Supplement B

to

Interior solution of azimuthally symmetric case of Laplace equation in orthogonal Similar Oblate Spheroidal coordinates

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Content:

B.1. Procedure leading to variables separation in Laplace equation in SOS coordinates

The Laplace equation in SOS coordinates in azimuthally symmetric case is

$$\frac{\partial}{\partial R}\left(\frac{\mathfrak{J}}{h_R^2}\frac{\partial V(R,\nu)}{\partial R}\right)+\frac{\partial}{\partial \nu}\left(\frac{\mathfrak{J}}{h_\nu^2}\frac{\partial V(R,\nu)}{\partial \nu}\right)=0 \quad , \tag{B1}$$

where $\mathfrak{J}=h_R h_\nu h_\lambda$ denotes the Jacobian.

When we insert the model potential (23)

$$V_2(R,\nu)=r(R)F(W)+V_0 \quad , \tag{B2}$$

into the Laplace equation, we obtain

$$\frac{\partial}{\partial R}\left(\frac{\mathfrak{J}}{h_R^2}\frac{\partial}{\partial R}(r(R)F(W)+V_0)\right)+\frac{\partial}{\partial \nu}\left(\frac{\mathfrak{J}}{h_\nu^2}\frac{\partial}{\partial \nu}(r(R)F(W)+V_0)\right)=0 \quad . \tag{B3}$$

$$\frac{\partial}{\partial R}\left(\frac{\mathfrak{J}}{h_R^2}\left[\frac{\partial r(R)}{\partial R}F(W)+r(R)\frac{\partial F(W)}{\partial R}\right]\right)+\frac{\partial}{\partial \nu}\left(\frac{\mathfrak{J}}{h_\nu^2}\left[\frac{\partial r(R)}{\partial \nu}F(W)+r(R)\frac{\partial F(W)}{\partial \nu}\right]\right)=0 \quad . \tag{B4}$$

$$\frac{\partial}{\partial R}\left(\frac{\mathfrak{J}}{h_R^2}\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\partial W}{\partial R}\right]\right)+\frac{\partial}{\partial \nu}\left(\frac{\mathfrak{J}}{h_\nu^2}\left[0+r(R)\frac{dF(W)}{dW}\frac{\partial W}{\partial \nu}\right]\right)=0 \quad . \tag{B5}$$

In order to make the factors with the Jacobian divided by the square of the metric scale factor (see expressions (12) and (13)) dependent only on R and on the composed variable W, not explicitly on ν, we either divide or multiply them by the derivative of W with respect to ν:

$$\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\partial W}{\partial R}\right]\right)+r(R)\frac{\partial}{\partial \nu}\left(\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\left[\frac{dF(W)}{dW}\right]\right)=0 \quad . \qquad \text{(B6)}$$

Then the product rule for derivatives is used:

$$\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial \nu}\right)\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\partial W}{\partial R}\right]+\frac{\partial W}{\partial \nu}\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\partial W}{\partial R}\right]+$$
$$\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\frac{\partial}{\partial R}\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\partial W}{\partial R}\right]+r(R)\frac{\partial}{\partial \nu}\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\left[\frac{dF(W)}{dW}\right]+r(R)\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\frac{\partial}{\partial \nu}\left[\frac{dF(W)}{dW}\right]=0 \quad .$$

(B7)

As $\frac{\partial W}{\partial R}=\frac{\mu W}{R}$, see (10), and as the order of partial derivatives can be inversed, we further obtain

$$\frac{\partial}{\partial \nu}\left(\frac{\mu W}{R}\right)\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+\frac{\partial W}{\partial \nu}\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+$$
$$\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\frac{\partial}{\partial R}\left\{\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]\right\}+r(R)\frac{\partial}{\partial \nu}\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\left[\frac{dF(W)}{dW}\right]+r(R)\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\frac{\partial}{\partial \nu}\left[\frac{dF(W)}{dW}\right]=0$$

. (B8)

As $\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)$ depend only on the composed parameter W (deduce it easily from (13)), not separately on ν, we can use the chain rule and write (when simultaneously using product rule in some other terms)

$$\frac{\mu}{R}\left(\frac{\Im}{h_R^2}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+\frac{\partial W}{\partial \nu}\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+$$
$$\left(\frac{\Im}{h_R^2}\right)\left\{\left[\frac{d^2r(R)}{dR^2}F(W)+\frac{dr(R)}{dR}\frac{\partial F(W)}{\partial R}+\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\mu W}{R}+r(R)\frac{d^2F(W)}{dW^2}\frac{\partial W}{\partial R}\frac{\mu W}{R}+r(R)\frac{dF(W)}{dW}\frac{\partial(\mu W)}{\partial R}\frac{1}{R}+\right.\right.$$
$$\left.\left.r(R)\frac{dF(W)}{dW}\mu W\frac{\partial(R^{-1})}{\partial R}\right]\right\}+r(R)\frac{\partial}{\partial W}\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\frac{\partial W}{\partial \nu}\left[\frac{dF(W)}{dW}\right]+r(R)\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\frac{\partial}{\partial \nu}\left[\frac{dF(W)}{dW}\right]=0 \quad . \qquad \text{(B9)}$$

Now, we can use the following identities (A96) and (A97) from **Appendix A**:

$$\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)=\frac{\Im}{R\frac{\partial W}{\partial \nu}}\left[(\mu+3)-\frac{1+\mu}{h_R^2}\right] \qquad \text{(B10)}$$

and

$$\frac{\partial}{\partial W}\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)=\frac{(1+\mu)}{\mu W}(1-h_R^2)\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right) \quad . \qquad \text{(B11)}$$

Then

$$\frac{\mu}{R}\left(\frac{\Im}{h_R^2}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+\frac{\partial W}{\partial \nu}\frac{\Im}{R\frac{\partial W}{\partial \nu}}\left[(\mu+3)-\frac{1+\mu}{h_R^2}\right]\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+$$
$$\left(\frac{\Im}{h_R^2}\right)\left\{\left[\frac{d^2r(R)}{dR^2}F(W)+\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\partial W}{\partial R}+\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\mu W}{R}+r(R)\frac{d^2F(W)}{dW^2}\frac{\partial W}{\partial R}\frac{\mu W}{R}+r(R)\frac{dF(W)}{dW}\frac{\partial(\mu W)}{\partial R}\frac{1}{R}+\right.\right.$$
$$\left.\left.r(R)\frac{dF(W)}{dW}\mu W\frac{d(R^{-1})}{dR}\right]\right\}+r(R)\frac{(1+\mu)}{\mu W}(1-h_R^2)\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\frac{\partial W}{\partial \nu}\left[\frac{dF(W)}{dW}\right]+r(R)\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\left[\frac{d^2F(W)}{dW^2}\right]\frac{\partial W}{\partial \nu}=0 \quad ,$$

(B12)

The Jacobian is canceled as it is present in all terms, and other simple algebraic changes are applied:

$$\frac{\mu}{R}\left(\frac{1}{h_R^2}\right)\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+\frac{1}{R}\left[(\mu+3)-\frac{1+\mu}{h_R^2}\right]\left[\frac{dr(R)}{dR}F(W)+r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right]+$$
$$\left(\frac{1}{h_R^2}\right)\left\{\left[\frac{d^2r(R)}{dR^2}F(W)+\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\mu W}{R}+\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\mu W}{R}+r(R)\frac{d^2F(W)}{dW^2}\frac{\mu W}{R}\frac{\mu W}{R}+r(R)\frac{dF(W)}{dW}\frac{\mu\mu W}{R}\frac{1}{R}-\right.\right.$$

$$\frac{1}{R^2} r(R)\frac{dF(W)}{dW}\mu W\Big]\Big\} + r(R)\frac{(1+\mu)}{\mu W}(1-h_R^2)\left(\frac{1}{h_\nu^2}\right)\left(\frac{\partial W}{\partial \nu}\right)^2\left[\frac{dF(W)}{dW}\right] + r(R)\left(\frac{1}{h_\nu^2}\right)\left(\frac{\partial W}{\partial\nu}\right)^2\left[\frac{d^2F(W)}{dW^2}\right]=0 \quad , \tag{B13}$$

The use of (A38) relation from **Appendix A** rewritten to the form

$$\frac{1}{h_\nu^2}\left(\frac{\partial W}{\partial \nu}\right)^2 = \frac{W^2}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2 f_S^2} \tag{B14}$$

(f_C and f_S being the generalized cosine and sine) for the terms in the equation containing h_ν, and expanding of the other terms leads to

$$\left[\frac{\mu}{R}\frac{1}{h_R^2}\frac{dr(R)}{dR}F(W) + \frac{\mu}{R}\frac{1}{h_R^2}r(R)\frac{dF(W)}{dW}\frac{\mu W}{R}\right] + \left[(\mu+3) - \frac{1+\mu}{h_R^2}\right]\left[\frac{1}{R}\frac{dr(R)}{dR}F(W) + r(R)\frac{dF(W)}{dW}\frac{\mu W}{R^2}\right] +$$
$$\left[\frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \frac{1}{h_R^2}\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\mu W}{R} + \frac{1}{h_R^2}\frac{dr(R)}{dR}\frac{dF(W)}{dW}\frac{\mu W}{R} + \frac{1}{h_R^2}r(R)\frac{d^2F(W)}{dW^2}\frac{\mu^2W^2}{R^2} + \frac{1}{h_R^2}r(R)\frac{dF(W)}{dW}\frac{\mu^2 W}{R^2} -\right.$$
$$\left.\frac{1}{h_R^2}\frac{1}{R^2}r(R)\frac{dF(W)}{dW}\mu W\right] + r(R)\frac{(1+\mu)}{\mu W}(1-h_R^2)\frac{W^2}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\left[\frac{dF(W)}{dW}\right] + r(R)\frac{W^2}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\left[\frac{d^2F(W)}{dW^2}\right] = 0 \quad . \tag{B15}$$

By putting together the same-derivative-order terms, we get

$$\frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \frac{\mu}{R}\frac{1}{h_R^2}\frac{dr(R)}{dR}F(W) + \left[(\mu+3) - \frac{1+\mu}{h_R^2}\right]\frac{1}{R}\frac{dr(R)}{dR}F(W) + \frac{\mu}{R}\frac{1}{h_R^2}r(R)\frac{dF(W)}{dW}\frac{\mu W}{R} +$$
$$\left[(\mu+3) - \frac{1+\mu}{h_R^2}\right]r(R)\frac{dF(W)}{dW}\frac{\mu W}{R^2} + \frac{1}{h_R^2}r(R)\frac{dF(W)}{dW}\frac{\mu^2W}{R^2} - \frac{1}{h_R^2}\frac{1}{R^2}r(R)\frac{dF(W)}{dW}\mu W + r(R)\frac{(1+\mu)}{\mu W}(1-$$
$$h_R^2)\frac{W^2}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\left[\frac{dF(W)}{dW}\right] + 2\frac{1}{R}\frac{dr(R)}{dR}\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{h_R^2}r(R)\frac{d^2F(W)}{dW^2}\frac{\mu^2W^2}{R^2} + r(R)\frac{W^2}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\left[\frac{d^2F(W)}{dW^2}\right] = 0 \quad . \tag{B16}$$

When we further simplify and use (17) from the main text, we get

$$\frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \left[(\mu+3) - \frac{1+\mu}{h_R^2} + \frac{\mu}{h_R^2}\right]\frac{1}{R}\frac{dr(R)}{dR}F(W) + \frac{1}{R^2}r(R)\frac{\mu^2W}{h_R^2}\frac{dF(W)}{dW} + \left[(\mu+3)r(R)\frac{dF(W)}{dW}\frac{\mu W}{R^2} -\right.$$
$$\left.\frac{1+\mu}{h_R^2}r(R)\frac{dF(W)}{dW}\frac{\mu W}{R^2}\right] + \frac{1}{R^2}r(R)\frac{\mu^2W}{h_R^2}\frac{dF(W)}{dW} - \frac{1}{R^2}r(R)\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{R^2}r(R)f_S^2W\frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\left[\frac{dF(W)}{dW}\right] +$$
$$2\frac{1}{R}\frac{dr(R)}{dR}\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{R^2}r(R)\frac{\mu^2W^2}{h_R^2}\frac{d^2F(W)}{dW^2} + \frac{1}{R^2}r(R)W^2\frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\left[\frac{d^2F(W)}{dW^2}\right] = 0 \quad , \tag{B17}$$

When we still further simplify, we get

$$\frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \left[(\mu+3) - \frac{1}{h_R^2}\right]\frac{1}{R}\frac{dr(R)}{dR}F(W) + \frac{1}{R^2}r(R)\frac{\mu^2W}{h_R^2}\frac{dF(W)}{dW} - \frac{1}{R^2}r(R)\frac{(1+\mu)\mu W}{h_R^2}\frac{dF(W)}{dW} +$$
$$\frac{1}{R^2}r(R)\frac{\mu^2W}{h_R^2}\frac{dF(W)}{dW} - \frac{1}{R^2}r(R)\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \left[\frac{1}{R^2}r(R)(\mu+3)\mu W\frac{dF(W)}{dW}\right] + \frac{1}{R^2}r(R)W\frac{h_R^2(1+\mu)^2}{f_C^2}\left[\frac{dF(W)}{dW}\right] +$$
$$2\frac{1}{R}\frac{dr(R)}{dR}\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{R^2}r(R)\left(\frac{\mu^2}{h_R^2} + \frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\right)W^2\frac{d^2F(W)}{dW^2} = 0 \quad , \tag{B18}$$

and thus

$$\frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \left[(\mu+3) - \frac{1}{h_R^2}\right]\frac{1}{R}\frac{dr(R)}{dR}F(W) + \frac{1}{R^2}r(R)\frac{(\mu-2)\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{R^2}r(R)\left[(\mu+3)\mu +\right.$$
$$\left.\frac{h_R^2(1+\mu)^2}{f_C^2}\right]W\frac{dF(W)}{dW} + 2\frac{1}{R}\frac{dr(R)}{dR}\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{R^2}r(R)\left(\frac{\mu^2}{h_R^2} + \frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\right)W^2\frac{d^2F(W)}{dW^2} = 0 \quad . \tag{B19}$$

Still one algebraic manipulation leads to

$$\frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \left[\frac{1}{R}\frac{dr(R)}{dR}(\mu+3)F(W) - \frac{1}{R}\frac{dr(R)}{dR}\frac{1}{h_R^2}F(W)\right] + \frac{1}{R^2}r(R)\left[\frac{(\mu-2)\mu}{h_R^2} + (\mu+3)\mu +\right.$$
$$\left.\frac{h_R^2(1+\mu)^2}{f_C^2}\right]W\frac{dF(W)}{dW} + 2\frac{1}{R}\frac{dr(R)}{dR}\frac{\mu W}{h_R^2}\frac{\partial F(W)}{\partial W} + \frac{1}{R^2}r(R)\left(\frac{\mu^2}{h_R^2} + \frac{h_R^2(1+\mu)^2}{f_C^2f_S^2}\right)W^2\frac{d^2F(W)}{dW^2} = 0 \quad . \tag{B20}$$

Now divide by $r(R)F(W)$ and multiply by R^2 and by the squared h_R:

$$\frac{R^2}{r(R)}\frac{d^2r(R)}{dR^2} + \frac{R}{r(R)}\frac{dr(R)}{dR}(\mu+3)h_R^2 - \frac{R}{r(R)}\frac{dr(R)}{dR} + \left[(\mu-2)\mu + (\mu+3)\mu h_R^2 + \frac{h_R^4(1+\mu)^2}{f_C^2}\right]\frac{W}{F(W)}\frac{dF(W)}{dW} +$$
$$2\frac{R}{r(R)}\frac{dr(R)}{dR}\mu\frac{W}{F(W)}\frac{dF(W)}{dW} + \left[\mu^2 + \frac{h_R^4(1+\mu)^2}{f_C^2f_S^2}\right]\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2} = 0 \quad . \tag{B21}$$

We reorganize still a bit the terms (and also change the highlighting of the terms in order to enhance various parts important in the present stage of derivation):

$$\left[(\mu-2)\mu+(\mu+3)\mu h_R^2+\frac{h_R^4(1+\mu)^2}{f_C^2}\right]\frac{W}{F(W)}\frac{dF(W)}{dW}+\left[\mu^2+\frac{h_R^4(1+\mu)^2}{f_C^2 f_S^2}\right]\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+\frac{R^2}{r(R)}\frac{d^2r(R)}{dR^2}-\frac{R}{r(R)}\frac{dr(R)}{dR}+\left[(\mu+3)h_R^2+2\mu\frac{W}{F(W)}\frac{dF(W)}{dW}\right]\frac{R}{r(R)}\frac{dr(R)}{dR}=0 \quad . \qquad \text{(B22)}$$

For the solution of the equation (B22), i.e. the general SOS case, we can apply a special variable separation approach. We see that the yellow terms (the first two) in (B22) depend only on *W*, while the blue ones (the middle ones) depend only on *R*. The gray-and-green term (the last one) is a product of a pure *W*-function and a pure *R*–function. The equation can be thus written in the following simplified form:

$$a(W)+b(R)+c(W)d(R)=0 \quad . \qquad \text{(B23)}$$

In order this equation being fulfilled for any *W* and for any *R*, either *b*(*R*) and *d*(*R*) has to be equal to constants for any *R*, or *a*(*W*) and *c*(*W*) has to be equal to constants for all *W*. Otherwise, different equations would arise when keeping $R=R_1$ and when keeping $R=R_2$, or when keeping $W=W_1$ and when keeping $W=W_2$. If e.g. $b(R_1)$ is not equal to $b(R_2)$, then equations

$$a(W)+b(R_1)+c(W)d(R_1)=0 \quad , \qquad \text{(B24)}$$

$$a(W)+b(R_2)+c(W)d(R_2)=0 \quad , \qquad \text{(B25)}$$

would be different equations for *a*(*W*) and *c*(*W*), and they could not be fulfilled at once (for any *W*). It cannot be compensated by different *d*(*R*). Similarly for *d*(*R*), and in other cases.

When the possibility with *a*(*W*) and *c*(*W*) being constants is tested, no solution of Laplace equation is found (although derivation of this finding is quite extensive). Therefore, the second option, i.e. *b*(*R*) and *d*(*R*) being equal to constants, is to be tested.

B.2. Test of *b*(*R*) equal to a constant, and *d*(*R*) equal to a constant.

When we test the second option for separation of (B22), i.e. *b*(*R*) and *d*(*R*) are constants, we arrive at the equations

$$d(R)=\frac{R}{r(R)}\frac{dr(R)}{dR}=K_d \quad , \qquad \text{(B26)}$$

$$b(R)=\frac{R^2}{r(R)}\frac{d^2r(R)}{dR^2}-\frac{R}{r(R)}\frac{dr(R)}{dR}=K_b \quad . \qquad \text{(B27)}$$

where K_d, K_b are separation constants. The first equation, which can be rewritten to the form

$$\frac{dr(R)}{dR}=K_d\frac{r(R)}{R} . \qquad \text{(B28)}$$

has solution

$$r(R)=C_dR^{K_d} , \qquad \text{(B29)}$$

C_d being an arbitrary constant. The term in the second equation (B27) containing the first derivative is thus

$$\frac{R}{r(R)}\frac{dr(R)}{dR}=\frac{R}{r(R)}K_d\frac{r(R)}{R}=K_d \quad . \qquad \text{(B30)}$$

The second derivative is then

$$\frac{d^2r(R)}{dR^2}=\frac{d}{dR}\left(K_d\frac{C_dR^{K_d}}{R}\right)=\frac{d}{dR}\left(K_dC_dR^{K_d-1}\right)=C_dK_d(K_d-1)R^{K_d-2} \qquad \text{(B31)}$$

and the term in the second equation (B27) containing the second derivative is thus

$$\frac{R^2}{r(R)}\frac{d^2r(R)}{dR^2} = \frac{R^2}{r(R)}C_dK_d(K_d-1)R^{K_d-2} = K_d(K_d-1)\frac{C_dR^{K_d}}{r(R)} = K_d(K_d-1)\frac{r(R)}{r(R)} = K_d(K_d-1) \quad . \tag{B32}$$

Therefore, we can write the second equation (B27) in the following form:

$$b(R) = K_d(K_d-1) - K_d = K_b \quad . \tag{B33}$$

This leads to the following equation connecting the separation constants K_b and K_d:

$$K_b = K_d[K_d-2] \quad . \tag{B34}$$

The remaining part of the Laplace equation (B22) can be thus written in the form

$$\left[(\mu-2)\mu+(\mu+3)\mu h_R^2+\frac{h_R^4(1+\mu)^2}{f_C^2}\right]\frac{W}{F(W)}\frac{dF(W)}{dW}+\left[\mu^2+\frac{h_R^4(1+\mu)^2}{f_C^2f_S^2}\right]\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+K_b+\left[(\mu+3)h_R^2+2\mu\frac{W}{F(W)}\frac{dF(W)}{dW}\right]K_d=0 \quad . \tag{B35}$$

where K_b and K_d are connected by the relation (B34). Then

$$\left[\mu^2+\frac{h_R^4(1+\mu)^2}{f_C^2f_S^2}\right]\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+\left[(\mu-2)\mu+(\mu+3)\mu h_R^2+\frac{h_R^4(1+\mu)^2}{f_C^2}+2K_d\mu\right]\frac{W}{F(W)}\frac{dF(W)}{dW}+K_d(\mu+3)h_R^2+K_d(K_d-2)=0 \quad . \tag{B36}$$

The strategy is now to modify algebraically (B36) to the form containing only h_R, not f_C or f_S. First, multiply the equation by f_Cf_S squared:

$$[\mu^2f_C^2f_S^2+h_R^4(1+\mu)^2]\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+K_d[(\mu-2)\mu f_C^2f_S^2+(\mu+3)\mu h_R^2f_C^2f_S^2+h_R^4(1+\mu)^2f_S^2+2K_d\mu f_C^2f_S^2]\frac{W}{F(W)}\frac{dF(W)}{dW}+(\mu+3)h_R^2f_C^2f_S^2+K_d[K_d-2]f_C^2f_S^2=0 \quad . \tag{B37}$$

Further, use the relations (17) from the main text and (A22) from **Appendix A** to express f_Cf_S squared and f_S squared in terms of h_R, i.e.

$$\left\{\mu^2\frac{1+\mu}{\mu^2}[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+h_R^4(1+\mu)^2\right\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+\left\{(\mu-2)\mu\frac{1+\mu}{\mu^2}[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+(\mu+3)\mu h_R^2\frac{1+\mu}{\mu^2}[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+h_R^4(1+\mu)^2\frac{1+\mu}{\mu}(1-h_R^2)+2K_d\mu\frac{1+\mu}{\mu^2}[(2+\mu)h_R^2-(1+\mu)h_R^4-1]\right\}\frac{W}{F(W)}\frac{dF(W)}{dW}+K_d(\mu+3)h_R^2\frac{1+\mu}{\mu^2}[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+K_d[K_d-2]\frac{1+\mu}{\mu^2}[(2+\mu)h_R^2-(1+\mu)h_R^4-1]=0 \quad . \tag{B38}$$

Multiplication by μ^2 leads to

$$\{(1+\mu)\mu^2[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+(1+\mu)^2\mu^2h_R^4\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+\{(\mu-2)(1+\mu)\mu[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+(1+\mu)(\mu+3)\mu h_R^2[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+(1+\mu)^2\mu h_R^4(1+\mu)(1-h_R^2)+2K_d(1+\mu)\mu[(2+\mu)h_R^2-(1+\mu)h_R^4-1]\}\frac{W}{F(W)}\frac{dF(W)}{dW}+K_d(1+\mu)(\mu+3)h_R^2[(2+\mu)h_R^2-(1+\mu)h_R^4-1]+K_d[K_d-2](1+\mu)[(2+\mu)h_R^2-(1+\mu)h_R^4-1]=0 \quad . \tag{B39}$$

We expand the terms

$$\{[(1+\mu)(2+\mu)\mu^2h_R^2-(1+\mu)^2\mu^2h_R^4-(1+\mu)\mu^2]+(1+\mu)^2\mu^2h_R^4\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2}+\{[(\mu-2)(1+\mu)(2+\mu)\mu h_R^2-(\mu-2)(1+\mu)^2\mu h_R^4-(\mu-2)(1+\mu)\mu]+[(1+\mu)(2+\mu)(\mu+3)\mu h_R^4-(1+\mu)^2(\mu+3)\mu h_R^6-(1+\mu)(\mu+3)\mu h_R^2]+((1+\mu)^3\mu h_R^4-(1+\mu)^3\mu h_R^6)+[2K_d(1+\mu)(2+\mu)\mu h_R^2-2K_d(1+\mu)^2\mu h_R^4-2K_d(1+\mu)\mu]\}\frac{W}{F(W)}\frac{dF(W)}{dW}+[K_d(1+\mu)(2+\mu)(\mu+3)h_R^4-K_d(1+\mu)^2(\mu+3)h_R^6-K_d(1+\mu)(\mu+3)h_R^2]+[K_d[K_d-2](1+\mu)(2+\mu)h_R^2-K_d[K_d-2](1+\mu)^2h_R^4-K_d[K_d-2](1+\mu)]=0 \tag{B40}$$

and mark the same-order (in h_R squared) terms with the same color

$$\{[(1+\mu)(2+\mu)\mu^2 h_R^2 - (1+\mu)^2\mu^2 h_R^4 - (1+\mu)\mu^2] + (1+\mu)^2\mu^2 h_R^4\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2} + \{[(\mu-2)(1+\mu)(2+\mu)\mu h_R^2 - (\mu-2)(1+\mu)^2\mu h_R^4 - (\mu-2)(1+\mu)\mu] + [(1+\mu)(2+\mu)(\mu+3)\mu h_R^4 - (1+\mu)^2(\mu+3)\mu h_R^6 - (1+\mu)(\mu+3)\mu h_R^2] + ((1+\mu)^3\mu h_R^4 - (1+\mu)^3\mu h_R^6) + [2K_d(1+\mu)(2+\mu)\mu h_R^2 - 2K_d(1+\mu)^2\mu h_R^4 - 2K_d(1+\mu)\mu]\}\frac{W}{F(W)}\frac{dF(W)}{dW} + [K_d(1+\mu)(2+\mu)(\mu+3)h_R^4 - K_d(1+\mu)^2(\mu+3)h_R^6 - K_d(1+\mu)(\mu+3)h_R^2] + [K_d[K_d-2](1+\mu)(2+\mu)h_R^2 - K_d[K_d-2](1+\mu)^2 h_R^4 - K_d[K_d-2](1+\mu)] = 0 \ . \tag{B41}$$

Put together the same-order terms within the pre-factors of derivatives and also the remaining terms:

$$\{(1+\mu)(2+\mu)\mu^2 h_R^2 - (1+\mu)^2\mu^2 h_R^4 + (1+\mu)^2\mu^2 h_R^4 - (1+\mu)\mu^2\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2} + \{[(\mu-2)(1+\mu)(2+\mu)\mu h_R^2 - (1+\mu)(\mu+3)\mu h_R^2 + 2K_d(1+\mu)(2+\mu)\mu h_R^2] + [-(\mu-2)(1+\mu)^2\mu h_R^4 + (1+\mu)(2+\mu)(\mu+3)\mu h_R^4 + (1+\mu)^3\mu h_R^4 - 2K_d(1+\mu)^2\mu h_R^4] + (-(1+\mu)^2(\mu+3)\mu h_R^6 - (1+\mu)^3\mu h_R^6) + [-(\mu-2)(1+\mu)\mu - 2K_d(1+\mu)\mu]\}\frac{W}{F(W)}\frac{dF(W)}{dW} + [-K_d(1+\mu)(\mu+3)h_R^2 + K_d[K_d-2](1+\mu)(2+\mu)h_R^2] + [K_d(1+\mu)(2+\mu)(\mu+3)h_R^4 - K_d[K_d-2](1+\mu)^2 h_R^4] - K_d(1+\mu)^2(\mu+3)h_R^6 - K_d[K_d-2](1+\mu) = 0 \ . \tag{B42}$$

Combine the same-order terms within the pre-factors of derivatives:

$$\{(1+\mu)(2+\mu)\mu^2 h_R^2 - \cancel{(1+\mu)^2\mu^2 h_R^4} + \cancel{(1+\mu)^2\mu^2 h_R^4} - (1+\mu)\mu^2\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2} + \{[(\mu-2)(2+\mu) - (\mu+3) + 2K_d(2+\mu)](1+\mu)\mu h_R^2 + [-(\mu-2)(1+\mu) + (2+\mu)(\mu+3) + (1+\mu)^2 - 2K_d(1+\mu)](1+\mu)\mu h_R^4 + (-(\mu+3) - (1+\mu))(1+\mu)^2\mu h_R^6 + [-(\mu-2) - 2K_d](1+\mu)\mu\}\frac{W}{F(W)}\frac{dF(W)}{dW} + [-(\mu+3) + [K_d-2](2+\mu)]K_d(1+\mu)h_R^2 + [(2+\mu)(\mu+3) - [K_d-2](1+\mu)]K_d(1+\mu)h_R^4 - K_d(1+\mu)^2(\mu+3)h_R^6 - K_d[K_d-2](1+\mu) = 0 \tag{B43}$$

and simplify them:

$$\{(1+\mu)(2+\mu)\mu^2 h_R^2 - (1+\mu)\mu^2\}\frac{W^2}{F(W)}\frac{d^2F(W)}{dW^2} + \{[(\mu\mu - \mu - 7) + 2K_d(2+\mu)](1+\mu)\mu h_R^2 + [(\mu\mu + 8\mu + 9) - 2K_d(1+\mu)](1+\mu)\mu h_R^4 + (-2\mu - 4)(1+\mu)^2\mu h_R^6 + [2 - \mu - 2K_d](1+\mu)\mu\}\frac{W}{F(W)}\frac{dF(W)}{dW} + [-3\mu - 7 + (2+\mu)K_d]K_d(1+\mu)h_R^2 + [(7\mu + 8 + \mu\mu) - (1+\mu)K_d]K_d(1+\mu)h_R^4 - K_d(1+\mu)^2(\mu+3)h_R^6 - K_d[K_d-2](1+\mu) = 0 \ . \tag{B44}$$

Still multiply the equation by $F(W)$, and divide by the factor which is present by the second derivative. We obtain

$$\frac{d^2F(W)}{dW^2} + \frac{\{[(\mu\mu-\mu-7)+2K_d(2+\mu)](1+\mu)\mu h_R^2 + [(\mu\mu+8\mu+9)-2K_d(1+\mu)](1+\mu)\mu h_R^4 + (-2\mu-4)(1+\mu)^2\mu h_R^6 + [2-\mu-2K_d](1+\mu)\mu\}}{\{(1+\mu)(2+\mu)\mu^2 h_R^2 - (1+\mu)\mu^2\}W}\frac{dF(W)}{dW} + \frac{\{[-3\mu-7+(2+\mu)K_d]K_d(1+\mu)h_R^2 + [(7\mu+8+\mu\mu)-(1+\mu)K_d]K_d(1+\mu)h_R^4 - K_d(1+\mu)^2(\mu+3)h_R^6 - K_d[K_d-2](1+\mu)\}}{\{(1+\mu)(2+\mu)\mu^2 h_R^2 - (1+\mu)\mu^2\}W^2}F(W) = 0 \ . \tag{B45}$$

In the factor by the first derivative, $(1+\mu)\mu$ is mutually cancelled in the numerator and in the denominator. In the factor by *F*(*W*), $(1+\mu)$ is mutually cancelled in the numerator and the denominator. Then the equation takes the form

$$\frac{d^2F(W)}{dW^2} + \frac{1}{\mu W}\frac{\{[(\mu\mu-\mu-7)+2K_d(2+\mu)]h_R^2 + [(\mu\mu+8\mu+9)-2K_d(1+\mu)]h_R^4 + (-2\mu-4)(1+\mu)h_R^6 + [2-\mu-2K_d]\}}{\{(2+\mu)h_R^2 - 1\}}\frac{dF(W)}{dW} + \frac{K_d}{\mu^2 W^2}\frac{\{[-3\mu-7+(2+\mu)K_d]h_R^2 + [(7\mu+8+\mu\mu)-(1+\mu)K_d]h_R^4 - (1+\mu)(\mu+3)h_R^6 - [K_d-2]\}}{\{(2+\mu)h_R^2 - 1\}}F(W) = 0 \ . \tag{B46}$$

B.3. Riccati equation solution.

The equation (B46) can be possibly written also with a help of (unknown) function $g(W)$ as follows (see Polyanin and Zaitsev 2003, Eq. 46):

$$\frac{d^2F(W)}{dW^2}+f(W)\frac{dF(W)}{dW}+\left[f(W)g(W)-[g(W)]^2+\frac{dg(W)}{dW}\right]F(W)=0. \quad \text{(B47)}$$

where $f(W)$ is known (it is the factor by the first derivative in (B46)),

$$f(W)=\frac{1}{\mu}\frac{\{[(\mu\mu-\mu-7)+2K_d(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]h_R^4+(-2\mu-4)(1+\mu)h_R^6+[2-\mu-2K_d]\}}{\{(2+\mu)h_R^2-1\}W} \quad . \quad \text{(B48)}$$

Then, the factor (function) by $F(W)$ in (B46) can be possibly written with a help of (unknown) $g(W)$ function. The factor (function) by $F(W)$ would also contain the function $f(W)$ appearing by the first derivative of $F(W)$. By comparison of (B46) and (B47), we obtain the equation

$$\frac{K_d}{\mu^2}\frac{\{[-3\mu-7+(2+\mu)K_d]h_R^2+[(7\mu+8+\mu\mu)-(1+\mu)K_d]h_R^4-(1+\mu)(\mu+3)h_R^6-[K_d-2]\}}{\{(2+\mu)h_R^2-1\}W^2}=\left[f(W)g(W)-[g(W)]^2+\frac{dg(W)}{dW}\right]=+\frac{1}{\mu}\frac{\{[(\mu\mu-\mu-7)+2K_d(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]h_R^4+(-2\mu-4)(1+\mu)h_R^6+[2-\mu-2K_d]\}}{\{(2+\mu)h_R^2-1\}W}g(W)-[g(W)]^2+\frac{dg(W)}{dW} \quad . \quad \text{(B49)}$$

If such $g(W)$ could be found which solves (B49), i.e. the first order nonlinear differential equation, then an exact solution of the partial Laplace equation (B46), which is the second order differential equation, would be possible to find as well. However, the above first order nonlinear differential equation (B49) for $g(W)$, which is the Ricatti general equation, has to be solved first for $g(W)$.

Note, that writing the Laplace equation (the separated part of it depending only on W, to be precise) in the form of (B47) equation is possibly not the only option how to find its solution. May be other solutions could be found in cases when the equation cannot be written in the form of (B47). Nevertheless, the (B47) form could probably lead to one of the simplest solutions. The search for another/others could be simplified when at least one solution is known.

Let's rewrite (B49) to a standard form of Ricatti equation:

$$\frac{dg(W)}{dW}=[g(W)]^2-\frac{1}{\mu}\frac{\{[(\mu\mu-\mu-7)+2K_d(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2K_d(1+\mu)]h_R^4+(-2\mu-4)(1+\mu)h_R^6+[2-\mu-2K_d]\}}{\{(2+\mu)h_R^2-1\}W}g(W)+\frac{K_d}{\mu^2}\frac{\{[-3\mu-7+(2+\mu)K_d]h_R^2+[(7\mu+8+\mu\mu)-(1+\mu)K_d]h_R^4-(1+\mu)(\mu+3)h_R^6-[K_d-2]\}}{\{(2+\mu)h_R^2-1\}W^2} \quad . \quad \text{(B50)}$$

To find a particular non-trivial but simple enough solution, we test $K_d=1$ option. For $K_d=1$, (B50) is

$$\frac{dg(W)}{dW}=[g(W)]^2-\frac{1}{\mu}\frac{\{[\mu\mu+\mu-3]h_R^2+[\mu\mu+6\mu+7]h_R^4-2(\mu+2)(1+\mu)h_R^6-\mu\}}{\{(2+\mu)h_R^2-1\}W}g(W)+\frac{1}{\mu^2}\frac{\{[-2\mu-5]h_R^2+[\mu\mu+6\mu+7]h_R^4-(1+\mu)(\mu+3)h_R^6+1\}}{\{(2+\mu)h_R^2-1\}W^2} \quad . \quad \text{(B51)}$$

Several test with model functions for $g(W)$, particularly with various powers of h_R divided by W, lead to a conclusion that a model in the shape

$$g(W)=C_1\frac{h_R^2}{\mu W}+C_2\frac{1}{\mu W}=\frac{1}{\mu W}(C_1h_R^2+C_2) \quad , \quad \text{(B52)}$$

with C_1 and C_2 being unknown constants which are to be found, could solve the Riccati equation (B51). For this model function, the pre-factors by the individual powers of h_R in (B51) could combine in order to be finally zero. (B52) seems to be an appropriate model as it contains only even powers of h_R. Also, using this form of the function, the explicitly expressed W in the denominators of the terms in the equation (B51) mutually cancel out. We can find, with the help of (A75) from **Appendix A**, that

$$\frac{dg(W)}{dW} = -\frac{1}{\mu}\frac{(C_1 h_R^2 + C_2)}{W^2} + \frac{1}{\mu W}\left(C_1 \frac{dh_R^2}{dW} + 0\right) = -\frac{1}{\mu}\frac{(C_1 h_R^2 + C_2)}{W^2} + C_1 \frac{1}{\mu W}\left(\frac{2}{\mu W}(h_R^2 - (2+\mu)h_R^4 + (1+\mu)h_R^6)\right) = \frac{1}{\mu^2 W^2}\left[-\mu(C_1 h_R^2 + C_2) + C_1\left(2(h_R^2 - (2+\mu)h_R^4 + (1+\mu)h_R^6)\right)\right] = \frac{1}{\mu^2 W^2}[-C_1 \mu h_R^2 - C_2 \mu + 2C_1 h_R^2 - 2C_1(2+\mu)h_R^4 + 2C_1(1+\mu)h_R^6] = \frac{C_1}{\mu^2 W^2}\left[(2-\mu)h_R^2 - 2(2+\mu)h_R^4 + 2(1+\mu)h_R^6 - \frac{C_2}{C_1}\mu\right] \tag{B53}$$

and

$$[g(W)]^2 = \frac{1}{\mu^2 W^2}(C_1^2 h_R^4 + 2C_1 C_2 h_R^2 + C_2^2) \quad . \tag{B54}$$

We then fill these relations into the Riccati equation (B51) and obtain

$$\frac{C_1}{\mu^2 W^2}\left[(2-\mu)h_R^2 - 2(2+\mu)h_R^4 + 2(1+\mu)h_R^6 - \frac{C_2}{C_1}\mu\right] = \frac{1}{\mu^2 W^2}(C_1^2 h_R^4 + 2C_1 C_2 h_R^2 + C_2^2) - \frac{1}{\mu}\frac{\{[\mu\mu+\mu-3]h_R^2 + [\mu\mu+6\mu+7]h_R^4 - 2(\mu+2)(1+\mu)h_R^6 - \mu\}}{\{(2+\mu)h_R^2 - 1\}W}\frac{1}{\mu W}(C_1 h_R^2 + C_2) + \frac{1}{\mu^2}\frac{\{[-2\mu-5]h_R^2 + [\mu\mu+6\mu+7]h_R^4 - (1+\mu)(\mu+3)h_R^6 + 1\}}{\{(2+\mu)h_R^2 - 1\}W^2} \quad . \tag{B55}$$

By multiplication by the denominator we obtain

$$C_1\left[(2-\mu)h_R^2 - 2(2+\mu)h_R^4 + 2(1+\mu)h_R^6 - \frac{C_2}{C_1}\mu\right]\{(2+\mu)h_R^2 - 1\} = (C_1^2 h_R^4 + 2C_1 C_2 h_R^2 + C_2^2)\{(2+\mu)h_R^2 - 1\} - \{[\mu\mu + \mu - 3]h_R^2 + [\mu\mu + 6\mu + 7]h_R^4 - 2(\mu+2)(1+\mu)h_R^6 - \mu\}(C_1 h_R^2 + C_2) + \{[-2\mu - 5]h_R^2 + [\mu\mu + 6\mu + 7]h_R^4 - (1+\mu)(\mu+3)h_R^6 + 1\} \quad . \tag{B56}$$

We expand the terms

$$[(2-\mu)(2+\mu)C_1 h_R^4 - 2(2+\mu)(2+\mu)C_1 h_R^6 + 2(1+\mu)(2+\mu)C_1 h_R^8 - \mu(2+\mu)C_2 h_R^2] - [(2-\mu)C_1 h_R^2 - 2(2+\mu)C_1 h_R^4 + 2(1+\mu)C_1 h_R^6 - C_2\mu] = (C_1^2(2+\mu)h_R^6 + 2C_1 C_2(2+\mu)h_R^4 + C_2^2(2+\mu)h_R^2) - (C_1^2 h_R^4 + 2C_1 C_2 h_R^2 + C_2^2) - \{[\mu\mu + \mu - 3]C_1 h_R^4 + [\mu\mu + 6\mu + 7]C_1 h_R^6 - 2(\mu+2)(1+\mu)C_1 h_R^8 - \mu C_1 h_R^2\} - \{[\mu\mu + \mu - 3]C_2 h_R^2 + [\mu\mu + 6\mu + 7]C_2 h_R^4 - 2(\mu+2)(1+\mu)C_2 h_R^6 - \mu C_2\} + \{[-2\mu - 5]h_R^2 + [\mu\mu + 6\mu + 7]h_R^4 - (1+\mu)(\mu+3)h_R^6 + 1\} \quad . \tag{B57}$$

Now, the same order terms in h_R are marked by an identical color:

$$[(2-\mu)(2+\mu)C_1 h_R^4 - 2(2+\mu)(2+\mu)C_1 h_R^6 + 2(1+\mu)(2+\mu)C_1 h_R^8 - \mu(2+\mu)C_2 h_R^2] + [-(2-\mu)C_1 h_R^2 + 2(2+\mu)C_1 h_R^4 - 2(1+\mu)C_1 h_R^6 + C_2\mu] = (C_1^2(2+\mu)h_R^6 + 2C_1 C_2(2+\mu)h_R^4 + C_2^2(2+\mu)h_R^2) + (-C_1^2 h_R^4 - 2C_1 C_2 h_R^2 - C_2^2) + \{-[\mu\mu + \mu - 3]C_1 h_R^4 - [\mu\mu + 6\mu + 7]C_1 h_R^6 + 2(\mu+2)(1+\mu)C_1 h_R^8 + \mu C_1 h_R^2\} + \{-[\mu\mu + \mu - 3]C_2 h_R^2 - [\mu\mu + 6\mu + 7]C_2 h_R^4 + 2(\mu+2)(1+\mu)C_2 h_R^6 + \mu C_2\} + \{[-2\mu - 5]h_R^2 + [\mu\mu + 6\mu + 7]h_R^4 - (1+\mu)(\mu+3)h_R^6 + 1\} \ . \qquad \text{B58)}$$

and they are put together (keeping in mind that some were on the opposite side of the equation):

$$0 = [C_2^2(2+\mu)h_R^2 + \mu(2+\mu)C_2 h_R^2 + (2-\mu)C_1 h_R^2 - 2C_1 C_2 h_R^2 + \mu C_1 h_R^2 - (\mu\mu + \mu - 3)C_2 h_R^2 + (-2\mu - 5)h_R^2] + [2C_1 C_2(2+\mu)h_R^4 - (2-\mu)(2+\mu)C_1 h_R^4 - 2(2+\mu)C_1 h_R^4 - C_1^2 h_R^4 - (\mu\mu + \mu - 3)C_1 h_R^4 - (\mu\mu + 6\mu + 7)C_2 h_R^4 + (\mu\mu + 6\mu + 7)h_R^4] + [C_1^2(2+\mu)h_R^6 + 2(2+\mu)(2+\mu)C_1 h_R^6 + 2(1+\mu)C_1 h_R^6 - (\mu\mu + 6\mu + 7)C_1 h_R^6 + 2(\mu+2)(1+\mu)C_2 h_R^6 - (1+\mu)(\mu+3)h_R^6] + [2(\mu+2)(1+\mu)C_1 h_R^8 - 2(1+\mu)(2+\mu)C_1 h_R^8] + [\mu C_2 + 1 - C_2\mu - C_2^2] \quad . \tag{B59}$$

We immediately see that the pre-factors of the term with eight power of h_R mutually cancel. Further, we combine the same order terms and get

$$0 = [C_2^2(2+\mu) + \mu(2+\mu)C_2 + (2-\mu)C_1 - 2C_1C_2 + \mu C_1 - (\mu\mu+\mu-3)C_2 + (-2\mu-5)]h_R^2 + [2C_1C_2(2+\mu) - (2-\mu)(2+\mu)C_1 - 2(2+\mu)C_1 - C_1^2 - (\mu\mu+\mu-3)C_1 - (\mu\mu+6\mu+7)C_2 + (\mu\mu+6\mu+7)]h_R^4 + [C_1^2(2+\mu) + 2(2+\mu)(2+\mu)C_1 + 2(1+\mu)C_1 - (\mu\mu+6\mu+7)C_1 + 2(\mu+2)(1+\mu)C_2 - (1+\mu)(\mu+3)]h_R^6 + [0]h_R^8 + [1-C_2^2] \quad . \qquad \text{(B60)}$$

We now simplify the individual pre-factors by the powers of h_R:

$$0 = [(\mu+2)C_2^2 + (\mu+3)C_2 + 2C_1 - 2C_1C_2 - 2\mu - 5]h_R^2 + [2(2+\mu)C_1C_2 - (5+3\mu)C_1 - C_1^2 - (\mu\mu+6\mu+7)C_2 + (\mu\mu+6\mu+7)]h_R^4 + [C_1^2(2+\mu) + (3+4\mu+\mu\mu)C_1 + 2(\mu+2)(1+\mu)C_2 - (1+\mu)(\mu+3)]h_R^6 + [1-C_2^2] \; . \qquad \text{(B61)}$$

Each pre-factor by the powers of h_R has to be zero. From the last term, it is thus clear that

$$C_2 = 1 \quad \text{or} \quad C_2 = -1 \quad . \qquad \text{(B62)}$$

If C_2=1, then

$$0 = [(\mu+2) + (\mu+3) + 2C_1 - 2C_1 - 2\mu - 5]h_R^2 + [2(2+\mu)C_1 - (5+3\mu)C_1 - C_1^2 - (\mu\mu+6\mu+7) + (\mu\mu+6\mu+7)]h_R^4 + [C_1^2(2+\mu) + (3+4\mu+\mu\mu)C_1 + 2(\mu+2)(1+\mu) - (1+\mu)(\mu+3)]h_R^6 + [0] \quad . \qquad \text{(B63)}$$

and thus

$$0 = [0]h_R^2 + [-(1+\mu) - C_1]C_1h_R^4 + [C_1^2(2+\mu) + (1+\mu)(\mu+3)C_1 + (1+\mu)(1+\mu)]h_R^6 + [0] \quad . \qquad \text{(B64)}$$

Finally, we have two equations for the pre-factors of the remaining powers of h_R, which has to be simultaneously equal to zero:

$$[-(1+\mu) - C_1] = 0 \quad , \qquad \text{(B65)}$$

$$[C_1^2(2+\mu) + (1+\mu)(\mu+3)C_1 + (1+\mu)(1+\mu)] = 0, \qquad \text{(B66)}$$

which lead to

$$C_1 = -(1+\mu) \; . \qquad \text{(B67)}$$

$$C_1 = \frac{-(1+\mu)(\mu+3) \pm \sqrt{(1+\mu)(\mu+3)(1+\mu)(\mu+3) - 4(2+\mu)(1+\mu)(1+\mu)}}{2(2+\mu)} = \frac{-(1+\mu)(\mu+3) \pm (1+\mu)\sqrt{(\mu+3)(\mu+3) - 4(2+\mu)}}{2(2+\mu)} = \frac{-(1+\mu)(\mu+3) \pm (1+\mu)\sqrt{(\mu\mu+6\mu+9) - (8+4\mu)}}{2(2+\mu)} = \frac{-(\mu+3) \pm \sqrt{(\mu\mu+2\mu+1)}}{2(2+\mu)}(1+\mu) = \frac{-\mu-3 \pm (1+\mu)}{2(2+\mu)}(1+\mu) = \frac{-2}{2(2+\mu)}(1+\mu) \quad \text{or} \quad \frac{-2\mu-4}{2(2+\mu)}(1+\mu) = -\frac{1+\mu}{2+\mu} \quad \text{or} \quad -(1+\mu) \quad . \qquad \text{(B68)}$$

The solution $C_1=-(1+\mu)$ is common for both equations, and it thus leads to $g(W)$ in the form

$$g_0(W) = -(1+\mu)\frac{h_R^2}{\mu W} + \frac{1}{\mu W} = \frac{1}{\mu W}[1-(1+\mu)h_R^2] = \frac{1}{\mu W}[-\mu f_1^2] = -\frac{f_C^2}{W} \quad . \qquad \text{(B69)}$$

We now test also C_2=–1 possibility. Then

$$0 = [(\mu+2) - (\mu+3) + 2C_1 + 2C_1 - 2\mu - 5]h_R^2 + [-2(2+\mu)C_1 - (5+3\mu)C_1 - C_1^2 + (\mu\mu+6\mu+7) + (\mu\mu+6\mu+7)]h_R^4 + [C_1^2(2+\mu) + (3+4\mu+\mu\mu)C_1 - 2(\mu+2)(1+\mu) - (1+\mu)(\mu+3)]h_R^6 + [0] \quad . \qquad \text{(B70)}$$

and thus

$$0 = [4C_1 - 2\mu - 6]h_R^2 + [-(9+5\mu)C_1 - C_1^2 + (2\mu\mu + 12\mu + 14)]h_R^4 + [C_1^2(2+\mu) + (3+4\mu+\mu\mu)C_1 - (3\mu+7)(1+\mu)]h_R^6 + [0] \quad . \qquad \text{(B71)}$$

Finally, we have three equations,

$$[4C_1 - 2\mu - 6] = 0 \quad , \qquad \text{(B72)}$$

$$[-C_1^2 - (9+5\mu)C_1 + (2\mu\mu + 12\mu + 14)] = 0 \; , \qquad \text{(B73)}$$

$$[C_1^2(2+\mu) + (3+4\mu+\mu\mu)C_1 - (3\mu+7)(1+\mu)] = 0 \quad , \qquad \text{(B74)}$$

Where the first one leads to

$$C_1 = \frac{3+\mu}{2} \quad . \tag{B75}$$

When we fill this result to the left-hand side of the second equation, we see that

$$\left[-\left(\frac{3+\mu}{2}\right)^2 - (9+5\mu)\frac{3+\mu}{2} + (2\mu\mu + 12\mu + 14)\right] = \left[-\left(\frac{9+6\mu+\mu\mu}{4}\right) - \frac{(27+24\mu+5\mu\mu)}{2} + (2\mu\mu + 12\mu + 14)\right] = -\frac{9+6\mu+\mu\mu+54+48\mu+10\mu\mu-8\mu\mu-48\mu-56}{4} = -\frac{7+6\mu+3\mu\mu}{4} \neq 0 \, . \tag{B76}$$

Thus, this case (i.e. C_2=–1) does not lead to a solution for $g(W)$.
The only possibility for solution is then (B69):

$$g(W) = \frac{1}{\mu W}\left[1-(1+\mu)h_R^2\right] = -\frac{f_C^2}{W} \quad . \tag{B77}$$

The equation (B47), in which we now know the function $g(W)$, has the particular solution (Polyanin and Zaitsev 2003, Eq. 46]:

$$F_0(W) = \exp\{-\int g(W)dW\} = \exp\left\{-\int \frac{1}{\mu W}\left[1-(1+\mu)h_R^2\right]dW\right\} = \exp\left\{\frac{\mu+1}{\mu}\int \frac{h_R^2}{W}dW - \frac{1}{\mu}\int \frac{1}{W}dW\right\}. \tag{B78}$$

To express it in an analytical form, we have to know the integral

$$\int W^{-1}h_R^2\, dW = \int W^{-1}\sum_{k=0}^{\infty}\binom{-\mu k}{k}(W^2)^k dW \tag{B79}$$

(the last expression containing power series is valid in the small-v region; nevertheless the final result is the same in the large-v region).
This can be expressed in the following form (see **Appendix A**, relation (A111)):

$$\int W^{-1}h_R^2\, dW = \ln\left[\frac{f_S}{f_C}\right] \quad . \tag{B80}$$

Having this integral expressed in a simple form, we can write the particular solution of the equation as

$$F_0(W) = \exp\left\{\frac{\mu+1}{\mu}\ln\left[\frac{f_S}{f_C}\right] - \frac{1}{\mu}\ln W\right\} = W^{-\frac{1}{\mu}}\left(\frac{f_S}{f_C}\right)^{\frac{\mu+1}{\mu}} \quad . \tag{B81}$$

Using the identity (A29) from **Appendix A**, we can write the particular solution of the angular part of the Laplace equation in the form

$$F_0(W) = \frac{f_S}{h_R} \quad . \tag{B82}$$

where f_S/h_R is generalized sine divided by the metric scale factor of the coordinate R.
According to (A7) from **Appendix A**, we can write for f_S/h_R the analytical expressions

$$\frac{f_S}{h_R} = W\sqrt{1+\mu}\sqrt{\frac{\sum_{k=0}^{\infty}\binom{-(1+\mu)-\mu k}{k}(W^2)^k}{\sum_{k=0}^{\infty}\binom{-\mu k}{k}(W^2)^k}} = W\sqrt{1+\mu}\sqrt{\sum_{k=0}^{\infty}\frac{-(1+\mu)}{-(1+\mu)-\mu k}\binom{-(1+\mu)-\mu k}{k}(W^2)^k} \tag{B83}$$

in the small-v region and

$$\frac{f_S}{h_R} = \frac{\sqrt{\sum_{k=0}^{\infty}\binom{-1+\frac{\mu}{1+\mu}k}{k}\left(W^{-\frac{2}{1+\mu}}\right)^k}}{\frac{1}{\sqrt{1+\mu}}\sqrt{\sum_{k=0}^{\infty}\binom{\frac{\mu}{1+\mu}k}{k}\left(W^{-\frac{2}{1+\mu}}\right)^k}} = \sqrt{1+\mu}\sqrt{\sum_{k=0}^{\infty}\frac{-1}{-1+\frac{\mu}{1+\mu}k}\binom{-1+\frac{\mu}{1+\mu}k}{k}\left(W^{-\frac{2}{1+\mu}}\right)^k} \tag{B84}$$

in the large-v region.
For μ=0, f_S/h_R simplifies (with the help of the binomial theorem) in the small-v region to

$$\left.\frac{f_S}{h_R}\right|_{\mu=0} = \left(\frac{R}{R_0}\right)^0 \frac{\sin\nu}{\cos^{(1+0)}\nu}\sqrt{1+0}\sqrt{\sum_{k=0}^{\infty}\frac{-(1+0)}{-(1+0)-0\cdot k}\binom{-(1+0)-0\cdot k}{k}\left(\left[\left(\frac{R}{R_0}\right)^0 \frac{\sin\nu}{\cos^{(1+0)}\nu}\right]^2\right)^k} =$$
$$\frac{\sin\nu}{\cos\nu}\sqrt{\sum_{k=0}^{\infty}\frac{-1}{-1}\binom{-1}{k}\left(\left[\frac{\sin\nu}{\cos\nu}\right]^2\right)^k} = \frac{\sin\nu}{\cos\nu}\sqrt{\left(1+\left[\frac{\sin\nu}{\cos\nu}\right]^2\right)^{-1}} = \frac{\sin\nu}{\cos\nu}\sqrt{\left(\frac{\cos^2\nu+\sin^2\nu}{\cos^2\nu}\right)^{-1}} = \frac{\sin\nu}{\cos\nu}\sqrt{\cos^2\nu} =$$
$$\sin\nu \qquad \text{(B85)}$$

and in the large-*v* region to

$$\left.\frac{f_S}{h_R}\right|_{\mu=0} = \sqrt{1+0}\sqrt{\sum_{k=0}^{\infty}\frac{-1}{-1+\frac{0}{1+0}k}\binom{-1+\frac{0}{1+0}k}{k}\left(\left[\left(\frac{R}{R_0}\right)^0 \frac{\sin\nu}{\cos^{(1+0)}\nu}\right]^{-\frac{2}{1+0}}\right)^k} = \sqrt{\sum_{k=0}^{\infty}\frac{-1}{-1}\binom{-1}{k}\left(\left[\frac{\sin\nu}{\cos\nu}\right]^{-2}\right)^k} =$$
$$\sqrt{\left(1+\left[\frac{\sin\nu}{\cos\nu}\right]^{-2}\right)^{-1}} = \sqrt{\left(\frac{\sin^2\nu+\cos^2\nu}{\sin^2\nu}\right)^{-1}} = \sin\nu \quad . \qquad \text{(B86)}$$

In both regions – in the limit μ=0 – the solution is thus

$$F_0(W)|_{\mu\to 0} = \sin\nu \quad . \qquad \text{(B87)}$$

Note, that this function is basically the solution of the Laplace equation (its separated angular part) in the spherical coordinates, i.e. the first-degree Legendre polynomial (considering that we do not have polar coordinates – the origin of *v* which is for $\mu = 0$ equivalent to the geocentric latitude – is at the equator, and it is thus a complement of the polar angle to π/2).

The first determined solution of the angular part of the Laplace equation is thus

$$F_A(W) = c_{A1}\frac{f_S}{h_R} \quad . \qquad \text{(B88)}$$

The f_S/h_R function (B82) is here multiplied by an arbitrary constant c_{A1}, as such multiplication evidently also leads to a solution of (B46).

B.4. Second kind solution.

(B88) is only a particular solution of the equation (B47). Nevertheless, the general solution for K_d=1 can be obtained as well. The general solution (for K_d=1) is given by

$$F(W) = c_{A1}F_0(W) + c_{B1}F_0(W)\int\frac{1}{[F_0(W)]^2}\exp\left[-\int f(W)dW\right]dW = F_A(W) + F_B(W) \qquad \text{(B89)}$$

The first part, i.e. by c_{A1}, is a solution of its own, as it is the already found solution (B88). The second part, i.e. by c_{B1}, is thus a new, separate, solution. It contains two nested integrals. To find the solution in an analytic form, we have to determine the integrals (inner and outer) in (B89).

The function *f*(*W*), which is in the inner integral, is given by (B48), which is, for K_d=1, simplified to

$$f(W) = \frac{1}{\mu}\frac{\{[(\mu\mu-\mu-7)+2.1(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2.1(1+\mu)]h_R^4+(-2\mu-4)(1+\mu)h_R^6+[2-\mu-2.1]\}}{\{(2+\mu)h_R^2-1\}W} =$$
$$\frac{1}{\mu}\frac{\{[(\mu\mu-\mu-7)+2(2+\mu)]h_R^2+[(\mu\mu+8\mu+9)-2(1+\mu)]h_R^4-(2\mu+4+2\mu\mu+4\mu)h_R^6+[-\mu]\}}{\{(2+\mu)h_R^2-1\}W} =$$
$$\frac{1}{\mu}\frac{\{[(\mu\mu-\mu-7)+(4+2\mu)]h_R^2+[(\mu\mu+6\mu+7)]h_R^4-(6\mu+4+2\mu\mu)h_R^6-\mu\}}{\{(2+\mu)h_R^2-1\}W} = \frac{\{(\mu\mu+\mu-3)h_R^2+[(\mu\mu+6\mu+7)]h_R^4-(2\mu\mu+6\mu+4)h_R^6-\mu\}}{\{(2+\mu)h_R^2-1\}\mu W} \qquad \text{(B90)}$$

As (see (A75) from **Appendix A**)

$$\frac{d(h_R^2)}{dW} = \frac{2}{\mu W}[h_R^2-(2+\mu)h_R^4+(1+\mu)h_R^6] = -\frac{2}{\mu W}h_R^2(1-h_R^2)[(1+\mu)h_R^2-1] \quad , \qquad \text{(B91)}$$

we can write the following differential relation:

$$\frac{dW}{\mu W} = \frac{d(h_R^2)}{2h_R^2[1-(2+\mu)h_R^2+(1+\mu)h_R^4]} = -\frac{d(h_R^2)}{2h_R^2(1-h_R^2)[(1+\mu)h_R^2-1]} \quad . \tag{B92}$$

Then

$$\int f(W)dW = \int \frac{\{(\mu\mu+\mu-3)h_R^2+[(\mu\mu+6\mu+7)]h_R^4-(2\mu\mu+6\mu+4)h_R^6-\mu\}}{\{(2+\mu)h_R^2-1\}}\frac{dW}{\mu W} =$$
$$\int \frac{\{(\mu\mu+\mu-3)h_R^2+[(\mu\mu+6\mu+7)]h_R^4-(2\mu\mu+6\mu+4)h_R^6-\mu\}}{\{(2+\mu)h_R^2-1\}}\frac{d(h_R^2)}{2h_R^2[1-(2+\mu)h_R^2+(1+\mu)h_R^4]} \quad . \tag{B93}$$

This integral can be solved:

$$\int f(W)dW = \frac{1}{2}\{\, \mu \ln[h_R^2] - (\mu+1)\ln[1-(\mu+1)h_R^2] - \ln[1-(\mu+2)h_R^2]\,\} + \ln\left[\frac{1}{C_F}\right] , \tag{B94}$$

where C_F is an arbitrary complex constant, and the complex-valued logarithm is assumed.
Then

$$\int f(W)dW = \frac{1}{2}\{\, \ln[h_R^{2\mu}] - \ln\left[(1-(\mu+1)h_R^2)^{(\mu+1)}\right] - \ln[1-(\mu+2)h_R^2]\,\} + \ln\left[\frac{1}{C_F}\right] = \left\{\, \ln[h_R^{\mu}] - \ln\left[(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}\right] - \ln\left[(1-(\mu+2)h_R^2)^{\frac{1}{2}}\right]\right\} + \ln\left[\frac{1}{C_F}\right] = \ln\left[\frac{1}{C_F}\frac{h_R^{\mu}}{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}\right] \quad . \tag{B95}$$

Thus

$$\exp[-\int f(W)dW] = \exp\left[-\ln\left[\frac{1}{C_F}\frac{h_R^{\mu}}{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}\right]\right] =$$
$$\exp\left[\ln\left[\left(\frac{1}{C_F}\frac{h_R^{\mu}}{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}\right)^{-1}\right]\right] = \left(\frac{1}{C_F}\frac{h_R^{\mu}}{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}\right)^{-1} =$$
$$C_F\frac{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}{h_R^{\mu}} \tag{B96}$$

Further, (B82) can be used for $F_0(W)$ and thus

$$\frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW] = \frac{1}{\left[\frac{f_S}{h_R}\right]^2}C_F\frac{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}{h_R^{\mu}} =$$
$$\frac{1}{\frac{1+\mu}{\mu}\frac{(1-h_R^2)}{h_R^2}}C_F\frac{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}{h_R^{\mu}} = C_F\frac{\mu}{(1+\mu)}\frac{(1-(\mu+1)h_R^2)^{\frac{(\mu+1)}{2}}(1-(\mu+2)h_R^2)^{\frac{1}{2}}}{(1-h_R^2)h_R^{\mu-2}} \quad . \tag{B97}$$

The last factor of the product in the numerator is purely complex number because the minimum value of h_R^2 is 1/(1+μ), on polar axis, and the term $(1-(\mu+2)h_R^2)$ is thus always negative. The term $(1-(\mu+1)h_R^2)$ is zero on polar axis and negative otherwise. The imaginary constant can be separated, leaving the real functions, and we then have:

$$\frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW] = C_F\frac{\mu}{(1+\mu)}\frac{(-1)^{\frac{(\mu+1)}{2}}\left((\mu+1)h_R^2-1\right)^{\frac{(\mu+1)}{2}}(-1)^{\frac{1}{2}}\left((\mu+2)h_R^2-1\right)^{\frac{1}{2}}}{(1-h_R^2)h_R^{\mu-2}} =$$
$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\frac{\left((\mu+1)h_R^2-1\right)^{\frac{(\mu+1)}{2}}\left((\mu+2)h_R^2-1\right)^{\frac{1}{2}}}{(1-h_R^2)h_R^{\mu-2}} \quad . \tag{B98}$$

We have a real function (multiplied by a complex constant) which can be further integrated (the outer integral in (B89)) to obtain

$$\int \frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW]dW = \int\left[-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\frac{\left((\mu+1)h_R^2-1\right)^{\frac{(\mu+1)}{2}}\left((\mu+2)h_R^2-1\right)^{\frac{1}{2}}}{(1-h_R^2)h_R^{\mu-2}}\right]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\int\left[\frac{h_R^2\left((\mu+1)h_R^2-1\right)^{\frac{(\mu+1)}{2}}\left((\mu+2)h_R^2-1\right)^{\frac{1}{2}}}{(1-h_R^2)(h_R^2)^{\frac{\mu}{2}}}\right]dW \quad . \tag{B99}$$

We now use Pólya and Szegö relations (see Eqs. (A1), (A2), (A3), (A4), (A5) in **Appendix A**) in the small-v region, i.e.

$$\frac{p^{a+1}}{(1+\mu)p-\mu} = \sum_{k=0}^{\infty}\binom{a-\mu k}{k}(W^2)^k \quad , \tag{B100}$$

$$p^a = \sum_{k=0}^{\infty}\frac{a}{a-\mu k}\binom{a-\mu k}{k}(W^2)^k \quad , \tag{B101}$$

for which the following equation is fulfilled

$$-(W^2)p^{-\mu}+p=1 \qquad \Rightarrow \qquad W^{-1}=(p-1)^{-\frac{1}{2}}p^{-\frac{\mu}{2}} \quad . \tag{B102}$$

Then, thanks to the relation (7) for the metric scale factor h_R in the small-v region, and the identity (B100) above, we obtain (B99) in the form

$$\int \frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\int\left[\frac{\frac{p^{0+1}}{(1+\mu)p-\mu}\left((\mu+1)\frac{p^{0+1}}{(1+\mu)p-\mu}-1\right)^{\frac{(\mu+1)}{2}}\left((\mu+2)\frac{p^{0+1}}{(1+\mu)p-\mu}-1\right)^{\frac{1}{2}}}{\left(1-\frac{p^{0+1}}{(1+\mu)p-\mu}\right)\left(\frac{p^{0+1}}{(1+\mu)p-\mu}\right)^{\frac{\mu}{2}}}\right]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\int\left[\frac{\frac{p}{(1+\mu)p-\mu}\left(\frac{(\mu+1)p}{(1+\mu)p-\mu}-1\right)^{\frac{(\mu+1)}{2}}\left(\frac{(\mu+2)p}{(1+\mu)p-\mu}-1\right)^{\frac{1}{2}}}{\left(1-\frac{p}{(1+\mu)p-\mu}\right)\left(\frac{p}{(1+\mu)p-\mu}\right)^{\frac{\mu}{2}}}\right]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\int\left[\frac{\frac{p}{(1+\mu)p-\mu}\left(\frac{(\mu+1)p-(1+\mu)p+\mu}{(1+\mu)p-\mu}\right)^{\frac{(\mu+1)}{2}}\left(\frac{(\mu+2)p-(1+\mu)p+\mu}{(1+\mu)p-\mu}\right)^{\frac{1}{2}}}{\left(\frac{(1+\mu)p-\mu-p}{(1+\mu)p-\mu}\right)\left(\frac{p}{(1+\mu)p-\mu}\right)^{\frac{\mu}{2}}}\right]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\int\left[\frac{\frac{p}{1}\left(\frac{\mu}{(1+\mu)p-\mu}\right)^{\frac{(\mu+1)}{2}}\left(\frac{p+\mu}{(1+\mu)p-\mu}\right)^{\frac{1}{2}}}{\left(\frac{\mu p-\mu}{1}\right)\left(\frac{p}{(1+\mu)p-\mu}\right)^{\frac{\mu}{2}}}\right]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{\mu}{(1+\mu)}\int\left[\frac{p\left(\frac{1}{(1+\mu)p-\mu}\right)^{\frac{\mu}{2}+\frac{1}{2}}(\mu)^{\frac{(\mu+1)}{2}}\left(\frac{1}{(1+\mu)p-\mu}\right)^{\frac{1}{2}}(p+\mu)^{\frac{1}{2}}}{\mu(p-1)\left(\frac{p}{(1+\mu)p-\mu}\right)^{\frac{\mu}{2}}}\right]dW =$$

$$-(-1)^{\frac{\mu}{2}}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{(1+\mu)}\int\left[\frac{p\left(\frac{1}{(1+\mu)p-\mu}\right)^{\frac{1}{2}}\left(\frac{1}{(1+\mu)p-\mu}\right)^{\frac{1}{2}}(p+\mu)^{\frac{1}{2}}}{(p-1)(p)^{\frac{\mu}{2}}}\right]dW = -(-1)^{\frac{\mu}{2}}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{(1+\mu)}\int\left[\frac{p^{1-\frac{\mu}{2}}\left(\frac{1}{(1+\mu)p-\mu}\right)(p+\mu)^{\frac{1}{2}}}{(p-1)}\right]dW \quad . \tag{B103}$$

As (see (B102))

$$p-1=W^2p^{-\mu} \quad , \tag{B104}$$

then

$$\int \frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW]dW = -(-1)^{\frac{\mu}{2}}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{(1+\mu)}\int\left[\frac{p^{1-\frac{\mu}{2}}\left(\frac{1}{(1+\mu)p-\mu}\right)(p+\mu)^{\frac{1}{2}}}{W^2p^{-\mu}}\right]dW = -(-1)^{\frac{\mu}{2}}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{(1+\mu)}\int\left[\frac{p^{1+\frac{\mu}{2}}(p+\mu)^{\frac{1}{2}}}{(1+\mu)p-\mu}\right]\frac{dW}{W^2} \quad . \tag{B105}$$

We further differentiate W^{-1} using (B102) with the aim of finding the element dp in terms of the element dW, moreover in a suitable form for W^2 term disappearing in the final integral of the solution:

$$\frac{d(W^{-1})}{dW} = -W^{-2} = \frac{d\left[(p-1)^{-\frac{1}{2}}p^{-\frac{\mu}{2}}\right]}{dW} = -\tfrac{1}{2}(p-1)^{-\frac{3}{2}}\frac{dp}{dW}p^{-\frac{\mu}{2}} - (p-1)^{-\frac{1}{2}}\frac{\mu}{2}p^{-\frac{\mu}{2}-1}\frac{dp}{dW} = -\frac{1}{2}\left[(p-1)^{-\frac{3}{2}}p^{-\frac{\mu}{2}} + \mu(p-1)^{-\frac{1}{2}}p^{-\frac{\mu}{2}-1}\right]\frac{dp}{dW} = -\frac{1}{2}p^{-\frac{\mu}{2}}\left[(p-1)^{-\frac{1}{2}-1} + \mu(p-1)^{-\frac{1}{2}}p^{-1}\right]\frac{dp}{dW} = -\frac{1}{2}p^{-\frac{\mu}{2}}\left[\frac{1}{(p-1)\sqrt{p-1}} + \mu\frac{1}{p\sqrt{p-1}}\right]\frac{dp}{dW} = -\frac{1}{2}\frac{p^{-\frac{\mu}{2}}}{\sqrt{p-1}}\left[\frac{1}{(p-1)} + \frac{\mu}{p}\right]\frac{dp}{dW} = -\frac{1}{2}\frac{p^{-\frac{\mu}{2}}}{\sqrt{p-1}}\left[\frac{p+\mu(p-1)}{p(p-1)}\right]\frac{dp}{dW} = -\frac{1}{2}\frac{p^{-\frac{\mu}{2}}}{\sqrt{p-1}}\left[\frac{(1+\mu)p-\mu}{p(p-1)}\right]\frac{dp}{dW} = -\frac{(p-1)^{-\frac{3}{2}}}{2}\frac{(1+\mu)p-\mu}{p^{\frac{\mu}{2}+1}}\frac{dp}{dW} \quad . \tag{B106}$$

Then

$$\frac{dW}{W^2} = \frac{(p-1)^{-\frac{3}{2}}}{2}\frac{(1+\mu)p-\mu}{p^{\frac{\mu}{2}+1}}dp \quad . \tag{B107}$$

After the substitution, (B105) becomes the form

$$\int \frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW]dW = -(-1)^{\frac{\mu}{2}}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{(1+\mu)}\int\left[\frac{p^{1+\frac{\mu}{2}}(p+\mu)^{\frac{1}{2}}}{(1+\mu)p-\mu}\right]\frac{(p-1)^{-\frac{3}{2}}}{2}\frac{(1+\mu)p-\mu}{p^{\frac{\mu}{2}+1}}dp = -i^{\mu}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{(1+\mu)}\int\left[\frac{(p+\mu)^{\frac{1}{2}}}{1}\right]\frac{(p-1)^{-\frac{3}{2}}}{2}\frac{1}{1}dp = -i^{\mu}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{2(1+\mu)}\int\left[\frac{(p+\mu)^{\frac{1}{2}}}{(p-1)^{\frac{3}{2}}}\right]dp \quad . \tag{B108}$$

Assuming p and μ are positive, we arrive for the integral in the above expression to

$$\int\frac{(p+\mu)^{\frac{1}{2}}}{(p-1)^{\frac{3}{2}}}dp = 2\sinh^{-1}\left(\sqrt{\frac{p-1}{\mu+1}}\right) - 2\sqrt{\frac{\mu+p}{p-1}} \quad . \tag{B109}$$

According to (B101),

$$p^1 = \sum_{k=0}^{\infty}\frac{1}{1-\mu k}\binom{1-\mu k}{k}(W^2)^k \tag{B110}$$

in the small-v region. When we use the identity (A7) from **Appendix A**, we can write it also in the form

$$p^1 = \frac{\sum_{k=0}^{\infty}\binom{-\mu k}{k}(W^2)^k}{\sum_{k=0}^{\infty}\binom{-1-\mu k}{k}(W^2)^k} = \frac{h_R^2}{f_C^2} \tag{B111}$$

where the last relation follows from (14) and (7). Then

$$\int\frac{(p+\mu)^{\frac{1}{2}}}{(p-1)^{\frac{3}{2}}}dp = 2\sinh^{-1}\left(\sqrt{\frac{\frac{h_R^2}{f_C^2}-1}{\mu+1}}\right) - 2\sqrt{\frac{h_R^2+\mu f_C^2}{h_R^2-f_C^2}} \quad . \tag{B112}$$

When using the relations between the generalized sine and cosine and the metric scale factor h_R (17), we can write for the argument of $\sinh^{-1}$:

$$\sqrt{\frac{\frac{h_R^2}{f_C^2}-1}{\mu+1}}=\sqrt{\frac{\frac{\frac{1}{1+\mu}+\frac{\mu}{1+\mu}f_C^2}{f_C^2}-1}{\mu+1}}=\sqrt{\frac{\frac{\frac{1}{1+\mu}+\frac{\mu}{1+\mu}f_C^2-f_C^2}{f_C^2}}{\mu+1}}=\sqrt{\frac{\frac{\frac{1}{1+\mu}-\frac{1}{1+\mu}f_C^2}{f_C^2}}{\mu+1}}=\sqrt{\frac{1}{1+\mu}\frac{\frac{1-f_C^2}{f_C^2}}{\mu+1}}=\sqrt{\frac{1}{1+\mu}\frac{\frac{f_S^2}{f_C^2}}{\mu+1}}=\frac{\frac{f_S}{f_C}}{1+\mu} \quad , \tag{B113}$$

and thus

$$\int\frac{(p+\mu)^{\frac{1}{2}}}{(p-1)^{\frac{3}{2}}}dp=2\sinh^{-1}\left(\frac{f_S}{(1+\mu)f_C}\right)-2\sqrt{1+\frac{(1+\mu)^2f_C^2}{f_S^2}}=$$
$$2\ln\left(\frac{f_S}{(1+\mu)f_C}+\sqrt{1+\frac{f_S^2}{(1+\mu)^2f_C^2}}\right)-2\sqrt{1+\frac{(1+\mu)^2f_C^2}{f_S^2}} \tag{B114}$$

where the last expression comes from the relation between the inverse hyperbolic sine and the logarithm function. When we insert (B114) relation to (B108), we obtain

$$\int\frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW]dW=-i^{\mu}C_F\frac{(\mu)^{\frac{(\mu+1)}{2}}}{2(1+\mu)}\left[2\ln\left(\frac{f_S}{(1+\mu)f_C}+\sqrt{1+\frac{f_S^2}{(1+\mu)^2f_C^2}}\right)-2\sqrt{1+\frac{(1+\mu)^2f_C^2}{f_S^2}}\right] \tag{B115}$$

Further, we select an arbitrary complex constant C_F in such way that

$$C_F=-\frac{(1+\mu)}{i^{\mu}(\mu)^{\frac{(\mu+1)}{2}}} \quad , \tag{B116}$$

and we then have the second possible real solution of the angular part of the separated Laplace equation for the given model (K_d=1) in the form (see (B89))

$$F_B(W)=F_0(W)c_{B1}\int\frac{1}{[F_0(W)]^2}\exp[-\int f(W)dW]dW=c_{B1}\frac{f_S}{h_R}\left[\ln\left(\frac{f_S}{(1+\mu)f_C}+\sqrt{1+\frac{f_S^2}{(1+\mu)^2f_C^2}}\right)-\sqrt{1+\frac{(1+\mu)^2f_C^2}{f_S^2}}\right] \quad , \tag{B117}$$

or, equivalently, with inverse hyperbolic sine,

$$F_B(W)=c_{B1}\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{f_S}{(1+\mu)f_C}\right)-\sqrt{1+\frac{(1+\mu)^2f_C^2}{f_S^2}}\right] \quad . \tag{B118}$$

B.5. Test of a correctness of the found second kind solution with full Laplace equation

The correctness of the found second kind solution with K_d=1 in the full Laplace equation should be tested. It is done in the following paragraphs. We assume the radial part of the solution in the form (B29) with K_d=1, i.e.

$$r(R)=C_dR^1 \quad . \tag{B119}$$

The Laplace equation (axially symmetric case) in the SOS coordinates with the inserted model function (23), in which the angular part $F(W)$ is substituted by $F_B(W)$ from (B118) and $r(R)$ is taken from (B119), has the form

$$\frac{\partial}{\partial R}\left(\frac{\mathfrak{J}}{h_R^2}\frac{\partial}{\partial R}\big(r(R)F_B(W)\big)\right)+\frac{\partial}{\partial \nu}\left(\frac{\mathfrak{J}}{h_\nu^2}\frac{\partial}{\partial \nu}\big(r(R)F_B(W)\big)\right)=0 \quad , \tag{B120}$$

where $\mathfrak{J}=h_Rh_\nu h_\lambda$ denotes the Jacobian determinant.

Now, insert the abovementioned functions into the azimuthally symmetric Laplace equation (B120), and we obtain

$$\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2}\frac{\partial}{\partial R}\left(R\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]\right)\right)+\frac{\partial}{\partial \nu}\left(\frac{\Im}{h_\nu^2}\frac{\partial}{\partial \nu}\left(R\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]\right)\right)=0 \quad , \tag{B121}$$

and thus (product rule for the inner derivatives)

$$\frac{\partial}{\partial R}\left\{\frac{\Im}{h_R^2}\left(\frac{\partial R}{\partial R}\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+R\frac{\partial\left(\frac{f_S}{h_R}\right)}{\partial R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+R\frac{f_S}{h_R}\frac{\partial}{\partial R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]\right)\right\}+R\frac{\partial}{\partial \nu}\left\{\frac{\Im}{h_\nu^2}\left(\frac{\partial\left(\frac{f_S}{h_R}\right)}{\partial \nu}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\frac{f_S}{h_R}\frac{\partial}{\partial \nu}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]\right)\right\}=0 \quad . \tag{B122}$$

As f_C, f_S, h_R are dependent on W (not separately on R and/or ν), we can use the chain rule for the inner derivatives:

$$\frac{\partial}{\partial R}\left\{\frac{\Im}{h_R^2}\left(\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+R\frac{d\left(\frac{f_S}{h_R}\right)}{dW}\frac{\partial W}{\partial R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+R\frac{f_S}{h_R}\frac{d}{dW}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]\frac{\partial W}{\partial R}\right)\right\}+R\frac{\partial}{\partial \nu}\left\{\frac{\Im}{h_\nu^2}\left(\frac{d\left(\frac{f_S}{h_R}\right)}{dW}\frac{\partial W}{\partial \nu}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\frac{f_S}{h_R}\frac{d}{dW}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]\frac{\partial W}{\partial \nu}\right)\right\}=0 \quad . \tag{B123}$$

The inner derivative of $\sinh^{-1}$ minus the square root is

$$\frac{d}{dW}\left\{\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right\}=\frac{d}{dW}\left\{\ln\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}+\sqrt{1+\frac{1}{(1+\mu)^2}\frac{f_S^2}{f_C^2}}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right\}=\frac{d}{d(f_C/f_S)}\left\{\sinh^{-1}\left(\frac{1}{(1+\mu)}\left(\frac{f_C}{f_S}\right)^{-1}\right)-\sqrt{1+(1+\mu)^2\left(\frac{f_C}{f_S}\right)^2}\right\}\frac{d(f_C/f_S)}{dW}=-\frac{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}{\frac{f_C}{f_S}}\left(-\frac{h_R^2}{W}\frac{f_C}{f_S}\right)=\frac{h_R^2}{W}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} \quad , \tag{B124}$$

where we used the derivative of generalized trigonometric functions ratio (A91) from **Appendix A**. By inserting (B124), the inner derivative $\frac{\partial W}{\partial R}$ (according to (10)), and $\frac{d\left(\frac{f_S}{h_R}\right)}{dW}$ (according to Eq. (A93) from **Appendix A**), (B123) can be rewritten to the form

$$\frac{\partial}{\partial R}\left\{\frac{\Im}{h_R^2}\left(\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+R\frac{f_C^2}{W}\frac{f_S}{h_R}\left[\frac{\mu}{R}W\right]\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+R\frac{f_S}{h_R}\frac{h_R^2}{W}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\left[\frac{\mu}{R}W\right]\right)\right\}+$$

$$R\frac{\partial}{\partial v}\left\{\frac{\Im}{h_v^2}\left(\frac{f_C^2}{W}\frac{f_S}{h_R}\frac{\partial W}{\partial v}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\frac{f_S}{h_R}\frac{h_R^2}{W}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\frac{\partial W}{\partial v}\right)\right\}=0 \quad , \tag{B125}$$

For the further derivation, it is of advantage to multiply/divide the terms inside the outer derivatives by *W* derivatives with respect to *v* at proper places. After some simplification, we obtain:

$$\frac{\partial}{\partial R}\left\{\frac{\partial W}{\partial v}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial v}}\right)\left(\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\right.\right.$$
$$\left.\left.\mu f_C^2\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\mu f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$
$$R\frac{\partial}{\partial v}\left\{\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\frac{1}{\frac{\partial W}{\partial v}}\left(\frac{f_C^2}{W}\frac{f_S}{h_R}\frac{\partial W}{\partial v}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\frac{f_S}{h_R}\frac{h_R^2}{W}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\frac{\partial W}{\partial v}\right)\right\}=0 \quad , \tag{B126}$$

Still using (17) in the first term (particularly for μf_c^2 and combining the first two parts of the first term), we arrive at

$$\frac{\partial}{\partial R}\left\{\frac{\partial W}{\partial v}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial v}}\right)\left(f_S(1+\mu)h_R\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+\mu f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$
$$R\frac{\partial}{\partial v}\left\{\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(\frac{f_C^2}{W}\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right]+f_S\frac{h_R}{W}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}=0 \quad , \tag{B127}$$

The parts containing the square root can be put together and then

$$\frac{\partial}{\partial R}\left\{\frac{\partial W}{\partial v}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial v}}\right)\left(f_S(1+\mu)h_R\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+(\mu f_S h_R-f_S(1+\mu)h_R)\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$
$$R\frac{\partial}{\partial v}\left\{\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(\frac{f_C^2}{W}\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\left(f_S\frac{h_R}{W}-\frac{f_C^2}{W}\frac{f_S}{h_R}\right)\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}=0 \quad , \tag{B128}$$

After some algebra, we have

$$\frac{\partial}{\partial R}\left\{\frac{\partial W}{\partial v}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial v}}\right)\left(f_S(1+\mu)h_R\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$
$$R\frac{\partial}{\partial v}\left\{\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(\frac{f_C^2}{W}\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{h_R^2-f_C^2}{Wh_R}f_S\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}=0 \quad . \tag{B129}$$

When we employ (A18) from **Appendix A** on the relation of the generalized trigonometric functions and the metric scale factor, $h_R^2-f_C^2=\frac{f_S^2}{1+\mu}$, we obtain

$$\frac{\partial}{\partial R}\left\{\frac{\partial W}{\partial v}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial v}}\right)\left(f_S(1+\mu)h_R\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$
$$R\frac{\partial}{\partial v}\left\{\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(\frac{1-f_S^2}{W}\frac{f_S}{h_R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{1}{(1+\mu)}\frac{f_S^2}{W}\frac{f_S}{h_R}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}=0, \tag{B130}$$

and further

$$\frac{\partial}{\partial R}\left\{\frac{\partial W}{\partial v}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial v}}\right)f_S h_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+R\frac{\partial}{\partial v}\left\{\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(\frac{1}{W}\frac{f_S}{h_R}\right)\left((1-\right.\right.$$
$$\left.\left.f_S^2)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}=0 \quad . \tag{B131}$$

Now, we can start with the outer derivatives using the product rule:

$$\left\{\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial \nu}\right)\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)f_S h_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{\partial W}{\partial \nu}\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)f_S h_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left(\frac{\partial f_S}{\partial R}h_R+f_S\frac{\partial h_R}{\partial R}\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)f_S h_R\left((1+\mu)\frac{\partial}{\partial R}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\frac{\partial}{\partial R}\left(\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right)\right\}+$$
$$R\left\{\frac{\partial}{\partial \nu}\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\left(\frac{1}{W}\frac{f_S}{h_R}\right)\left(\left(1-f_S^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\frac{\partial}{\partial \nu}\left(\frac{1}{W}\frac{f_S}{h_R}\right)\left(\left(1-f_S^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{\Im}{h_\nu^2}\frac{\partial W}{\partial \nu}\right)\left(\frac{1}{W}\frac{f_S}{h_R}\right)\left(\frac{\partial}{\partial W}\left(1-f_S^2\right)\frac{\partial W}{\partial \nu}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\left(1-f_S^2\right)\frac{d}{dW}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\frac{\partial W}{\partial \nu}+\frac{d}{dW}\left(\frac{f_S^2}{1+\mu}\right)\frac{\partial W}{\partial \nu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}+\left(\frac{f_S^2}{1+\mu}\right)\frac{d}{dW}\left(\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\frac{\partial W}{\partial \nu}\right)\right\}=0\ , \qquad \text{(B132)}$$

The derivative of $\sinh^{-1}$ function is (assuming f_S/f_C and μ are positive and also thanks to (A91) from **Appendix A**)

$$\frac{d}{dW}\left\{\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right\}=\frac{d}{d(f_C/f_S)}\left\{\sinh^{-1}\left(\frac{1}{(1+\mu)}\left(\frac{f_C}{f_S}\right)^{-1}\right)\right\}\frac{d(f_C/f_S)}{dW}=-\frac{1}{\frac{f_C}{f_S}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left(-\frac{h_R^2}{W}\frac{f_C}{f_S}\right)=\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\ , \qquad \text{(B133)}$$

The derivative of the square root function is (assuming f_S/f_C and μ are positive and also thanks to (A91) from **Appendix A**)

$$\frac{d}{dW}\left\{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right\}=\frac{d}{d(f_C/f_S)}\left\{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right\}\frac{d(f_C/f_S)}{dW}=\frac{(1+\mu)^2\frac{f_C}{f_S}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left(-\frac{h_R^2}{W}\frac{f_C}{f_S}\right)=-\frac{(1+\mu)^2\frac{f_C^2}{f_S^2}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\ . \qquad \text{(B134)}$$

We can now insert these relations for derivatives, (B133) and (B134), into (B132). Further, we can use the relations (A96) and (A98) from **Appendix A** for the derivatives containing Jacobian, and also the relations (A78), (A93) from **Appendix A**:

$$\left\{\frac{\partial}{\partial \nu}\left[\frac{\partial W}{\partial R}\right]\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)f_S h_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{\partial W}{\partial \nu}\frac{\Im}{R\frac{\partial W}{\partial \nu}}\left\{(\mu+3)-\frac{1+\mu}{h_R^2}\right\}f_S h_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)\left(\frac{1}{W}h_R^2 f_S f_C^2\frac{\partial W}{\partial R}h_R-f_S\frac{\mu}{1+\mu}\frac{h_R}{W}f_S^2 f_C^2\frac{\partial W}{\partial R}\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{\partial W}{\partial \nu}\left(\frac{\Im}{h_R^2\frac{\partial W}{\partial \nu}}\right)f_S h_R\left((1+\right.\right.$$

$$\mu)\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\frac{\partial W}{\partial R}+\frac{(1+\mu)^2\frac{f_C^2}{f_S^2}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\frac{\partial W}{\partial R}\Bigg)\Bigg)\Bigg\}+R\Bigg\{\frac{f_S^2}{W}\frac{\Im}{h_v^2}\left(\frac{\partial W}{\partial v}\right)^2\left(\frac{1}{W}\frac{f_S}{h_R}\right)\Bigg(\left(1-f_S^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+$$

$$\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\Bigg)+$$

$$\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(-\frac{1}{W^2}\frac{f_S}{h_R}+\frac{1}{W}\frac{\partial\left(\frac{f_S}{h_R}\right)}{\partial W}\right)\frac{\partial W}{\partial v}\left(\left(1-f_S^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+$$

$$\left(\frac{\Im}{h_v^2}\frac{\partial W}{\partial v}\right)\left(\frac{1}{W}\frac{f_S}{h_R}\right)\Bigg(\left(-\frac{2}{W}h_R^2f_S^2f_C^2\right)\frac{\partial W}{\partial v}\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\left(1-f_S^2\right)\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\frac{\partial W}{\partial v}+$$

$$\frac{1}{1+\mu}\frac{2}{W}h_R^2f_S^2f_C^2\frac{\partial W}{\partial v}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}-\left(\frac{f_S^2}{1+\mu}\right)\frac{(1+\mu)^2\frac{f_C^2}{f_S^2}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\frac{\partial W}{\partial v}\Bigg)\Bigg\}=0 \quad . \tag{B135}$$

Further use (10) for the W derivative with respect to R, and then simplify (e.g. Jacobian can be canceled out as it is in all the terms, $\frac{\partial W}{\partial v}$ can be mutually cancelled in the denominator and numerator of some terms):

$$\Bigg\{\frac{\partial}{\partial v}\left[\frac{\mu}{R}W\right]\left(\frac{1}{h_R^2\frac{\partial W}{\partial v}}\right)f_Sh_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+$$

$$\frac{1}{R}\left\{(\mu+3)-\frac{1+\mu}{h_R^2}\right\}f_Sh_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{1}{h_R^2}\right)\left(\frac{1}{W}h_R^2f_Sf_C^2\frac{\mu}{R}Wh_R-\right.$$

$$\left.f_S\frac{\mu}{1+\mu}\frac{h_R}{W}f_S^2f_C^2\frac{\mu}{R}W\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+$$

$$\left(\frac{f_S}{h_R}\right)\left((1+\mu)\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\frac{\mu}{R}W+\frac{(1+\mu)^2\frac{f_C^2}{f_S^2}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\frac{h_R^2}{W}\frac{\mu}{R}W\right)\Bigg\}+$$

$$R\Bigg\{\frac{f_S^2}{W}\frac{1}{h_v^2}\left(\frac{\partial W}{\partial v}\right)^2\left(\frac{1}{W}\frac{f_S}{h_R}\right)\left(\left(1-f_S^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{1}{h_v^2}\left(\frac{\partial W}{\partial v}\right)^2\right)\left(-\frac{1}{W^2}\frac{f_S}{h_R}+\right.$$

$$\left.\frac{1}{W}\frac{f_C^2}{W}\frac{f_S}{h_R}\right)\left(\left(1-f_S^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+$$

$$\left(\frac{1}{h_v^2}\left(\frac{\partial W}{\partial v}\right)^2\right)\frac{h_R f_S}{W^2}\left(\left(-2f_S^2 f_C^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\left(1-f_S^2\right)\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}+\frac{1}{1+\mu}2f_S^2 f_C^2\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}-\left(\frac{f_S^2}{1+\mu}\right)\frac{(1+\mu)^2\frac{f_C^2}{f_S^2}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\right)\Bigg\}=0 \quad , \tag{B136}$$

Further cancellation of terms in numerators and denominators, and the use of (A38) relation from **Appendix A** rewritten to the form

$$\frac{1}{h_v^2}\left(\frac{\partial W}{\partial v}\right)^2=\frac{W^2}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2 f_S^2} \tag{B137}$$

for the terms containing h_v leads to

$$\Bigg\{\frac{\mu}{R}\left(\frac{f_S}{h_R}\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\frac{1}{R}\left\{(\mu+3)-\frac{1+\mu}{h_R^2}\right\}f_S h_R\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{f_S}{h_R}\right)f_C^2\frac{\mu}{R}\left(h_R^2-\frac{\mu}{1+\mu}f_S^2\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+h_R f_S\frac{\mu}{R}\left(\frac{(1+\mu)}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}+\frac{(1+\mu)^2\frac{f_C^2}{f_S^2}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\right)\Bigg\}+$$

$$R\Bigg\{\frac{1}{R^2}\frac{(1+\mu)^2}{f_C^2}h_R f_S\left(f_C^2\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)-\frac{1}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2 f_S^2}\frac{f_S}{h_R}f_S^2\left(f_C^2\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{1+\mu}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{1}{R^2}\frac{h_R^2(1+\mu)^2}{f_C^2 f_S^2}\right)h_R f_S\left(\left(-2f_S^2 f_C^2\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+f_C^2\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}+\frac{1}{1+\mu}2f_S^2 f_C^2\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}-\frac{(1+\mu)f_C^2}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\right)\Bigg\}=0 \quad . \tag{B138}$$

Cancel mutually all R and do further simplifications:

$$\left\{\mu\left(\frac{f_S}{h_R}\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left\{(\mu+3)f_S h_R-(1+\mu)\frac{f_S}{h_R}\right\}\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(\frac{f_S}{h_R}\right)f_C^2\mu(2h_R^2-1)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+h_R f_S\mu\left(\frac{\mu}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}+\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$

$$\left\{\left((1+\mu)^2 h_R f_S\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+\frac{f_S^2}{f_C^2}(1+\mu)h_R f_S\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)-\left((1+\mu)^2 h_R f_S\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+(1+\mu)h_R f_S\frac{f_S^2}{f_C^2}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left((-2)(1+\mu)^2 h_R^2 h_R f_S\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\mu h_R^2(1+\mu)^2\frac{h_R}{f_S}\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}+2(1+\mu)h_R^2 h_R f_S\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}=0 \quad . \qquad \text{(B139)}$$

The two before-last terms (in the green part) mutually cancel, and we expand the other parts:

$$\left\{\left((1+\mu)\mu\left(\frac{f_S}{h_R}\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\mu\left(\frac{f_S}{h_R}\right)\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+(\mu+3)f_S h_R(1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-(\mu+3)f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}-(1+\mu)\frac{f_S}{h_R}(1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]+(1+\mu)\frac{f_S}{h_R}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}+\left(2\mu h_R f_S f_C^2-\mu\left(\frac{f_2}{h_R}\right)f_C^2\right)\left((1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]-\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)+\left(h_R f_S\mu\frac{\mu}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}+h_R f_S\mu\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right\}+$$

$$\left\{(-2)(1+\mu)^2 h_R^2 h_R f_S \left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - \mu h_R^2 (1+\mu)^2 \frac{h_R}{f_S}\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}} + 2(1+\mu)h_R^2 h_R f_S\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right\} = 0 \quad , \qquad \text{(B140)}$$

Combination of two terms and further expansion of other terms leads to

$$\left\{\left((1+\mu)\mu\left(\frac{f_S}{h_R}\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - \mu\left(\frac{f_S}{h_R}\right)\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right) + (\mu+3) f_S h_R (1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - 3 f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} - (1+\mu)\frac{f_S}{h_R}(1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] + (1+\mu)\frac{f_S}{h_R}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} + \left[\left((1+\mu)2\mu h_R f_S f_C^2\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - 2\mu h_R f_S f_C^2\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right) - \left((1+\mu)\mu\left(\frac{f_S}{h_R}\right) f_C^2\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - \mu\left(\frac{f_S}{h_R}\right) f_C^2\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right)\right] + \left(h_R f_S \mu\frac{\mu}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\right)\right\} + \left\{(-2)(1+\mu)^2 h_R^2 h_R f_S\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - \mu h_R^2(1+\mu)^2\frac{h_R}{f_S}\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}} + 2(1+\mu)h_R^2 h_R f_S\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right\} = 0 \quad , \qquad \text{(B141)}$$

Now, move together the terms containing the same function (either sinh^{-1}, or square root, or square root in the denominator):

$$(1+\mu)\mu\left(\frac{f_S}{h_R}\right)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] + (\mu+3) f_S h_R (1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - (1+\mu)\frac{f_S}{h_R}(1+\mu)\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] + (1+\mu)2\mu h_R f_S f_C^2\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - (1+\mu)\mu\left(\frac{f_S}{h_R}\right) f_C^2\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] - 2(1+\mu)^2 h_R^2 h_R f_S\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right] + \left\{\left(-\mu\left(\frac{f_S}{h_R}\right)\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\right) - 3 f_S h_R\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} + (1+\mu)\frac{f_S}{h_R}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} - 2h_R f_S \mu f_C^2\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} + \left(\frac{f_S}{h_R}\right)\mu f_C^2\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}} + 2(1+\right.$$

$$\mu)h_R^2 h_R f_S\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\Bigg\} + \left\{ + \left(h_R f_S \mu \frac{\mu}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}} \right) - \mu h_R^2 (1+\mu)^2 \frac{h_R}{f_S}\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}} \right\} = 0 \quad , \tag{B142}$$

and simplify (also use (17) to substitute f_C with the term containing h_R at some occurrences):

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\Big\{ (1+\mu)\mu\left(\frac{f_S}{h_R}\right) + (\mu+3)f_S h_R(1+\mu) - (1+\mu)\frac{f_S}{h_R}(1+\mu) + (1+\mu)2\mu h_R f_S(1 - f_S^2) - (1+\mu)\mu\left(\frac{f_S}{h_R}\right)(1-f_S^2) - 2(1+\mu)^2 h_R^2 h_R f_S\Big\} + \sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\left\{\left(-\mu\left(\frac{f_S}{h_R}\right)\right) - 3f_S h_R + (1+\mu)\frac{f_S}{h_R} - 2h_R f_S\left((1+\mu)h_R^2 - 1\right) + \left(\frac{f_S}{h_R}\right)\left((1+\mu)h_R^2 - 1\right) + 2(1+\mu)h_R^2 h_R f_S\right\} + \frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left\{+(h_R f_S \mu\mu) - \mu h_R^2(1+\mu)^2\frac{h_R}{f_S}\right\} = 0 \tag{B143}$$

Further simplification leads to:

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\Big\{ (\mu+3)f_S h_R(1+\mu) - (1+\mu)\frac{f_S}{h_R} + \left(2(1+\mu)\mu h_R f_S - 2(1+\mu)\mu h_R f_S f_S^2\right) + \left(-(1+\mu)\mu\left(\frac{f_S}{h_R}\right) + (1+\mu)\mu\left(\frac{f_S}{h_R}\right)f_S^2\right) - 2(1+\mu)^2\left(1 - \frac{\mu}{1+\mu}f_S^2\right)h_R f_S\Big\} + \sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\left\{-3f_S h_R + \frac{f_S}{h_R} - \left(2(1+\mu)h_R f_S h_R^2 - 2h_R f_S\right) + \left((1+\mu)\left(\frac{f_S}{h_R}\right)h_R^2 - \left(\frac{f_S}{h_R}\right)\right) + 2(1+\mu)h_R^2 h_R f_S\right\} + \frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left\{+(h_R f_S \mu\mu) - \mu\left(1 - \frac{\mu}{1+\mu}f_S^2\right)(1+\mu)^2\frac{h_R}{f_S}\right\} = 0 \quad . \tag{B144}$$

Still more algebraic manipulations bring the following relation:

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\Big\{ (\mu+3+2\mu)f_S h_R(1+\mu) - (1+\mu)\frac{f_S}{h_R}(1+\mu) + \left(-2(1+\mu)\mu h_R f_S f_S^2\right) + \left(+(1+\mu)\mu\left(\frac{f_S}{h_R}\right)f_S^2\right) - \left(2(1+\mu)^2 h_R f_S - 2(1+\mu)\mu h_R f_S f_S^2\right)\Big\} + \sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\left\{-f_S h_R + \left((1+\mu)\left(\frac{f_S}{h_R}\right)h_R^2\right)\right\} + \frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left\{+(h_R f_S \mu\mu) - \left(\mu(1+\mu)^2\frac{h_R}{f_S} - \mu\mu(1+\mu)h_R f_S\right)\right\} = 0 \quad , \tag{B145}$$

which still simplifies to

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\Big\{ (\mu+3+2\mu-2(1+\mu))f_S h_R(1+\mu) + (1+\mu)\left(\frac{f_S}{h_R}\right)\left(\mu f_S^2 - (1+\mu)\right)\Big\} + \sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\left\{-f_S h_R + \left((1+\mu)h_R f_S\right)\right\} + \frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left\{+2\mu\mu h_R f_S - \mu(1+\mu)^2\frac{h_R}{f_S} + \mu\mu\mu h_R f_S\right\} = 0 \quad . \tag{B146}$$

Then

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\Big\{ (\mu+1)f_S h_R(1+\mu) + (1+\mu)\left(\frac{f_S}{h_R}\right)\left((1-h_R^2)(1+\mu) - (1+\mu)\right)\Big\} + \sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\left\{\mu h_R f_S\right\} + \frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\mu h_R f_S\left\{+2\mu - (1+\mu)^2\frac{1}{f_S^2} + \mu\mu\right\} = 0 \quad . \tag{B147}$$

Another step brings us to

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\left\{(\mu+1)f_S h_R(1+\mu)+(1+\mu)\left(\frac{f_S}{h_R}\right)(1+\mu)\left((1-h_R^2)-1\right)\right\}+$$

$$\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\{\mu h_R f_S\}+\frac{1}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\mu h_R f_S\left\{+2\mu-(1+\mu)^2\frac{1}{f_S^2}+\mu\mu\right\}=0 \quad . \tag{B148}$$

When we put the last two terms to a common denominator, we have

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\left\{(\mu+1)f_S h_R(1+\mu)+(1+\mu)\left(\frac{f_S}{h_R}\right)(1+\mu)\left((1-h_R^2)-1\right)\right\}+$$

$$\mu h_R f_S\frac{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}+\left\{+2\mu-(1+\mu)^2\frac{1}{f_S^2}+\mu\mu\right\}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}=0 \quad , \tag{B149}$$

Some more algebraic manipulation leads to

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\left\{(\mu+1)f_S h_R(1+\mu)-(1+\mu)\left(\frac{f_S}{h_R}\right)(1+\mu)(h_R^2)\right\}+$$

$$\mu h_R f_S\frac{1+(1+\mu)^2\frac{f_C^2}{f_S^2}+\left\{2\mu-(1+\mu)^2\frac{1}{f_S^2}+\mu\mu\right\}}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}=0 \quad , \tag{B150}$$

which clearly shows that the factor by $\sinh^{-1}$ is zero:

$$\left[\sinh^{-1}\left(\frac{1}{(1+\mu)}\frac{f_S}{f_C}\right)\right]\{0\}+\frac{\mu h_R f_S}{\sqrt{1+(1+\mu)^2\frac{f_C^2}{f_S^2}}}\left[1+(1+\mu)^2\frac{f_C^2}{f_S^2}+\left\{2\mu-(1+\mu)^2\frac{1}{f_S^2}+\mu\mu\right\}\right]=0. \tag{B151}$$

What remains is still simplified with the basic relation of generalized trigonometric functions (16)

$$\left[1+(1+\mu)^2\frac{1-f_S^2}{f_S^2}+\left\{2\mu-(1+\mu)^2\frac{1}{f_S^2}+\mu\mu\right\}\right]=0 \quad , \tag{B152}$$

and we see that

$$1+(1+\mu)^2\frac{1}{f_S^2}-(1+\mu)^2+\left\{2\mu-(1+\mu)^2\frac{1}{f_S^2}+\mu\mu\right\}=0 \quad , \tag{B153}$$

which leads to

$$[1-(1+\mu)^2+\{2\mu+\mu\mu\}]=0 \quad , \tag{B154}$$

and thus

$$[1-(1+2\mu+\mu\mu)+\{2\mu+\mu\mu\}]=0. \tag{B155}$$

This is evidently fulfilled for any μ, and for any coordinates R, v, where the Laplace equation is valid.

Then, the solution of the Laplace equation of the second kind with K_d=1 is proved.