60[\mu\mu\mu+\mu\mu-\mu-1]E-16[\mu\mu+5\mu+3]G+12[\mu\mu\mu\mu+8\mu\mu\mu+22\mu\mu+24\mu+9]\}$$
$$\{60[\mu\mu(\mu+1)-(\mu+1)]E-16[\mu\mu+5\mu+3]G+12[\mu\mu\mu\mu+\mu\mu\mu+7\mu\mu\mu+7\mu\mu+15\mu\mu+15\mu+9\mu+9]\}$$
$$\{60(\mu+1)[\mu\mu-1]E-16[\mu\mu+5\mu+3]G+12[\mu\mu\mu(\mu+1)+7\mu\mu(\mu+1)+15\mu(\mu+1)+9(\mu+1)]\}$$
$$\{60(\mu+1)(\mu+1)(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)[\mu\mu\mu+7\mu\mu+15\mu+9]\}$$
$$\{60(\mu+1)^2(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)[\mu\mu\mu+\mu\mu+6\mu\mu+6\mu+9\mu+9]\}$$
$$\{60(\mu+1)^2(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)[\mu\mu(\mu+1)+6\mu(\mu+1)+9(\mu+1)]\}$$
$$\{60(\mu+1)^2(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)^2[\mu\mu+6\mu+9]\}$$
$$\{60(\mu+1)^2(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)^2[\mu\mu+3\mu+3\mu+9]\}$$
$$\{60(\mu+1)^2(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)^2[\mu(\mu+3)+3(\mu+3)]\}$$
$$[60(\mu+1)^2(\mu-1)E-16[\mu\mu+5\mu+3]G+12(\mu+1)^2(\mu+3)^2]$$

The power four part can be modified using internet-based algebraic machine to

$$\{[-30E\mu\mu\mu+30E\mu\mu+150E\mu+90E]+8G\mu\mu+24G\mu-6\mu\mu\mu\mu-60\mu\mu\mu-192\mu\mu-228\mu-90\}$$

which can be further modified in a series of the following steps:

$$\{30[-\mu\mu\mu+\mu\mu+5\mu+3]E+8G\mu\mu+24G\mu-6[\mu\mu\mu\mu+10\mu\mu\mu+32\mu\mu+38\mu+15]\}$$
$$\{30[-\mu\mu\mu-\mu\mu+2\mu\mu+2\mu+3\mu+3]E+8G\mu\mu+24G\mu-6[\mu\mu\mu\mu+3\mu\mu\mu+7\mu\mu\mu+21\mu\mu+11\mu\mu+33\mu+5\mu+15]\}$$
$$\{30[-\mu\mu(\mu+1)+2\mu(\mu+1)+3(\mu+1)]E+8G\mu(\mu+3)-6[\mu\mu\mu(\mu+3)+7\mu\mu(\mu+3)+11\mu(\mu+3)+5(\mu+3)]\}$$
$$\{30(\mu+1)[-\mu\mu+2\mu+3]E+8G\mu(\mu+3)-6(\mu+3)[\mu\mu\mu+7\mu\mu+11\mu+5]\}$$
$$\{30(\mu+1)[-\mu\mu-\mu+3\mu+3]E+8G\mu(\mu+3)-6(\mu+3)[\mu\mu\mu+\mu\mu+6\mu\mu+6\mu+5\mu+5]\}$$
$$\{30(\mu+1)[-\mu(\mu+1)+3(\mu+1)]E+8G\mu(\mu+3)-6(\mu+3)[\mu\mu(\mu+1)+6\mu(\mu+1)+5(\mu+1)]\}$$
$$\{30(\mu+1)(\mu+1)[-\mu+3]E+8G\mu(\mu+3)-6(\mu+1)(\mu+3)[\mu\mu+6\mu+5]\}$$
$$\{30(\mu+1)^2[3-\mu]E+8G\mu(\mu+3)-6(\mu+1)(\mu+3)[\mu\mu+\mu+5\mu+5]\}$$
$$\{30(\mu+1)^2[3-\mu]E+8G\mu(\mu+3)-6(\mu+1)(\mu+3)[\mu(\mu+1)+5(\mu+1)]\}$$
$$\big[30(\mu+1)^2(3-\mu)E+8G\mu(\mu+3)-6(\mu+1)(\mu+1)(\mu+3)[\mu+5]\big]$$
$$[30(\mu+1)^2(3-\mu)E+8G\mu(\mu+3)-6(\mu+1)^2(\mu+3)(\mu+5)]$$

The power zero part can be modified using internet-based algebraic machine to

$$[-30E(\mu\mu+2\mu+1)+8G(\mu+6)]-6\mu\mu\mu-30\mu\mu-42\mu-18$$

which can be further modified in a series of the following steps:

$$[-30E(\mu+1)^2+8G(\mu+6)]-6(\mu\mu\mu+5\mu\mu+7\mu+3)$$
$$[-30E(\mu+1)^2+8G(\mu+6)]-6(\mu\mu\mu+\mu\mu+4\mu\mu+4\mu+3\mu+3)$$
$$[-30E(\mu+1)^2+8G(\mu+6)]-6\big(\mu\mu(\mu+1)+4\mu(\mu+1)+3(\mu+1)\big)$$
$$[-30E(\mu+1)^2+8G(\mu+6)]-6(\mu+1)(\mu\mu+4\mu+3)$$
$$[-30E(\mu+1)^2+8G(\mu+6)]-6(\mu+1)(\mu\mu+\mu+3\mu+3)$$
$$[-30E(\mu+1)^2+8G(\mu+6)]-6(\mu+1)\big(\mu(\mu+1)+3(\mu+1)\big)$$
$$[-30E(\mu+1)^2+8G(\mu+6)]-6(\mu+1)(\mu+1)(\mu+3)$$
$$[-30E(\mu+1)^2+8G(\mu+6)-6(\mu+1)^2(\mu+3)]$$

Further, combine the same power terms according to the partial calculations shown above:

$$[0]E\mu\mu s^8+\{0\}\mu s^6+[30(\mu+1)^2(3-\mu)E+8G\mu(\mu+3)-6(\mu+1)^2(\mu+3)(\mu+5)](1+\mu)s^4+[60(\mu+1)^2(\mu-1)E-16(\mu\mu+5\mu+3)G+12(\mu+1)^2(\mu+3)^2](1+\mu)^2s^2+[-30E(\mu+1)^2+8G(\mu+6)-6(\mu+1)^2(\mu+3)](1+\mu)^4=0 \quad . \tag{E150}$$

The two highest power pre-factors are zero, and the remaining equation is:

$$[30(\mu+1)^2(3-\mu)E+8G\mu(\mu+3)-6(\mu+1)^2(\mu+3)(\mu+5)](1+\mu)s^4+[60(\mu+1)^2(\mu-1)E-16(\mu\mu+5\mu+3)G+12(\mu+1)^2(\mu+3)^2](1+\mu)^2s^2+[-30E(\mu+1)^2+8G(\mu+6)-6(\mu+1)^2(\mu+3)](1+\mu)^4=0 \; . \tag{E151}$$

All three pre-factors by the individual powers of *s* have to be zero. Thus

$$[30(\mu+1)^2(3-\mu)E+8G\mu(\mu+3)-6(\mu+1)^2(\mu+3)(\mu+5)](1+\mu)=0 \quad , \tag{E152}$$
$$[60(\mu+1)^2(\mu-1)E-16(\mu\mu+5\mu+3)G+12(\mu+1)^2(\mu+3)^2](1+\mu)^2=0 \quad , \tag{E153}$$
$$[-30E(\mu+1)^2+8G(\mu+6)-6(\mu+1)^2(\mu+3)](1+\mu)^4=0 \quad . \tag{E154}$$

Then, divide by constants appearing at the ends of the left sides of the equations, and divide the first equation also by $(3-\mu)$:

$$\left[30(\mu+1)^2E+8G\frac{\mu(\mu+3)}{(3-\mu)}-6\frac{(\mu+1)^2(\mu+3)(\mu+5)}{(3-\mu)}\right]=0 \quad , \tag{E155}$$
$$[60(\mu+1)^2(\mu-1)E-16(\mu\mu+5\mu+3)G+12(\mu+1)^2(\mu+3)^2]=0 \quad , \tag{E156}$$
$$[-30E(\mu+1)^2+8G(\mu+6)-6(\mu+1)^2(\mu+3)]=0 \quad . \tag{E157}$$

Sum the first (E152) and the third (E154) equations, and perform step by step simplifications:

$$\left[+8G(\mu+6)-6(\mu+1)^2(\mu+3)+8G\frac{\mu(\mu+3)}{(3-\mu)}-6\frac{(\mu+1)^2(\mu+3)(\mu+5)}{(3-\mu)}\right]=0 \quad , \tag{E158}$$

$$8G\left[(\mu+6)+\frac{\mu(\mu+3)}{(3-\mu)}\right]-6(\mu+1)^2(\mu+3)\left[1+\frac{(\mu+5)}{(3-\mu)}\right]=0 \quad , \tag{E159}$$

$$8G\left[\frac{(\mu+6)(3-\mu)+\mu(\mu+3)}{(3-\mu)}\right]-6(\mu+1)^2(\mu+3)\left[1+\frac{(\mu+5)}{(3-\mu)}\right]=0 \quad , \tag{E160}$$

$$8G\left[\frac{(3\mu-\mu\mu+18-6\mu)+(\mu\mu+3\mu)}{(3-\mu)}\right]-6(\mu+1)^2(\mu+3)\left[\frac{(3-\mu)+(\mu+5)}{(3-\mu)}\right]=0 \quad , \tag{E161}$$

$$8G\frac{18}{3-\mu}-6(\mu+1)^2(\mu+3)\frac{8}{3-\mu}=0 \quad , \tag{E162}$$

$$18G-6(\mu+1)^2(\mu+3)=0 \quad , \tag{E163}$$

$$3G-(\mu+1)^2(\mu+3)=0 \quad , \tag{E164}$$

$$G=(\mu+1)^2\frac{\mu+3}{3} \quad . \tag{E165}$$

Substitute the found G to the second (E156) equation, and perform step by step simplifications:

$$\left[60(\mu+1)^2(\mu-1)E-16(\mu\mu+5\mu+3)\left\{(\mu+1)^2\frac{\mu+3}{3}\right\}+12(\mu+1)^2(\mu+3)^2\right]=0 , \tag{E166}$$

$$\left[15(\mu-1)E-4(\mu\mu+5\mu+3)\left\{\frac{\mu+3}{3}\right\}+3(\mu+3)^2\right]=0 \quad , \tag{E167}$$

$$45(\mu-1)E-4(\mu\mu+5\mu+3)\{\mu+3\}+9(\mu+3)^2=0 , \tag{E168}$$

$$45(\mu-1)E+[-4(\mu\mu+5\mu+3)+9(\mu+3)](\mu+3)=0 \quad , \tag{E169}$$

$$45(\mu-1)E+[-(4\mu\mu+20\mu+12)+(9\mu+27)](\mu+3)=0 \quad , \tag{E170}$$

$$45(\mu-1)E+(-4\mu\mu-11\mu+15)(\mu+3)=0 \quad , \tag{E171}$$

$$45(\mu-1)E+(-4\mu\mu+4\mu-15\mu+15)(\mu+3)=0 \quad , \tag{E172}$$

$$45(\mu-1)E+\big(-4\mu(\mu-1)-15(\mu-1)\big)(\mu+3)=0 \quad , \tag{E173}$$

$$45(\mu-1)E+(\mu-1)(-4\mu-15)(\mu+3)=0 \quad , \tag{E174}$$

$$45E-(4\mu+15)(\mu+3)=0 \quad , \tag{E175}$$

$$E=\frac{(4\mu+15)(\mu+3)}{45} \quad . \tag{E176}$$

We see, that there exist the solution for the given model function with G and E given by (E165) and by (E176). Thus, finally, we insert the found G and E factors to the model function (E87):

$$F_{test3}(s)=$$
$$\frac{1}{(1+\mu)^6}\frac{(\mu+3)}{6}\left((5+3\mu)s^3-3(1+\mu)^2 s\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)+\frac{1}{(1+\mu)^6}\left(-\frac{5}{2}\frac{(4\mu+15)(\mu+3)}{45}s^2+\right.$$
$$\left.\frac{2}{3}(\mu+1)^2\frac{\mu+3}{3}\right)\sqrt{(1+\mu)^2-\mu s^2} \quad . \tag{E177}$$

Algebraically slightly modified in the second part, it leads to the solution

$$Q_3^S(s)\sim F_{test3}(s)=$$
$$\frac{1}{(1+\mu)^6}\frac{(\mu+3)}{3}\frac{1}{2}\left((5+3\mu)s^3-3(1+\mu)^2 s\right)\frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right)-\frac{1}{(1+\mu)^6}\frac{\mu+3}{3}\frac{1}{2}\left(\frac{4\mu+15}{3}s^2-\right.$$
$$\left.\frac{4}{3}(1+\mu)^2\right)\sqrt{(1+\mu)^2-\mu s^2} \quad . \tag{E178}$$

which is thus proportional to the second-kind third degree generalized Legendre function $Q_3^S(s)$. Proportionality rather than equality is declared here for the generalized Legendre function as the proper normalization factor is still a question connected with a recurrence formula to be found later.

When μ=0, then the function simplifies to the Legendre function of the second kind of third degree, as can be easily found by substitution μ=0 into (QS3):

$$Q_3^S(s)\Big|_{\mu=0}=Q_3(s)=\frac{1}{2}(5s^3-3s)\frac{1}{2}\ln\left(\frac{1+s}{1-s}\right)-\left(\frac{5}{2}s^2-\frac{2}{3}\right) \quad . \tag{E179}$$

E.3. Tests of the recursion formula for the second-kind generalized Legendre functions

The recursion formula

$$Q_{n+1}^{SI}(s) = \frac{2n+1}{n+1}\frac{1}{1+\mu}s\,Q_n^{SI}(s) - \frac{n}{n+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)Q_{n-1}^{SI}(s) \qquad \text{(E180)}$$

is to be tested for validity for the second kind generalized Legendre functions. When we observe the formula (100) in the main text, i.e.

$$Q_n^{SI}(s) = P_n^{SI}(s)\,Q_0^{SI}(s) - T_n(s)\sqrt{(1+\mu)^2 - \mu s^2} \quad , \qquad \text{(E181)}$$

it is clear that the recursion formula is fulfilled for the first part, as it is fulfilled for $P_n^{SI}(s)$ (see (91) of the main text), and a multiplication of both parts of the recursion formula by

$$Q_0^{S}(s) = \frac{1}{2}\ln\left(\frac{\left[s+\sqrt{(1+\mu)^2-\mu s^2}\right]^2}{(1+\mu)\,[(1+\mu)-s^2]}\right) \qquad \text{(E182)}$$

does not change the result.

Then, we have to check the recursion for $T_n(s)$ from the second part of (E181) (note that multiplication of both parts of the recursion formula by $\sqrt{(1+\mu)^2 - \mu s^2}$) does not change the result). Then, the test of the recursion formula reduces to testing the relation

$$T_{n+1}(s) = \frac{2n+1}{n+1}\frac{1}{1+\mu}s\,T_n(s) - \frac{n}{n+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_{n-1}(s) \quad , \qquad \text{(E183)}$$

where $T_0(s)$=0 and $T_1(s)=1/(1+\mu)$.

First, the formula is tested for n=1 (set so in order to derive the function of the second degree):

$$T_{1+1}(s) = \frac{2\cdot 1+1}{1+1}\frac{1}{1+\mu}s\,T_1(s) - \frac{1}{1+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_{1-1}(s) \quad , \qquad \text{(E184)}$$

$$T_2(s) = \frac{3}{2}\frac{1}{1+\mu}s\,T_1(s) - \frac{1}{2}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_0(s) \quad . \qquad \text{(E185)}$$

By inserting zero and first degree functions, we receive

$$T_2(s) = \frac{3}{2}\frac{1}{1+\mu}s\cdot\frac{1}{1+\mu} - \frac{1}{2}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)\cdot 0 \quad , \qquad \text{(E186)}$$

and after simplification we get

$$T_2(s) = \frac{3}{2}\frac{1}{(1+\mu)^2}s \quad . \qquad \text{(E187)}$$

Therefore, according to (E181) and (E187), we have

$$Q_2^{SI}(s) = P_2^{SI}(s)\,Q_0^{SI}(s) - \frac{3}{2}\frac{1}{(1+\mu)^2}s\sqrt{(1+\mu)^2 - \mu s^2} \quad , \qquad \text{(E188)}$$

or equivalently

$$Q_2^{SI}(s) = \frac{1}{(1+\mu)^2}\cdot\frac{1}{2}\cdot[(\mu+3)s^2 - (1+\mu)^2]\,Q_0^{SI}(s) - \frac{3}{2}\frac{1}{(1+\mu)^2}s\sqrt{(1+\mu)^2 - \mu s^2} = \frac{1}{(1+\mu)^2}\cdot\frac{1}{2}\cdot \left\{[(\mu+3)s^2 - (1+\mu)^2]\,Q_0^{SI}(s) - 3s\sqrt{(1+\mu)^2 - \mu s^2}\right\} \quad , \qquad \text{(E189)}$$

which is (except the normalization factor) the same as the second kind solution (94) in the main text.

Further, the formula (E183) is tested for n=2 (set so in order to derive the function of the third degree):

$$T_{2+1}(s) = \frac{2\cdot 2+1}{2+1}\frac{1}{1+\mu}s\,T_2(s) - \frac{2}{2+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_{2-1}(s) \quad , \qquad \text{(E190)}$$

$$T_3(s) = \frac{5}{3}\frac{1}{1+\mu} s\, T_2(s) - \frac{2}{3}\left(1 - \frac{\mu}{(1+\mu)^2} s^2\right) T_1(s) \quad . \tag{E191}$$

By inserting first and second (see (E187)) degree functions, we receive

$$T_3(s) = \frac{5}{3}\frac{1}{1+\mu} s \cdot \frac{3}{2}\frac{1}{(1+\mu)^2} s - \frac{2}{3}\left(1 - \frac{\mu}{(1+\mu)^2} s^2\right) \cdot \frac{1}{1+\mu} \quad , \tag{E192}$$

and after simplification we get

$$T_3(s) = \frac{5}{3} \cdot \frac{3}{2}\frac{1}{(1+\mu)^3} s^2 - \left(\frac{2}{3}\frac{1}{1+\mu} - \frac{2}{3}\frac{\mu}{(1+\mu)^3} s^2\right) \quad , \tag{E193}$$

$$T_3(s) = \frac{5}{3} \cdot \frac{3}{2}\frac{1}{(1+\mu)^3} s^2 + \frac{2}{3}\frac{2}{2}\frac{\mu}{(1+\mu)^3} s^2 - \frac{2}{3}\frac{1}{1+\mu} \quad , \tag{E194}$$

$$T_3(s) = \frac{1}{2}\frac{1}{(1+\mu)^3}\frac{4\mu+15}{3} s^2 - \frac{1}{2}\frac{1}{1+\mu}\frac{4}{3} \quad , \tag{E195}$$

$$T_3(s) = \frac{1}{2}\frac{1}{(1+\mu)^3}\left[\frac{4\mu+15}{3} s^2 - \frac{4}{3}(1+\mu)^2\right]. \tag{E196}$$

Therefore, according to (E181) and (E196), we have

$$Q_3^{SI}(s) = P_3^{SI}(s)\, Q_0^{SI}(s) - \frac{1}{2}\frac{1}{(1+\mu)^3}\left[\frac{4\mu+15}{3} s^2 - \frac{4}{3}(1+\mu)^2\right]\sqrt{(1+\mu)^2 - \mu s^2} \quad , \tag{E197}$$

or equivalently

$$Q_3^{SI}(s) = \frac{1}{(1+\mu)^3} \cdot \frac{1}{2} \cdot [(3\mu+5)s^3 - 3(1+\mu)^2 s]\, Q_0^{SI}(s) - \frac{1}{2}\frac{1}{(1+\mu)^3}\left[\frac{4\mu+15}{3} s^2 - \frac{4}{3}(1+\mu)^2\right]\sqrt{(1+\mu)^2 - \mu s^2} = \frac{1}{(1+\mu)^3} \cdot \frac{1}{2} \cdot \left\{[(3\mu+5)s^3 - 3(1+\mu)^2 s]\, Q_0^{SI}(s) - \left[\frac{4\mu+15}{3} s^2 - \frac{4}{3}(1+\mu)^2\right]\sqrt{(1+\mu)^2 - \mu s^2}\right\} \quad , \tag{E198}$$

which is (except the normalization factor) the same as the second kind solution (95) in the main text.

Next, the formula (E183) is tested for $n=3$ (set so in order to derive the function of the fourth degree):

$$T_{3+1}(s) = \frac{2\cdot 3+1}{3+1}\frac{1}{1+\mu} s\, T_3(s) - \frac{3}{3+1}\left(1 - \frac{\mu}{(1+\mu)^2} s^2\right) T_{3-1}(s) \quad , \tag{E199}$$

$$T_4(s) = \frac{7}{4}\frac{1}{1+\mu} s\, T_3(s) - \frac{3}{4}\left(1 - \frac{\mu}{(1+\mu)^2} s^2\right) T_2(s) \quad . \tag{E200}$$

By inserting the second (see (E187)) and the third (see (E196)) degree functions, we receive

$$T_4(s) = \frac{7}{4}\frac{1}{1+\mu} s \cdot \frac{1}{2}\frac{1}{(1+\mu)^3}\left[\frac{4\mu+15}{3} s^2 - \frac{4}{3}(1+\mu)^2\right] - \frac{3}{4}\left(1 - \frac{\mu}{(1+\mu)^2} s^2\right)\frac{3}{2}\frac{1}{(1+\mu)^2} s \quad . \tag{E201}$$

and after expanding and simplifications we get

$$T_4(s) = \left[\frac{7}{4}\frac{1}{2}\frac{1}{(1+\mu)^4}\frac{4\mu+15}{3} s^3 - \frac{7}{4}\frac{1}{2}\frac{1}{(1+\mu)^4}\frac{4}{3}(1+\mu)^2 s\right] - \left(\frac{3}{4}\frac{3}{2}\frac{1}{(1+\mu)^2} s - \frac{3}{4}\frac{\mu}{(1+\mu)^2}\frac{3}{2}\frac{1}{(1+\mu)^2} s^3\right) , \tag{E202}$$

$$T_4(s) = \left[\frac{7}{4}\frac{1}{2}\frac{1}{(1+\mu)^4}\frac{4\mu+15}{3} s^3 + \frac{3}{4}\frac{3\mu}{2}\frac{1}{(1+\mu)^4} s^3\right] - \left(\frac{7}{4}\frac{1}{2}\frac{1}{(1+\mu)^2}\frac{4}{3} s + \frac{3}{4}\frac{3}{2}\frac{1}{(1+\mu)^2} s\right) \quad , \tag{E203}$$

$$T_4(s) = \frac{1}{2}\frac{1}{(1+\mu)^4}\left[\frac{7}{4}\frac{4\mu+15}{3} + \frac{3}{4}\frac{3\mu}{1}\right] s^3 - \frac{1}{2}\frac{1}{(1+\mu)^2}\left(\frac{7}{4}\frac{4}{3} + \frac{3}{4}\frac{3}{1}\right) s \quad , \tag{E204}$$

$$T_4(s) = \frac{1}{2}\frac{1}{(1+\mu)^4}\frac{1}{4}\left[\frac{7}{1}\frac{4\mu+15}{3} + \frac{3}{1}\frac{3\mu}{1}\right] s^3 - \frac{1}{2}\frac{1}{(1+\mu)^2}\frac{1}{4}\left(\frac{7}{1}\frac{4}{3} + \frac{3}{1}\frac{3}{1}\right) s \quad , \tag{E205}$$

$$T_4(s) = \frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{28\mu+105}{3} + \frac{27\mu}{3}\right] s^3 - \frac{1}{(1+\mu)^2}\frac{1}{2^3}\left(\frac{28}{3} + \frac{27}{3}\right) s \quad , \tag{E206}$$

$$T_4(s) = \frac{1}{(1+\mu)^4}\frac{1}{2^3}\frac{55\mu+105}{3} s^3 - \frac{1}{(1+\mu)^2}\frac{1}{2^3}\frac{55}{3} s \quad , \tag{E207}$$

$$T_4(s) = \frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{55\mu+105}{3} s^3 - (1+\mu)^2 \frac{55}{3} s\right] \quad . \tag{E208}$$

Therefore, according to (E181) and (E208), we have

$$Q_4^{SI}(s) = P_4^{SI}(s)\, Q_0^{SI}(s) - \frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{55\mu+105}{3} s^3 - (1+\mu)^2 \frac{55}{3} s\right]\sqrt{(1+\mu)^2 - \mu s^2} \quad , \tag{E209}$$

or equivalently

$$Q_4^{SI}(s)=\frac{1}{(1+\mu)^4}\cdot\frac{1}{2^3}\cdot\left[(3\mu^2+30\mu+35)s^4-2\cdot 3(\mu+5)(1+\mu)^2s^2+3(1+\mu)^4\right]Q_0^{SI}(s)-\frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{55\mu+105}{3}s^3-(1+\mu)^2\frac{55}{3}s\right]\sqrt{(1+\mu)^2-\mu s^2}=\frac{1}{(1+\mu)^4}\cdot\frac{1}{2^3}\cdot\left\{\left[(3\mu^2+30\mu+35)s^4-2\cdot 3(\mu+5)(1+\mu)^2s^2+3(1+\mu)^4\right]Q_0^{SI}(s)-\left[\frac{55\mu+105}{3}s^3-(1+\mu)^2\frac{55}{3}s\right]\sqrt{(1+\mu)^2-\mu s^2}\right\}\quad, \tag{E210}$$

which is newly derived generalized Legendre function of the second kind of the fourth degree, which is inserted into **Table 2**.

The fifth degree function can be determined using the formula (E183) when n=4 is set:

$$T_{4+1}(s)=\frac{2\cdot 4+1}{4+1}\frac{1}{1+\mu}s\,T_4(s)-\frac{4}{4+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_{4-1}(s)\quad, \tag{E211}$$

$$T_5(s)=\frac{9}{5}\frac{1}{1+\mu}s\,T_4(s)-\frac{4}{5}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_3(s)\quad. \tag{E212}$$

By inserting the third (see (E196)) and the fourth (see (E208)) degree functions, we receive

$$T_5(s)=\frac{9}{5}\frac{1}{1+\mu}s\frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{55\mu+105}{3}s^3-(1+\mu)^2\frac{55}{3}s\right]-\frac{4}{5}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)\frac{1}{2}\frac{1}{(1+\mu)^3}\left[\frac{4\mu+15}{3}s^2-\frac{4}{3}(1+\mu)^2\right]\quad, \tag{E213}$$

and – after expanding and simplifications – we get

$$T_5(s)=\frac{9}{5}\frac{1}{(1+\mu)^5}\frac{1}{2^3}\left[\frac{55\mu+105}{3}s^4-(1+\mu)^2\frac{55}{3}s^2\right]-\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\left[\frac{4\mu+15}{3}s^2-\frac{4}{3}(1+\mu)^2-\frac{\mu}{(1+\mu)^2}s^2\frac{4\mu+15}{3}s^2+\frac{\mu}{(1+\mu)^2}s^2\frac{4}{3}(1+\mu)^2\right]\quad, \tag{E214}$$

$$T_5(s)=\left[\frac{9}{5}\frac{1}{(1+\mu)^5}\frac{1}{2^3}\frac{55\mu+105}{3}s^4-\frac{9}{5}\frac{1}{(1+\mu)^5}\frac{1}{2^3}(1+\mu)^2\frac{55}{3}s^2\right]-\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\left[-\frac{\mu}{(1+\mu)^2}\frac{4\mu+15}{3}s^4+\frac{4\mu+15}{3}s^2+\frac{4\mu}{3}s^2-\frac{4}{3}(1+\mu)^2\right]\quad, \tag{E215}$$

$$T_5(s)=\left[\frac{9}{5}\frac{1}{(1+\mu)^5}\frac{1}{2^3}\frac{55\mu+105}{3}s^4-\frac{9}{5}\frac{1}{(1+\mu)^3}\frac{1}{2^3}\frac{55}{3}s^2\right]-\left[-\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{\mu}{(1+\mu)^2}\frac{4\mu+15}{3}s^4+\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{8\mu+15}{3}s^2-\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{4}{3}(1+\mu)^2\right]\quad, \tag{E216}$$

$$T_5(s)=\left[\frac{9}{5}\frac{1}{(1+\mu)^5}\frac{1}{2^3}\frac{55\mu+105}{3}s^4+\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{\mu}{(1+\mu)^2}\frac{4\mu+15}{3}s^4\right]-\left[\frac{9}{5}\frac{1}{(1+\mu)^3}\frac{1}{2^3}\frac{55}{3}s^2+\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{8\mu+15}{3}s^2-\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{4}{3}(1+\mu)^2\right]\quad, \tag{E217}$$

$$T_5(s)=\frac{1}{(1+\mu)^5}\frac{1}{2}\frac{1}{5}\frac{1}{3}\left[\frac{9}{1}\frac{1}{2^2}\frac{55\mu+105}{1}+\frac{4\mu}{1}\frac{4\mu+15}{1}\right]s^4-\frac{1}{(1+\mu)^3}\frac{1}{2}\frac{1}{5}\frac{1}{3}\left[\frac{9}{1}\frac{1}{2^2}\frac{55}{1}+\frac{4}{1}\frac{8\mu+15}{1}\right]s^2+\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{4}{3}(1+\mu)^2\quad, \tag{E218}$$

$$T_5(s)=\frac{1}{(1+\mu)^5}\frac{1}{2}\frac{1}{5}\frac{1}{3}\left[\frac{1}{2^2}\frac{9\cdot 55\mu+9\cdot 105}{1}+\frac{16\mu}{4}\frac{4\mu+15}{1}\right]s^4-\frac{1}{(1+\mu)^3}\frac{1}{2}\frac{1}{5}\frac{1}{3}\left[\frac{1}{2^2}\frac{9\cdot 55}{1}+\frac{16}{4}\frac{8\mu+15}{1}\right]s^2+\frac{4}{5}\frac{1}{2}\frac{1}{(1+\mu)^3}\frac{4}{3}(1+\mu)^2\quad, \tag{E219}$$

$$T_5(s)=\frac{1}{(1+\mu)^5}\frac{1}{2}\frac{1}{5}\frac{1}{3}\left[\frac{495\mu+945}{4}+\frac{64\mu^2+240\mu}{4}\right]s^4-\frac{1}{(1+\mu)^3}\frac{1}{2}\frac{1}{5}\frac{1}{3}\left[\frac{495}{4}+\frac{128\mu+240}{4}\right]s^2+\frac{4}{5}\frac{4}{2^3}\frac{1}{(1+\mu)^3}\frac{4}{3}(1+\mu)^2\quad, \tag{E220}$$

$$T_5(s)=\frac{1}{(1+\mu)^5}\frac{1}{2}\frac{1}{5}\frac{1}{3}\frac{64\mu^2+735\mu+945}{4}s^4-\frac{1}{(1+\mu)^3}\frac{1}{2}\frac{1}{5}\frac{1}{3}\frac{128\mu+735}{4}s^2+\frac{4}{5}\frac{4}{2^3}\frac{1}{(1+\mu)^3}\frac{4}{3}(1+\mu)^2\quad, \tag{E221}$$

$$T_5(s)=\frac{1}{(1+\mu)^5}\frac{1}{2^3}\frac{64\mu^2+735\mu+945}{15}s^4-\frac{1}{(1+\mu)^5}\frac{1}{2^3}\frac{128\mu+735}{15}(1+\mu)^2s^2+\frac{64}{15}\frac{1}{2^3}\frac{1}{(1+\mu)^5}(1+\mu)^4\quad, \tag{E222}$$

$$T_5(s)=\frac{1}{(1+\mu)^5}\frac{1}{2^3}\left[\left(\frac{64}{15}\mu^2+49\mu+63\right)s^4-\left(\frac{128}{15}\mu+49\right)(1+\mu)^2s^2+\frac{64}{15}(1+\mu)^4\right]\quad. \tag{E223}$$

Therefore, according to (E181) and (E223), we have

$$Q_5^{SI}(s) = P_5^{SI}(s)\,Q_0^{SI}(s) - \frac{1}{(1+\mu)^5}\frac{1}{2^3}\left[\left(\frac{64}{15}\mu^2 + 49\mu + 63\right)s^4 - \left(\frac{128}{15}\mu + 49\right)(1+\mu)^2 s^2 + \frac{64}{15}(1+\mu)^4\right]\sqrt{(1+\mu)^2 - \mu s^2} \quad , \qquad \text{(E224)}$$

or equivalently

$$Q_5^{SI}(s) = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2^3}\cdot[(15\mu^2+70\mu+63)s^5 - 10(3\mu+7)(1+\mu)^2 s^3 + 15(1+\mu)^4 s^1]\,Q_0^{SI}(s) - \frac{1}{(1+\mu)^5}\frac{1}{2^3}\left[\left(\frac{64}{15}\mu^2+49\mu+63\right)s^4 - \left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^2 + \frac{64}{15}(1+\mu)^4\right]\sqrt{(1+\mu)^2-\mu s^2} = \frac{1}{(1+\mu)^5}\cdot\frac{1}{2^3}\cdot\left\{[(15\mu^2+70\mu+63)s^5 - 10(3\mu+7)(1+\mu)^2 s^3 + 15(1+\mu)^4 s^1]\,Q_0^{SI}(s) - \left[\left(\frac{64}{15}\mu^2+49\mu+63\right)s^4 - \left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^2 + \frac{64}{15}(1+\mu)^4\right]\sqrt{(1+\mu)^2-\mu s^2}\right\} \quad , \qquad \text{(E225)}$$

which is newly derived generalized Legendre function of the second kind of the fifth degree, which is inserted into **Table 2**.

Finally, the sixth degree function can be determined using the formula (E183) when *n*=5 is set:

$$T_{5+1}(s) = \frac{2\cdot5+1}{5+1}\frac{1}{1+\mu}s\,T_5(s) - \frac{5}{5+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_{5-1}(s) \quad , \qquad \text{(E226)}$$

$$T_6(s) = \frac{11}{6}\frac{1}{1+\mu}s\,T_5(s) - \frac{5}{6}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)T_4(s) \ . \qquad \text{(E227)}$$

By inserting the fourth (see (E208)) and the fifth (see (E223)) degree functions, we receive

$$T_6(s) = \frac{11}{6}\frac{1}{1+\mu}s\frac{1}{(1+\mu)^5}\frac{1}{2^3}\left[\left(\frac{64}{15}\mu^2+49\mu+63\right)s^4 - \left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^2 + \frac{64}{15}(1+\mu)^4\right] - \frac{5}{6}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)\frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{55\mu+105}{3}s^3 - (1+\mu)^2\frac{55}{3}s\right] \quad , \qquad \text{(E228)}$$

and after expanding and simplifications we get

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left[\frac{11}{6}\left(\frac{64}{15}\mu^2+49\mu+63\right)s^5 - \frac{11}{6}\left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^3 + \frac{11}{6}\frac{64}{15}(1+\mu)^4 s\right] - \frac{5}{6}\frac{1}{(1+\mu)^4}\frac{1}{2^3}\left[\frac{55\mu+105}{3}s^3 - (1+\mu)^2\frac{55}{3}s - \frac{\mu}{(1+\mu)^2}\frac{55\mu+105}{3}s^5 + (1+\mu)^2\frac{55}{3}\frac{\mu}{(1+\mu)^2}s^3\right] \quad , \qquad \text{(E229)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left\{\left[\frac{11}{6}\left(\frac{64}{15}\mu^2+49\mu+63\right)s^5 - \frac{11}{6}\left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^3 + \frac{11}{6}\frac{64}{15}(1+\mu)^4 s\right] - \frac{5}{6}\left[\frac{55\mu+105}{3}(1+\mu)^2 s^3 - (1+\mu)^4\frac{55}{3}s - \mu\frac{55\mu+105}{3}s^5 + (1+\mu)^2\frac{55\mu}{3}s^3\right]\right\} \quad , \qquad \text{(E230)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left\{\left[\frac{11}{6}\left(\frac{64}{15}\mu^2+49\mu+63\right)s^5 - \frac{11}{6}\left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^3 + \frac{11}{6}\frac{64}{15}(1+\mu)^4 s\right] - \left[\frac{5}{6}\frac{110\mu+105}{3}(1+\mu)^2 s^3 - \frac{5}{6}(1+\mu)^4\frac{55}{3}s - \frac{5}{6}\mu\frac{55\mu+105}{3}s^5\right]\right\} \quad , \qquad \text{(E231)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left\{\left[\frac{11}{6}\left(\frac{64}{15}\mu^2+49\mu+63\right)s^5 + \frac{5}{6}\mu\frac{55\mu+105}{3}s^5\right] - \left[\frac{11}{6}\left(\frac{128}{15}\mu+49\right)(1+\mu)^2 s^3 + \frac{5}{6}\frac{110\mu+105}{3}(1+\mu)^2 s^3\right] + \left[\frac{11}{6}\frac{64}{15}(1+\mu)^4 s + \frac{5}{6}(1+\mu)^4\frac{55}{3}s\right]\right\} \ . \qquad \text{(E232)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left\{\frac{1}{6}\left[\frac{11}{1}\left(\frac{64\mu^2+735\mu+945}{15}\right) + \frac{5\cdot5\mu}{5}\frac{55\mu+105}{3}\right]s^5 - \left[\frac{11}{6}\left(\frac{128\mu+735}{15}\right) + \frac{5\cdot5}{6\cdot5}\frac{110\mu+105}{3}\right](1+\mu)^2 s^3 + \left[\frac{11}{6}\frac{64}{15} + \frac{5\cdot5}{6\cdot5}\frac{55}{3}\right](1+\mu)^4 s\right\} \quad , \qquad \text{(E233)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left\{\frac{1}{6}\frac{1}{3}\left[\frac{704\mu^2+8085\mu+10395}{5} + \frac{1375\mu\mu+2625\mu}{5}\right]s^5 - \frac{1}{3}\left[\frac{1408\mu+8085}{6\cdot5} + \frac{2750\mu+2625}{6\cdot5}\right](1+\mu)^2 s^3 + \frac{1}{6}\left[\frac{704}{15} + \frac{1375}{15}\right](1+\mu)^4 s\right\} \quad , \qquad \text{(E234)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^3}\left\{\frac{1}{6}\frac{1}{3}\left[\frac{2079\mu^2+10710\mu+10395}{5}\right]s^5 - \frac{1}{3}\left[\frac{4158\mu+10710}{6\cdot5}\right](1+\mu)^2 s^3 + \frac{1}{6}\left[\frac{2079}{15}\right](1+\mu)^4 s\right\} \quad , \qquad \text{(E235)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^4}\left\{\frac{1}{9}\left[\frac{2079}{5}\mu^2 + 2142\mu + 2079\right]s^5 - \frac{1}{3}\left[\frac{1386}{5}\mu + 714\right](1+\mu)^2 s^3 + \frac{693}{15}(1+\mu)^4 s\right\} \quad , \qquad \text{(E236)}$$

$$T_6(s) = \frac{1}{(1+\mu)^6}\frac{1}{2^4}\left\{\left[\frac{231}{5}\mu^2 + 238\mu + 231\right]s^5 - \left[\frac{462}{5}\mu + 238\right](1+\mu)^2 s^3 + \frac{231}{5}(1+\mu)^4 s\right\} \quad . \tag{E237}$$

Therefore, according to (E181) and (E237), we have

$$Q_6^{SI}(s) = P_6^{SI}(s)\,Q_0^{SI}(s) - \frac{1}{(1+\mu)^6}\frac{1}{2^4}\left[\left(\frac{231}{5}\mu^2 + 238\mu + 231\right)s^5 - \left(\frac{462}{5}\mu + 238\right)(1+\mu)^2 s^3 + \frac{231}{5}(1+\mu)^4 s\right]\sqrt{(1+\mu)^2 - \mu s^2} \quad , \tag{E238}$$

or equivalently

$$Q_6^{SI}(s) = \frac{1}{(1+\mu)^6}\cdot\frac{1}{2^4}\cdot[(5\mu^3 + 105\mu^2 + 315\mu + 231)s^6 - 5(3\mu^2 + 42\mu + 63)(1+\mu)^2 s^4 + 3\cdot 5(\mu+7)(1+\mu)^4 s^2 - 5(1+\mu)^6]\,Q_0^{SI}(s) - \frac{1}{(1+\mu)^6}\frac{1}{2^4}\left[\left(\frac{231}{5}\mu^2 + 238\mu + 231\right)s^5 - \left(\frac{462}{5}\mu + 238\right)(1+\mu)^2 s^3 + \frac{231}{5}(1+\mu)^4 s\right]\sqrt{(1+\mu)^2 - \mu s^2} = \frac{1}{(1+\mu)^6}\frac{1}{2^4}\cdot\left\{[(5\mu^3 + 105\mu^2 + 315\mu + 231)s^6 - 5(3\mu^2 + 42\mu + 63)(1+\mu)^2 s^4 + 3\cdot 5(\mu+7)(1+\mu)^4 s^2 - 5(1+\mu)^6]\,Q_0^{SI}(s) - \left[\left(\frac{231}{5}\mu^2 + 238\mu + 231\right)s^5 - \left(\frac{462}{5}\mu + 238\right)(1+\mu)^2 s^3 + \frac{231}{5}(1+\mu)^4 s\right]\sqrt{(1+\mu)^2 - \mu s^2}\right\} \quad , \tag{E239}$$

which is newly derived generalized Legendre function of the second kind of the sixth degree, which is inserted also into **Table 2**.