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## Research Article

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# Global well-posedness in a two-dimensional two-species chemotaxis-Navier-Stokes system with competitive kinetics

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## Abstract

In this article, we study the Cauchy problem for the two-species chemotaxis-Navier-Stokes system with Lotka-Volterra competitive kinetics

$$\begin{cases} u_t + (u \cdot \nabla)u = \Delta u + \nabla \mathbb{P} + (m + n)\nabla \phi, & \nabla \cdot u = 0, \\ n_t + (u \cdot \nabla)n = \Delta n - \nabla \cdot (n \nabla c) + n(1 - n - m), \\ m_t + (u \cdot \nabla)m = \Delta m - \nabla \cdot (m \nabla c) + m(1 - n - m), \\ c_t + u \cdot \nabla c = \Delta c - (n + m)c. \end{cases}$$

The system is a model that describes the dynamics of two species and comes from a problem on account of the influence of chemotaxis, the Lotka-Volterra kinetics and fluid. We prove the global-in-time existence and uniqueness of the weak solution to the system for a large class of initial data on the whole space  $\mathbb{R}^2$ .

**Keywords:** Chemotaxis; Navier-Stokes; Lotka-Volterra; Global well-posedness; Weak solution

# 1 Introduction and main results

Consider the following two species: chemotaxis-Navier-Stokes system with Lotka-Volterra competitive kinetics proposed in [6]:

$$\begin{cases} u_t + (u \cdot \nabla)u = \Delta u + \nabla \mathbb{P} + (m+n)\nabla \phi, & \nabla \cdot u = 0, & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ n_t + (u \cdot \nabla)n = \Delta n - \nabla \cdot (n\nabla c) + n(1-n-m), & & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ m_t + (u \cdot \nabla)m = \Delta m - \nabla \cdot (m\nabla c) + m(1-n-m), & & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ c_t + u \cdot \nabla c = \Delta c - (n+m)c, & & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ (u(x,0), n(x,0), m(x,0), c(x,0)) = (u_0, n_0, m_0, c_0), & & \text{in } \mathbb{R}^2. \end{cases} \quad (1)$$

where  $u : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$  denotes the velocity field of the fluid,  $\mathbb{P} : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$  denotes the pressure function,  $n, m : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$  stands for the densities of species, and  $c : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$  represents the chemical concentration. The time-independent function  $\phi = \phi(x)$  denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force.

In conclusion, we first obtain the following chemotaxis-fluid coupled model proposed in [23]:

$$\begin{cases} u_t + \kappa u \cdot \nabla u - \Delta u + \nabla \mathbb{P} = -n\nabla \phi, & \nabla \cdot u = 0, \\ n_t + u \cdot \nabla n - \Delta n = -\nabla \cdot (\chi(c)n\nabla c), \\ c_t + u \cdot \nabla c - \Delta c = -f(c)n. \end{cases} \quad (2)$$

The system (2) describes a biological process in which the swimming bacteria move toward higher concentrations of oxygen according to the mechanism of chemotaxis, and the movement of fluid is under the influence of the gravitational force generated by the bacteria themselves. Both the oxygen concentration and bacterial density are transported by the fluid and diffuse through the fluid. In recent years, the model (2) has been extensively studied by many authors. Duan, Lorz and Markowich [3] constructed the global existence of weak solutions to the Cauchy problem in two spatial dimensions,  $\mathbb{R}^2$ , Liu and Lorz [8] removed the smallness assumption and obtained the global existence of weak solutions with large data in  $\mathbb{R}^2$ . In bounded convex domains of  $\mathbb{R}^2$ , Winkler [28] proved that the initial-boundary value problem of (2) possesses unique global classical solutions. Lorz [9] studied local-in-time weak solutions in a bounded domain in  $\mathbb{R}^d, d = 2, 3$  with no-flux boundary conditions and in  $\mathbb{R}^2$  with inhomogeneous Dirichlet conditions for oxygen. Zhang and Zheng [31] showed that there exist global weak solutions to the Cauchy problem of (4) in  $\mathbb{R}^2$  for a large class of initial data. For other interesting results, one should also refer to [19–21, 24–27].

Meanwhile, equation (1) consists of a chemotaxis-Navier-Stokes equation, which is known as a challenging model, and a two-species chemotaxis system with Lotka-Volterra competitive kinetics, which has a complicated form. Here, chemotaxis is the biased migration of cells toward more favorable environmental conditions, e.g., higher concentration of oxygen, which plays an outstanding role in a large range of biological applications [5]. The

equation has received considerable attention in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  in the last few years. In a one-species context ( $m = 0$ ), the global existence of weak and classical solutions have been investigated in [28–30] without a logistic source and in [7, 21, 22] with a logistic source. In a two-species context, related studies first begin with fluid-free systems with signal production to understand the influence of chemotaxis and the Lotka-Volterra kinetics. For more details, one can refer to [1, 2, 12–15, 18] and the reference therein.

For the two-species chemotaxis-fluid system with competitive terms (1), the first global existence of solutions was very recently studied by Hirata et al. [6] under homogeneous Neumann boundary conditions in a bounded domain  $\Omega \in \mathbb{R}^2$  with a smooth boundary, who focused on the global classical solution. It remains an interesting question to investigate the global-in-time well-posedness of the weak solution to system (1) on the whole space  $\mathbb{R}^2$ . To answer this question, we introduce the notion of a weak solution of (1) and establish both the existence and uniqueness of such a weak solution for a large class of initial data. The first difficulty is that (1) consists of two-species chemotaxis with competitive kinetics. The second difficulty is the effect from fluid described by the Navier-Stokes equation. We will overcome these two difficulties by using recent techniques developed by Hirata et al. [6], Winkler [28] and Zhang and Zheng [31].

We first recall the definition of weak solutions to the system (1). For  $T > 0$ , let  $Q_T = \mathbb{R}^2 \times T$ . For initial data, we say that  $(n, m, c, u)$  is a weak solution of system (1) if the following conditions hold:

(i)  $n(x, t) > 0, m(x, t) > 0, c(x, t) > 0, t \geq 0, x \in \mathbb{R}^2$ , and for any  $T > 0$ ,

$$\begin{cases} (m, n, c, u) \in C(\mathbb{R}^+; L^2(\mathbb{R}^2)) \times C(\mathbb{R}^+; L^2(\mathbb{R}^2)) \times C(\mathbb{R}^+; H^1(\mathbb{R}^2)) \times C(\mathbb{R}^+; L^2(\mathbb{R}^2)), \\ (m, n) \in (L^\infty([0, T]; L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^2([0, T]; H^1(\mathbb{R}^2)))^2, \\ c \in L^\infty([0, T]; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)) \cap L^2([0, T]; H^2(\mathbb{R}^2)), \\ u \in L^\infty([0, T]; L^2(\mathbb{R}^2)) \cap L^2([0, T]; H^1(\mathbb{R}^2)). \end{cases}$$

(v) Moreover, for any  $\psi \in C_0^\infty(\mathbb{R}^2; [0, T])$ ,

$$\begin{aligned} \int_{Q_T} n_t \psi dxdt + \int_{Q_T} (u \nabla n - n(1 - n - m)) \psi dxdt &= \int_{Q_T} (n \nabla c - \nabla n) \nabla \psi dxdt, \\ \int_{Q_T} m_t \psi dxdt + \int_{Q_T} (u \nabla m - m(1 - m - n)) \psi dxdt &= \int_{Q_T} (m \nabla c - \nabla m) \nabla \psi dxdt, \\ \int_{Q_T} c_t \psi dxdt + \int_{Q_T} (u \nabla c + (n + m)c) \psi dxdt &= - \int_{Q_T} \nabla c \nabla \psi dxdt, \end{aligned}$$

and for any  $\tilde{\psi} \in C_0^\infty(\mathbb{R}^2; [0, T])$ ,

$$\int_{Q_T} u_t \tilde{\psi} dxdt + \int_{Q_T} (u \nabla u - (m + n) \nabla \phi) \tilde{\psi} dxdt = - \int_{Q_T} \nabla u \nabla \tilde{\psi} dxdt$$

as well as

$$\int_{Q_T} \nabla \tilde{\psi} \cdot u dxdt = 0.$$

If  $(m, n, c, u)$  is a weak solution of system (1) in  $\mathbb{R}^2 \times (0, T)$  for all  $T > 0$ , then  $(m, n, c, u)$  is called a global-in-time weak solution.

Before stating our results, we introduce two notations:

$$X_0 = \{n_0 \in L^2(\mathbb{R}^2), m_0 \in L^2(\mathbb{R}^2), \sqrt{1+|x|^2}n_0 \in L^1(\mathbb{R}^2), \sqrt{1+|x|^2}m_0 \in L^1(\mathbb{R}^2), \\ u_0 \in L^2(\mathbb{R}^2), \nabla \sqrt{c_0} \in L^2(\mathbb{R}^2), c_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), n_0 > 0, m_0 > 0, c_0 > 0\}. \quad (3)$$

Now, we state our main theorem.

**Theorem 1.1.** *Let  $(u_0, m_0, n_0, c_0) \in X_0$  satisfy (3); then the solution of (1) possesses a unique global-in-time weak solution*

$$(m, n, c, u) \in C(\mathbb{R}^+; L^2(\mathbb{R}^2)) \times C(\mathbb{R}^+; L^2(\mathbb{R}^2)) \times C(\mathbb{R}^+; H^1(\mathbb{R}^2)) \times C(\mathbb{R}^+; L^2(\mathbb{R}^2))$$

satisfying

$$(m, n) \in (L_{loc}^\infty(\mathbb{R}^+; L_{loc}^2(\mathbb{R}^2) \cap L_{loc}^1(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)))^2, \\ c \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^2(\mathbb{R}^2)), \\ u \in L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)).$$

**Remarks.**

(i) Our results may be generalized to a bounded domain by slightly modifying our proof and adding reasonable boundary conditions. Compared with [6], we can obtain the existence and uniqueness of weak solutions to (1) in the whole space  $\mathbb{R}^2$ .

(ii) It seems to be impossible to obtain a global estimate of  $n(t), m(t) \in L^2(\mathbb{R}^2)$  with aid of the energy method directly. To overcome this difficulty, by analyzing the structure of the equations, we establish the estimates of  $\int_{\mathbb{R}^2} n^\epsilon |\ln n^\epsilon| dx$  and  $\int_{\mathbb{R}^2} m^\epsilon |\ln m^\epsilon| dx$ .

(iii) The quadratic degradation in the equations of  $n$  and  $m$  to system (1) is a key point to prove the global-in-time regularity, which can provide a damping effect when the diffusion terms  $-\Delta n$  and  $-\Delta m$  do not prove the smoothing effect in the sense of  $L^1(\mathbb{R}^2)$ .

**Notation.** We adopt the convention that the nonessential constant  $C$  may change from line to line, and  $C_{a_1, a_2, \dots, a_k}$  means a constant  $C$  depending on  $a_1, a_2, \dots, a_k$ . Given two quantities  $A$  and  $B$ , we denote  $B \lesssim A$  as  $B \leq CA$ . We often label  $\|(a, b)\|_X = \|a\|_X + \|b\|_X$ .

## 2 Preliminaries

Before going further, we recall the functional spaces we are going to use. Let  $S(\mathbb{R}^2)$  be the Schwartz class of rapidly decreasing functions.  $S'(\mathbb{R}^2)$  is the space of tempered distributions. Here,  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote Fourier and inverse Fourier transforms of  $L^1(\mathbb{R}^2)$  functions, respectively, which are defined by

$$\mathcal{F}f = \hat{f}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}^{-1}f = \check{f}(x) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \hat{f}(\xi) d\xi.$$

More generally, the Fourier transform of any  $f \in S'(\mathbb{R}^2)$  is given by  $(\mathcal{F}f, g) = (f, \mathcal{F}g)$  for any  $g \in S(\mathbb{R}^2)$ . Let  $C$  be the annulus  $\{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  such that

$$\forall \xi \in \mathbb{R}^2 \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1.$$

For  $f \in S'(\mathbb{R}^2)$ , the inhomogeneous dyadic blocks are defined by

$$\Delta_{-1}f := \chi(D)f; \quad \Delta_jf := \varphi(2^{-j}D)f \quad \text{if } j \in \mathbb{N},$$

where

$$\chi(\xi) = 1 - \sum_{j \in \mathbb{N}} \varphi(2^{-j}\xi).$$

Formally,  $S_jf = \sum_{-1 \leq j' \leq j-1} \Delta_{j'}f$  is a low-frequency cutoff and  $\Delta_j = S_j - S_{j-1}$ . Recall that for any  $s \in \mathbb{R}$  and  $(p, r) \in [1, \infty] \times [1, \infty]$ , the inhomogeneous Besov space  $B_{p,r}^s(\mathbb{R}^2)$  is defined by

$$\|f\|_{B_{p,r}^s(\mathbb{R}^2)} := \left( \sum_{k \geq -1} 2^{rks} \|\Delta_k f\|_{L^p(\mathbb{R}^2)}^r \right)^{\frac{1}{r}} < \infty.$$

It is worthwhile to remark that  $B_{2,2}^s(\mathbb{R}^2)$  coincide with the usual Sobolev spaces  $H^s(\mathbb{R}^2)$ .

In the following, we recall some useful inequalities and technical results that will be of interest..

**Lemma 2.1** (Hajer et al. [4]). *Let  $C$  be an annulus and  $\mathcal{B}$  be a ball in  $\mathbb{R}^2$ . There exists a positive constant  $C$  such that for any nonnegative integer  $k$ , any couple  $(p, q) \in [1, \infty]^2$  with  $q \geq p \geq 1$ , and any function  $u$  of  $L^p(\mathbb{R}^2)$ ,*

(i) *if  $\text{supp } \hat{u} \subseteq \lambda\mathcal{B}$ , then*

$$\|D^k u\|_{L^q(\mathbb{R}^2)} := \sup_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^q(\mathbb{R}^2)} \leq C^{k+1} \lambda^{k+2(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p(\mathbb{R}^2)};$$

(ii) *if  $\text{supp } \hat{u} \subseteq \lambda C$ , then*

$$C^{-k-1} \lambda^k \|u\|_{L^p(\mathbb{R}^2)} \leq \|D^k u\|_{L^p(\mathbb{R}^2)} \leq C^{k+1} \lambda^k \|u\|_{L^p(\mathbb{R}^2)}.$$

**Lemma 2.2** (Hajer et al. [4]). *Let  $s > 0$ . Then, there exists a constant  $C > 0$  such that for all  $u, v \in L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$ ,*

$$\|uv\|_{H^s(\mathbb{R}^2)} \leq C(\|v\|_{H^s(\mathbb{R}^2)}\|u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{H^s(\mathbb{R}^2)}\|v\|_{L^\infty(\mathbb{R}^2)}).$$

**Lemma 2.3** (Aubin–Lions [16, 17]). *Let  $1 < \alpha < \infty$ ,  $1 \leq \beta \leq \infty$  and  $X_0, X_1, X_2$  be reflexive separable Banach spaces such that  $X_0 \hookrightarrow X_1$  and  $X_1 \hookrightarrow X_2$ . Then,*

$$\{u \in L^\alpha(0, T; X_0); \partial_t u \in L^\beta(0, T; X_2)\} \hookrightarrow L^\alpha(0, T; X_1).$$

### 3 Global-in-time well-posedness for a regular problem

Let  $\rho(x) \in C_0^\infty(\mathbb{R}^2)$  be a nonnegative function, vanishing outside the unit ball  $B_1(0) = \{x \in \mathbb{R}^2; |x| < 1\}$  and satisfying  $\int_{\mathbb{R}^2} \rho(x) dx = 1$ . A typical example is

$$\rho(x) := \frac{1}{A} e^{1/(|x|^2-1)}, \quad |x| < 1; \quad \rho(x) := 0, \quad |x| \geq 1,$$

where  $A = \int_{B_1(0)} e^{1/(|x|^2-1)} dx$ . For any  $\epsilon > 0$ ,  $\rho^\epsilon$  denotes the standard mollifier given by  $\rho^\epsilon(x) = \epsilon^{-2} \rho(\epsilon^{-1}x)$ , which vanishes outside the ball  $B_\epsilon(0) = \{x \in \mathbb{R}^2; |x| < \epsilon\}$  and  $\int_{\mathbb{R}^2} \rho^\epsilon(x) dx = 1$ .

To obtain the global existence of weak solutions of (1), we first consider the following regularized problem:

$$\begin{cases} n_t^\epsilon + (u^\epsilon \cdot \nabla) n^\epsilon = \Delta n^\epsilon - \nabla \cdot (n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) + n^\epsilon (1 - n^\epsilon * \rho^\epsilon - m^\epsilon * \rho^\epsilon), \\ m_t^\epsilon + (u^\epsilon \cdot \nabla) m^\epsilon = \Delta m^\epsilon - \nabla \cdot (m^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) + m^\epsilon (1 - n^\epsilon * \rho^\epsilon - m^\epsilon * \rho^\epsilon), \\ c_t^\epsilon + u^\epsilon \cdot \nabla c^\epsilon = \Delta c^\epsilon - (n^\epsilon * \rho^\epsilon + m^\epsilon * \rho^\epsilon) c^\epsilon, \\ u_t^\epsilon + (u^\epsilon \cdot \nabla) u^\epsilon = \Delta u^\epsilon + \nabla \mathbb{P}^\epsilon + ((m^\epsilon + n^\epsilon) \nabla \phi) * \rho^\epsilon, \quad \nabla \cdot u^\epsilon = 0, \\ (u_0^\epsilon, n_0^\epsilon, m_0^\epsilon, c_0^\epsilon) = (u_0 * \rho^\epsilon, n_0 * \rho^\epsilon, m_0 * \rho^\epsilon, c_0 * \rho^\epsilon). \end{cases} \quad (4)$$

For  $k \geq 1$ , let  $J_k$  be the Friedrich projection defined by

$$\widehat{J_k f}(\xi) = \chi_{[0,k]}(|\xi|) \hat{f}(\xi), \quad \xi \in \mathbb{R}^2.$$

To suitably regularize problem (4), on the basis of the space

$$H_k^s(\mathbb{R}^2) := \{f \in H^s(\mathbb{R}^2) : \text{supp } \hat{f} \subseteq B(0, k), \quad s > 1\},$$

we consider the following system:

$$\begin{cases} n_t^{k,\epsilon} + J_k(J_k u^{k,\epsilon} \cdot \nabla(J_k n^{k,\epsilon})) = \Delta J_k^2 n^{k,\epsilon} - \nabla J_k \cdot (J_k n^{k,\epsilon} \nabla (c^{k,\epsilon} * \rho^\epsilon)) \\ \quad + J_k^2 n^{k,\epsilon} - J_k(J_k n^{k,\epsilon} (n^{k,\epsilon} * \rho^\epsilon)) - J_k(J_k n^{k,\epsilon} (m^{k,\epsilon} * \rho^\epsilon)), \\ m_t^{k,\epsilon} + J_k(J_k u^{k,\epsilon} \cdot \nabla(J_k m^{k,\epsilon})) = \Delta J_k^2 m^{k,\epsilon} - \nabla J_k \cdot (J_k m^{k,\epsilon} \nabla (c^{k,\epsilon} * \rho^\epsilon)) \\ \quad + J_k^2 m^{k,\epsilon} - a_2 \mu_2 J_k(J_k m^{k,\epsilon} (n^{k,\epsilon} * \rho^\epsilon)) - J_k(J_k m^{k,\epsilon} (m^{k,\epsilon} * \rho^\epsilon)), \\ c_t^{k,\epsilon} + J_k(J_k u^{k,\epsilon} \cdot \nabla(J_k c^{k,\epsilon})) = \Delta J_k^2 c^{k,\epsilon} - J_k(J_k c^{k,\epsilon} (n^{k,\epsilon} * \rho^\epsilon + m^{k,\epsilon} * \rho^\epsilon)), \\ u_t^{k,\epsilon} + J_k(J_k u^{k,\epsilon} \cdot \nabla(J_k u^{k,\epsilon})) = J_k^2 u^{k,\epsilon} + \nabla J_k \mathbb{P}^{k,\epsilon} + J_k(((m^{k,\epsilon} + n^{k,\epsilon}) \nabla \phi) * \rho^\epsilon), \\ \nabla \cdot u^{k,\epsilon} = 0, \quad (u^{k,\epsilon}(x, 0), n^{k,\epsilon}(x, 0), m^{k,\epsilon}(x, 0), c^\epsilon(x, 0)) = (J_k u_0^\epsilon, J_k n_0^\epsilon, J_k m_0^\epsilon, J_k c_0^\epsilon). \end{cases} \quad (5)$$

Let

$$X^{k,\epsilon}(t) := (n^{k,\epsilon}(t), m^{k,\epsilon}(t), c^{k,\epsilon}(t), u^{k,\epsilon}(t))^T, \quad F^{k,\epsilon}(X^{k,\epsilon}) := (F_1^{k,\epsilon}, F_2^{k,\epsilon}, F_3^{k,\epsilon}, F_4^{k,\epsilon})^T.$$

Then, the approximating system (5) can be changed into the following ODE:

$$\frac{d}{dt}X^{k,\epsilon} = F^{k,\epsilon}(X^{k,\epsilon}). \quad (6)$$

For fixed  $s > 1$ , we denote  $O$  and  $O^M$  by

$$O := \{(n, m, c, u) \in (H^s(\mathbb{R}^2))^4, \nabla \cdot u = 0\}$$

and

$$O^M := \{(n, m, c, u) \in O : \|(n, m, c, u)\|_{H^s(\mathbb{R}^2)} \leq M, M > 0\}.$$

Now, we apply the Picard theorem to show the global-in-time existence and uniqueness of the smooth solution to system (5) for  $k \in \mathbb{N}^+$  and  $\epsilon > 0$ .

**Lemma 3.1.** *Let  $\phi \in \dot{W}^{1,\infty}(\mathbb{R}^2)$ ,  $(n_0^\epsilon, m_0^\epsilon, c_0^\epsilon, u_0^\epsilon) \in (H^s(\mathbb{R}^2))^4$  with  $s > 1$  and  $n_0^\epsilon, m_0^\epsilon, c_0^\epsilon > 0$ . Then, for fixed  $k \in \mathbb{N}^+$  and  $\epsilon > 0$ , system (5) has a unique global smooth solution  $(n^{k,\epsilon}, m^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon}) \in (C(\mathbb{R}^+; H^s(\mathbb{R}^2)))^4$ .*

*Proof.* Since  $s > 0$  can be any positive real number,  $(m^{k,\epsilon}, n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})$  is a classical solution to (5). Thanks to the Cauchy-Lipschitz theorem, we need to show that  $F^{k,\epsilon}$  is locally Lipschitz continuous. Let

$$X_1^{k,\epsilon} = (n_1^{k,\epsilon}, m_1^{k,\epsilon}, c_1^{k,\epsilon}, u_1^{k,\epsilon}) \in O, \quad X_2^{k,\epsilon} = (n_2^{k,\epsilon}, m_2^{k,\epsilon}, c_2^{k,\epsilon}, u_2^{k,\epsilon}) \in O.$$

For simplicity, we label the terms

$$\begin{aligned} I_1 &= \Delta J_k^2(n_1^{k,\epsilon} - n_2^{k,\epsilon}) + J_k^2(n_1^{k,\epsilon} - n_2^{k,\epsilon}), \\ I_2 &= -J_k(J_k(u_1^{k,\epsilon} - u_2^{k,\epsilon}) \cdot \nabla(J_k n_1^{k,\epsilon})) - J_k(J_k u_2^{k,\epsilon} \cdot \nabla(J_k n_1^{k,\epsilon} - J_k n_2^{k,\epsilon})), \\ I_3 &= -\nabla \cdot J_k(J_k(n_1^{k,\epsilon} - n_2^{k,\epsilon})\nabla(c_1^{k,\epsilon} * \rho^\epsilon)) - \nabla \cdot J_k(J_k n_2^{k,\epsilon}\nabla((c_1^{k,\epsilon} - c_2^{k,\epsilon}) * \rho^\epsilon)), \\ I_4 &= -J_k(J_k(n_1^{k,\epsilon} - n_2^{k,\epsilon})(n_1^{k,\epsilon} * \rho^\epsilon)) - J_k(J_k n_2^{k,\epsilon}((n_1^{k,\epsilon} - n_2^{k,\epsilon}) * \rho^\epsilon)), \\ I_5 &= -J_k(J_k(u_1^{k,\epsilon} - u_2^{k,\epsilon})(u_1^{k,\epsilon} * \rho^\epsilon)) - J_k(J_k u_2^{k,\epsilon}((u_1^{k,\epsilon} - u_2^{k,\epsilon}) * \rho^\epsilon)) \end{aligned}$$

and

$$I_5 = -\nabla \cdot J_k(J_k(n_1^{k,\epsilon} - n_2^{k,\epsilon})\nabla(m_1^{k,\epsilon} * \rho^\epsilon)) - \nabla \cdot J_k(J_k n_2^{k,\epsilon}\nabla((m_1^{k,\epsilon} - m_2^{k,\epsilon}) * \rho^\epsilon)).$$

By (6), a simple calculation yields that

$$F_1^{k,\epsilon}(X_1^{k,\epsilon}) - F_1^{k,\epsilon}(X_2^{k,\epsilon}) := I_1 + I_2 + I_3 + I_4 + I_5. \quad (7)$$

Next, we separately compute  $I_j$  for  $j = 1, 2, 3, 4, 5$ . Taking advantage of Lemma 2.1, we have

$$\begin{aligned} \|I_1\|_{H^s(\mathbb{R}^2)} &\leq \|\Delta J_k^2(n_1^{k,\epsilon} - n_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} + \|J_k^2(n_1^{k,\epsilon} - n_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} \\ &\leq C(k^2 + 1)\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)} \leq C_k\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}. \end{aligned} \quad (8)$$

On the other hand, applying Lemma 2.1, we observe that

$$\begin{aligned}
\|I_2\|_{H^s(\mathbb{R}^2)} &\leq \left( \|J_k(u_1^{k,\epsilon} - u_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} + \|J_k(u_1^{k,\epsilon} - u_2^{k,\epsilon})\|_{L^\infty(\mathbb{R}^2)} \right) \\
&\quad \times \left( \|\nabla(J_k n_1^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} + \|\nabla(J_k n_1^{k,\epsilon})\|_{L^\infty(\mathbb{R}^2)} \right) \\
&\quad + \left( \|J_k u_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)} + \|J_k u_2^{k,\epsilon}\|_{L^\infty(\mathbb{R}^2)} \right) \\
&\quad \times \left( \|\nabla(J_k n_1^{k,\epsilon} - J_k n_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} + \|\nabla(J_k n_1^{k,\epsilon} - J_k n_2^{k,\epsilon})\|_{L^\infty(\mathbb{R}^2)} \right) \\
&\leq C(1+k)k^{3+s}\|u_1^{k,\epsilon} - u_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|n_1^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\quad + C(k^2+k)(k^s+k)\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|u_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\leq C_k\|(u_1^{k,\epsilon} - u_2^{k,\epsilon}, n_1^{k,\epsilon} - n_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)}\|(n_1^{k,\epsilon}, u_2^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{9}$$

The terms  $I_3$ ,  $I_4$  and  $I_5$  can be treated similarly. Indeed, by Lemmas 2.1 and 2.2, one can deduce that

$$\begin{aligned}
\|I_3\|_{H^s(\mathbb{R}^3)} &\leq C(k^2+k)(\epsilon^{-2} + \epsilon^{-1-s})\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|c_1^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\quad + C(k^2+k^{s+1})(\epsilon^{-1} + \epsilon^{-2})\|c_1^{k,\epsilon} - c_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|n_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\leq C_{k,\epsilon}\|(n_1^{k,\epsilon} - n_2^{k,\epsilon}, c_1^{k,\epsilon} - c_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)}\|(c_1^{k,\epsilon}, n_2^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\|I_4\|_{H^s(\mathbb{R}^3)} &\leq C(1+k)(\epsilon^{-1} + \epsilon^{-s})\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|n_1^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\quad + C(k+k^s)(\epsilon^{-1} + 1)\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|n_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\leq C_{k,\epsilon}\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|(n_1^{k,\epsilon}, n_2^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{11}$$

For  $I_5$ , we similarly have that

$$\begin{aligned}
\|I_5\|_{H^s(\mathbb{R}^3)} &\leq C(k+k^2)(\epsilon^{-2} + \epsilon^{-1-s})\|n_1^{k,\epsilon} - n_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|m_1^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\quad + C(k^2+k^{s+1})(\epsilon^{-1} + \epsilon^{-2})\|m_1^{k,\epsilon} - m_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}\|n_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
&\leq C_{k,\epsilon}\|(n_1^{k,\epsilon} - n_2^{k,\epsilon}, m_1^{k,\epsilon} - m_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)}\|(m_1^{k,\epsilon}, n_2^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{12}$$

Hence, summing up (8)–(12), we conclude from (7) that

$$\begin{aligned}
&\|F_1^{k,\epsilon}(X_1^{k,\epsilon}) - F_1^{k,\epsilon}(X_2^{k,\epsilon})\|_{H^s(\mathbb{R}^3)} \\
&\leq C_{k,\epsilon}\|X_1^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}, \|X_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|X_1^{k,\epsilon} - X_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}.
\end{aligned} \tag{13}$$

Imitating the  $H^s$ -norm of  $\|F_1^{k,\epsilon}(X_1^{k,\epsilon}) - F_1^{k,\epsilon}(X_2^{k,\epsilon})\|_{H^s(\mathbb{R}^3)}$ , we proceed similarly

$$\begin{aligned}
&\sum_{i=2}^4 \|F_i^{k,\epsilon}(X_1^{k,\epsilon}) - F_i^{k,\epsilon}(X_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} \\
&\leq C_{k,\epsilon}\|X_1^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}, \|X_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|X_1^{k,\epsilon} - X_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}.
\end{aligned} \tag{14}$$

For each  $k \geq 1$  and  $X_1^{k,\epsilon}, X_2^{k,\epsilon} \in O^M$ , applying (13) and (14) gives

$$\|F^{k,\epsilon}(X_1^{k,\epsilon}) - F^{k,\epsilon}(X_2^{k,\epsilon})\|_{H^s(\mathbb{R}^2)} \leq C_{k,\epsilon} \|X_1^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|X_2^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|X_1^{k,\epsilon} - X_2^{k,\epsilon}\|_{H^s(\mathbb{R}^2)},$$

from which we know that  $F^{k,\epsilon}$  maps  $H^s(\mathbb{R}^2)$  into  $H^s(\mathbb{R}^2)$  and  $F^{k,\epsilon}$  is locally Lipschitz continuous on any set  $O^M$ . Hence, the Picard theorem of ODEs on Banach spaces ensures that for every  $(m_0, n_0, c_0, u_0) \in (H^s(\mathbb{R}^2))^4$ , there exists a unique solution  $(m^{k,\epsilon}, n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon}) \in (O^M \cap H^s(\mathbb{R}^2))^4$  with  $T_k > 0$  being the maximal existence time.

We are left to prove that  $(u^{k,\epsilon}, m^{k,\epsilon}, n^{k,\epsilon}, c^{k,\epsilon})$  is global-in-time based on the continuation of an autonomous ODE on a Banach space. By the same argument used in (3), we easily conclude that

$$\|X^\epsilon\|_{H^s(\mathbb{R}^2)}^2 + \int_0^t \|X^\epsilon\|_{H^{s+1}(\mathbb{R}^2)}^2 ds \leq C_{t,k,\epsilon} \|X_0^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)} \|X_0^\epsilon\|_{H^s(\mathbb{R}^2)}^2,$$

for all  $t \in [0, T_k)$ . From which, similar to [11, Theorem 3.3], by invoking the continuation property of ODEs on a Banach space, we can show that  $T_k = \infty$ . Here, we omit the details.  $\square$

In the following, we are concerned with an  $H^s(\mathbb{R}^2)$  estimate of  $(m^{k,\epsilon}, n^{k,\epsilon}, c^{k,\epsilon}, u^{k,\epsilon})$  that is uniform for  $k$ .

**Lemma 3.2.** *Let  $\phi \in \dot{W}^{1,\infty}(\mathbb{R}^2)$ ,  $(n_0^\epsilon, m_0^\epsilon, c_0^\epsilon, u_0^\epsilon) \in (H^s(\mathbb{R}^2))^4$  with  $s > 1$  and  $n_0^\epsilon, m_0^\epsilon, c_0^\epsilon > 0$ . Then, for fixed  $k \in \mathbb{N}^+$  and  $\epsilon > 0$ , the system (5) possesses the following estimate:*

$$\|X^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \int_0^t \|J_k X^{k,\epsilon}\|_{H^{s+1}(\mathbb{R}^2)}^2 ds \leq C_{t,\epsilon}. \quad (15)$$

*Proof.* First, we consider the  $L^2(\mathbb{R}^2)$  estimate of  $c^{k,\epsilon}$ . Multiplying (5)<sub>3</sub> by  $c^{k,\epsilon}$  and integrating the resulting equality with respect to the spatial variable  $x$ , an application of Hölder's inequality then asserts

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\ &= - \int_{\mathbb{R}^2} (J_k c^{k,\epsilon} (n^{k,\epsilon} * \rho^\epsilon)) J_k c^{k,\epsilon} dx - \int_{\mathbb{R}^2} (J_k c^{k,\epsilon} (m^{k,\epsilon} * \rho^\epsilon)) J_k c^{k,\epsilon} dx \\ &\leq C \|c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^4 + C \|(m^{k,\epsilon}, n^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (16)$$

Since

$$\int_{\mathbb{R}^2} J_k (J_k u^{k,\epsilon} \cdot \nabla (J_k n^{k,\epsilon})) n^{k,\epsilon} dx = \frac{1}{2} \int_{\mathbb{R}^2} J_k u^{k,\epsilon} \cdot \nabla (J_k n^{k,\epsilon})^2 dx = 0,$$

from which taking the  $L^2(\mathbb{R}^2)$ -inner product with  $n^{k,\epsilon}$  for (5)<sub>1</sub> and performing integration by parts gives

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \lesssim \|J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|n^{k,\epsilon} * \rho^\epsilon\|_{L^\infty(\mathbb{R}^2)} + \|J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|m^{k,\epsilon} * \rho^\epsilon\|_{L^\infty(\mathbb{R}^2)} \\
& \quad + \|J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|\nabla(c^{k,\epsilon} * \rho^\epsilon)\|_{L^\infty(\mathbb{R}^2)} \|\nabla J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \leq C\epsilon^{-1} \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^3 + C\epsilon^{-1} \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \quad + C\epsilon^{-2} \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|\nabla J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{17}$$

Similarly, testing (5)<sub>2</sub> against  $m^{k,\epsilon}$ , a straightforward computation yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C\epsilon^{-1} \|m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^3 + C\epsilon^{-1} \|m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \quad + C\epsilon^{-2} \|m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|\nabla J_k m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{18}$$

Mimicking an energy estimate for (5)<sub>4</sub>, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C \|(m^{k,\epsilon}, n^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2 + C \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2 \|J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{19}$$

Summing up inequalities (16)–(19), we have

$$\begin{aligned}
& \frac{d}{dt} \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \left\| \left( \nabla J_k m^{k,\epsilon}, \nabla J_k n^{k,\epsilon}, \nabla J_k c^{k,\epsilon}, \nabla J_k u^{k,\epsilon} \right) \right\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C_\epsilon \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^4 + C_\epsilon \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{20}$$

Now, we turn to showing the  $H^1(\mathbb{R}^2)$ -estimate of  $n^{k,\epsilon}$  is uniform. We multiply the first equation of (5) by  $-\Delta n^{k,\epsilon}$ , and a direct calculation along with integration by parts yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \lesssim \int_{\mathbb{R}^2} \nabla \cdot (J_k n^{k,\epsilon} \nabla (c^{k,\epsilon} * \rho^\epsilon)) J_k \Delta n^{k,\epsilon} dx + \int_{\mathbb{R}^2} (\nabla J_k n^{k,\epsilon} \nabla \cdot) J_k u^{k,\epsilon} \cdot \nabla J_k n^{k,\epsilon} dx \\
& \quad + \|\nabla J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} J_k n^{k,\epsilon} (n^{k,\epsilon} * \rho^\epsilon) \Delta J_k n^{k,\epsilon} dx \\
& \quad + \int_{\mathbb{R}^2} J_k n^{k,\epsilon} (m^{k,\epsilon} * \rho^\epsilon) \Delta J_k n^{k,\epsilon} dx \\
& \lesssim (\epsilon^{-2} + \epsilon^{-3}) \|(\nabla J_k n^{k,\epsilon}, J_k n^{k,\epsilon})\|_{L^2(\mathbb{R}^2)} \|c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|J_k \Delta n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \quad + \|\nabla J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|\Delta J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \quad + \epsilon^{-1} \|J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|J_k \Delta n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \quad + \epsilon^{-1} \|J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|J_k \Delta n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \leq C_\epsilon \left\| (\nabla J_k n^{k,\epsilon}, J_k n^{k,\epsilon}) \right\|_{L^2(\mathbb{R}^2)}^2 \left( \|c^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + 1 \right) \\
& \quad + \frac{1}{2} \|\Delta J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned}$$

which implies

$$\begin{aligned}
& \frac{d}{dt} \|\nabla n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C_\epsilon \left\| (\nabla J_k n^{k,\epsilon}, J_k n^{k,\epsilon}) \right\|_{L^2(\mathbb{R}^2)}^2 \left( \|c^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + 1 \right). \tag{21}
\end{aligned}$$

Similarly, another direct calculation using the second equation in (5) yields

$$\begin{aligned}
& \frac{d}{dt} \|\nabla m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta J_k m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C_\epsilon \left\| (\nabla J_k m^{k,\epsilon}, J_k m^{k,\epsilon}) \right\|_{L^2(\mathbb{R}^2)}^2 \left( \|c^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + 1 \right). \tag{22}
\end{aligned}$$

By the third equation of (5), we can conclude that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla c^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}^2 \\
& \lesssim \|\nabla J_k c^{k,\epsilon}\|_{L^4(\mathbb{R}^2)}^2 \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \\
& \quad + \|J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|J_k \Delta c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|n^{k,\epsilon} * \rho^\epsilon + m^{k,\epsilon} * \rho^\epsilon\|_{L^\infty(\mathbb{R}^2)} \\
& \leq \frac{1}{2} \|\Delta J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}^2 + C_\epsilon \|\nabla J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\
& \quad + C_\epsilon \|J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|n^{k,\epsilon}, m^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned}$$

and hence,

$$\begin{aligned} & \frac{d}{dt} \|\nabla c^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^3)}^2 \\ & \leq C_\epsilon \|\nabla J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla J_k u^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + C_\epsilon \|J_k c^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \|(n^{k,\epsilon}, m^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (23)$$

Applying the curl to the fourth equation in (5) and multiplying the resulting equation by  $w^{k,\epsilon} = \nabla^\perp u^{k,\epsilon}$  and then integrating with respect to the spatial variable  $x$ , we obtain that

$$\frac{1}{2} \frac{d}{dt} \|w^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k w^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \leq C_\epsilon \|(n^{k,\epsilon}, m^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2 + \|w^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2,$$

which combined with (20)–(25), allows us to see that

$$\frac{d}{dt} \|X^{k,\epsilon}\|_{H^1(\mathbb{R}^2)}^2 + \|J_k X^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 \leq C_\epsilon \left( \|X^{k,\epsilon}\|_{H^1(\mathbb{R}^2)}^4 + \|X^{k,\epsilon}\|_{H^1(\mathbb{R}^2)}^2 \right). \quad (24)$$

Hence, we conclude (3)

$$\|X^{k,\epsilon}\|_{H^1(\mathbb{R}^2)}^2 + \int_0^t \|J_k X^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 d\tau \leq \frac{1 + \|X_0^{k,\epsilon}\|_{H^1(\mathbb{R}^2)}}{1 - Ct(1 + \|X_0^{k,\epsilon}\|_{H^1(\mathbb{R}^2)})}. \quad (25)$$

Next, we consider  $(u^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}, c^{k,\epsilon}) \in (H^s(\mathbb{R}^2))^4$  by using the Fourier localization technique. Applying  $\Delta_q$  to the first equation in (5) for  $q \in \mathbb{Z}$ , and multiplying both sides of the resulting equality by  $\Delta_q n^{k,\epsilon}$  and integrating with respect to  $x$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla J_k \Delta_q n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\ & = \int_{\mathbb{R}^2} \Delta_q (J_k u^{k,\epsilon} J_k n^{k,\epsilon}) \cdot \nabla \Delta_q J_k n^{k,\epsilon} dx + \int_{\mathbb{R}^2} \Delta_q (J_k n^{k,\epsilon} \nabla (c^{k,\epsilon} * \rho^\epsilon)) \cdot \nabla \Delta_q J_k n^{k,\epsilon} dx \\ & \quad + \|\Delta_q \Delta_q n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 - \int_{\mathbb{R}^2} \Delta_q (J_k n^{k,\epsilon} \nabla (n^{k,\epsilon} * \rho^\epsilon)) \cdot \Delta_q J_k n^{k,\epsilon} dx \\ & \quad - \int_{\mathbb{R}^2} \Delta_q (J_k n^{k,\epsilon} \nabla (m^{k,\epsilon} * \rho^\epsilon)) \cdot \Delta_q J_k n^{k,\epsilon} dx \\ & \leq C \|\Delta_q (J_k u^{k,\epsilon} J_k n^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2 + C \|\Delta_q (J_k n^{k,\epsilon} \nabla (c^{k,\epsilon} * \rho^\epsilon))\|_{L^2(\mathbb{R}^2)}^2 \\ & \quad + \frac{1}{2} \|\nabla \Delta_q J_k n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + 2 \|J_k \Delta_q n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 \\ & \quad + \|\Delta_q (J_k n^{k,\epsilon} \nabla (m^{k,\epsilon} * \rho^\epsilon))\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta_q (J_k n^{k,\epsilon} \nabla (n^{k,\epsilon} * \rho^\epsilon))\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (26)$$

Multiplying  $2^{2qs}$  on both sides of (26) and then taking the  $l^1$ -norm, by virtue of Lemma 2.1 and the embedding  $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \|n^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|\nabla J_k n^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 \\ & \leq C\|(J_k u^{k,\epsilon}, J_k n^{k,\epsilon})\|_{H^s(\mathbb{R}^2)}^2 (\|(J_k u^{k,\epsilon}, J_k n^{k,\epsilon})\|_{H^2(\mathbb{R}^2)}^2 + 1) \\ & \quad + C_\epsilon \|(c^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2 (\|J_k n^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 + \|J_k n^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2). \end{aligned} \quad (27)$$

By the same argument used for (27), we know that

$$\begin{aligned} & \frac{d}{dt} \|m^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|\nabla J_k m^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 \\ & \leq C\|(J_k u^{k,\epsilon}, J_k m^{k,\epsilon})\|_{H^s(\mathbb{R}^2)}^2 (\|(J_k u^{k,\epsilon}, J_k m^{k,\epsilon})\|_{H^2(\mathbb{R}^2)}^2 + 1) \\ & \quad + C_\epsilon \|(c^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2 (\|J_k m^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 + \|J_k m^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2). \end{aligned} \quad (28)$$

Similarly, applying  $\Delta_q$  to (5)<sub>3</sub> for  $q \in \mathbb{Z}$ , and multiplying both sides of the resulting equality by  $\Delta_q c^{k,\epsilon}$  and integrating with respect to  $x$ , we have

$$\begin{aligned} & \frac{d}{dt} \|c^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|\nabla J_k c^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 \\ & \leq C\|J_k u^{k,\epsilon} \cdot J_k c^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + C\|J_k c^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|J_k c^{k,\epsilon} \nabla(n^{k,\epsilon} * \rho^\epsilon + m^{k,\epsilon} * \rho^\epsilon)\|_{H^s(\mathbb{R}^2)}^2 \\ & \leq C\|(J_k u^{k,\epsilon}, J_k c^{k,\epsilon})\|_{H^s(\mathbb{R}^2)}^2 (\|(J_k u^{k,\epsilon}, J_k c^{k,\epsilon})\|_{H^2(\mathbb{R}^2)}^2 + 1) \\ & \quad + C_\epsilon \|(n^{k,\epsilon}, m^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2 (\|J_k c^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 + \|J_k c^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2). \end{aligned} \quad (29)$$

Applying  $\Delta_q$  to the fourth equation in (5) for  $q \in \mathbb{Z}$ , we arrive at

$$\partial_t \Delta_q u^{k,\epsilon} + \mathbf{P} \Delta_q J_k (J_k u^{k,\epsilon} \cdot \nabla (J_k u^{k,\epsilon})) = \Delta_q J_k^2 u^{k,\epsilon} + \mathbf{P} \Delta_q J_k ((m^{k,\epsilon} + n^{k,\epsilon}) \nabla \phi) * \rho^\epsilon, \quad (30)$$

where  $\mathbf{P}$  is the Leray projection operator, which can be expressed as a  $2 \times 2$  matrix:  $\mathbf{P} = \{\mathbf{P}_{j,k}\}_{1 \leq j,k \leq 2} = \{\delta_{j,k} + R_j R_k\}_{1 \leq j,k \leq 2}$  with  $\delta_{j,k}$  being the Kronecker symbol and  $R_j = \partial_j (-\Delta)^{-\frac{1}{2}}$  being the Riesz transform.

Multiplying  $2^{2qs}$  on both sides of (30) and then taking the  $l^1$ -norm, it holds that

$$\begin{aligned} & \frac{d}{dt} \|u^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|\nabla J_k u^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 \\ & \lesssim \|J_k u^{k,\epsilon} \cdot J_k u^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|J_k u^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \|J_k ((m^{k,\epsilon} + n^{k,\epsilon}) \nabla \phi) * \rho^\epsilon\|_{H^s(\mathbb{R}^2)}^2 \\ & \leq C\|J_k u^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 (1 + \|J_k u^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2) + C_\epsilon \|(m^{k,\epsilon}, n^{k,\epsilon})\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (31)$$

Summing up (27), (28), (29) and (31),

$$\begin{aligned} & \frac{d}{dt} \|X^{k,\epsilon}\|_{H^s(\mathbb{R}^3)}^2 + \|\nabla J_k X^{k,\epsilon}\|_{H^s(\mathbb{R}^3)}^2 \\ & \leq C_\epsilon \left(1 + \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|J_k X^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2\right) \|X^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 \\ & \quad + C_\epsilon \left(1 + \|J_k X^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2\right) \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

from which, by (25), using the Grönwall inequality, we have

$$\begin{aligned} & \|X^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + \int_0^t \|J_k X^{k,\epsilon}\|_{H^{s+1}(\mathbb{R}^2)}^2 ds \\ & \leq \left( \|X_0^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + C_\epsilon \int_0^t \|J_k X^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 ds \right) \\ & \quad \times \exp \left( C(\epsilon) \int_0^t 1 + \|X^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|J_k X^{k,\epsilon}\|_{H^2(\mathbb{R}^2)}^2 ds \right) \\ & \leq \left( \|X_0^{k,\epsilon}\|_{H^s(\mathbb{R}^2)}^2 + C_\epsilon t H(t, \epsilon) + C_\epsilon H(t, \epsilon)^2 \right) \\ & \quad \times \exp (C_\epsilon t H(t, \epsilon) + C_\epsilon H(t, \epsilon) + C_\epsilon t) \leq C_{t,\epsilon}, \end{aligned}$$

where

$$H(t, \epsilon) = \frac{1 + \|X_0^{k,\epsilon}\|_{H^1(\mathbb{R}^2)}}{1 - Ct(1 + \|X_0^{k,\epsilon}\|_{H^1(\mathbb{R}^2)})}.$$

This completes the proof.  $\square$

**Lemma 3.3.** *Let  $\phi \in \dot{W}^{1,\infty}(\mathbb{R}^2)$ ,  $(n_0^\epsilon, m_0^\epsilon, c_0^\epsilon, u_0^\epsilon) \in (H^s(\mathbb{R}^2))^4$  with  $s > 1$  and  $n_0^\epsilon, m_0^\epsilon, c_0^\epsilon > 0$ . Then, there is a unique global smooth solution  $(u^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}, c^{k,\epsilon})$ , which converges to a limit  $(u^\epsilon, n^\epsilon, m^\epsilon, c^\epsilon)$  satisfying system (4) in the sense of distribution when  $k \rightarrow \infty$ .*

*Proof.* From Lemmas 3.1 and 3.2, we can deduce

$$(\partial_t u^{k,\epsilon}, \partial_t m^{k,\epsilon}, \partial_t n^{k,\epsilon}, \partial_t c^{k,\epsilon}) \in (L_{loc}^2(\mathbb{R}^+; H^{s-1}(\mathbb{R}^2)))^4.$$

Let  $0 \leq \chi_l \leq 1$  be a smooth cutoff function such that  $\chi_l = 1$  in  $B_l(0)$ ,  $\chi_l = 0$  outside  $B_{l+1}(0)$ . Due to the Moser estimate of Lemma 2.1, we infer that for any  $l \in \mathbb{N}$ ,

$$\begin{aligned} & \|\chi_l n^{k,\epsilon}\|_{H^{s-1}(\mathbb{R}^2)}^2 \leq \|n^{k,\epsilon}\|_{H^{s-1}(\mathbb{R}^2)} \|\chi_l\|_{L^\infty(\mathbb{R}^2)} + \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)} \|\chi_l\|_{W^{s-1,\infty}(\mathbb{R}^2)} \\ & \leq C(\|n^{k,\epsilon}\|_{H^{s-1}(\mathbb{R}^2)} + \|n^{k,\epsilon}\|_{L^2(\mathbb{R}^2)}) \end{aligned} \tag{32}$$

and for  $t_2 > t_1 > 0$ ,

$$\begin{aligned} \|\chi_l n^{k,\epsilon}(t_1) - \chi_l n^{k,\epsilon}(t_2)\|_{H^{s-1}(\mathbb{R}^2)} &= \left\| \chi_l \int_{t_1}^{t_2} \chi_l \partial_t n^{k,\epsilon}(\tau) d\tau \right\|_{H^{s-1}(\mathbb{R}^2)} \\ &\leq (t_1 - t_2)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \|\chi_l \partial_t n^{k,\epsilon}(\tau)\|_{H^{s-1}(\mathbb{R}^2)}^2 d\tau \right)^{\frac{1}{2}}. \end{aligned} \quad (33)$$

Summing up (32) and (33), we can obtain that  $\{(\chi_l u^{k,\epsilon}, \chi_l n^{k,\epsilon}, \chi_l m^{k,\epsilon}, \chi_l c^{k,\epsilon})\}$  is uniformly bounded and equicontinuous in  $C(\mathbb{R}^+, H^{s-1}(\mathbb{R}^2))$ . Since  $\{(u^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}, c^{k,\epsilon})\} \rightarrow \{(\chi_l u^{k,\epsilon}, \chi_l n^{k,\epsilon}, \chi_l m^{k,\epsilon}, \chi_l c^{k,\epsilon})\}$  is compact from  $H^s(\mathbb{R}^2)$  into  $H^{s'}(\mathbb{R}^2)$  as  $s > s'$ . The Aubin-Lions Lemma 2.3 in combination with Cantor's diagonal process implies a distribution  $\{(u^\epsilon, n^\epsilon, m^\epsilon, c^\epsilon)\}$  belonging to  $C(\mathbb{R}^+, H^{s'}(\mathbb{R}^2))$  and a subsequence (which we still denote by  $\{(u^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}, c^{k,\epsilon})\}_{k \in \mathbb{N}}$ ), for all  $l \in \mathbb{N}$ ; we have  $(\chi_l u^{k,\epsilon}, \chi_l n^{k,\epsilon}, \chi_l m^{k,\epsilon}, \chi_l c^{k,\epsilon}) \rightarrow (\chi_l u^\epsilon, \chi_l n^\epsilon, \chi_l m^\epsilon, \chi_l c^\epsilon)$  when  $k \rightarrow \infty$  in  $(C(\mathbb{R}^+, H^{s'}(\mathbb{R}^2)))^4$ .

Thus,  $(u^{k,\epsilon}, n^{k,\epsilon}, m^{k,\epsilon}, c^{k,\epsilon})$  tends to  $(u^\epsilon, n^\epsilon, m^\epsilon, c^\epsilon)$  in  $\mathcal{D}'(\mathbb{R}^+, \mathbb{R}^2)$ . Based on the Fatou Lemma, we have

$$(u^\epsilon, n^\epsilon, m^\epsilon, c^\epsilon) \in (L_{loc}^\infty(\mathbb{R}^+; H^s(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^{s+1}(\mathbb{R}^2)))^4.$$

Next, we still have to prove that  $(u^\epsilon, n^\epsilon, m^\epsilon, c^\epsilon)$  is continuous in  $(H^s(\mathbb{R}^2))^4$ . By applying the operator  $\Delta_q (q \geq 0)$  to the fourth equation of (4), we obtain

$$\partial_t \Delta_q u^\epsilon + \mathbf{P} \Delta_q (u^\epsilon \cdot \nabla u^\epsilon) = \Delta_q u^\epsilon + \mathbf{P} \Delta_q (((m^\epsilon + n^\epsilon) \nabla \phi) * \rho^\epsilon), \quad (34)$$

where  $\mathbf{P}$  is the Leray projection operator, which can be expressed as a  $2 \times 2$  matrix:  $\mathbf{P} = \{\mathbf{P}_{j,k}\}_{1 \leq j,k \leq 2} = \{\delta_{j,k} + R_j R_k\}_{1 \leq j,k \leq 2}$  with  $\delta_{j,k}$  being the Kronecker symbol and  $R_j = \partial_j (-\Delta)^{-\frac{1}{2}}$  being the Riesz transform. Taking the  $L^2(\mathbb{R}^2)$ -inner product with the above equation with  $\Delta_q u^\epsilon$  yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Delta_q u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C 2^{2q} \|\Delta_q u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq C \|\Delta_q (u^\epsilon)^2\|_{L^2(\mathbb{R}^d)}^2 + \frac{C}{2} 2^{2q} \|\Delta_q u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C 2^{-2q} \|\Delta_q (((m^\epsilon + n^\epsilon) \nabla \phi) * \rho^\epsilon)\|_{L^2(\mathbb{R}^2)}^2, \end{aligned} \quad (35)$$

from which it is easy to observe that

$$\begin{aligned} &\|\Delta_q u^\epsilon\|_{L_t^\infty L^2(\mathbb{R}^2)}^2 + \int_0^t 2^{2q} \|\Delta_q u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 d\tau \\ &\leq \|\Delta_q u_0^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C \int_0^t \|\Delta_q (u^\epsilon)^2\|_{L^2(\mathbb{R}^d)}^2 + 2^{-2q} \|\Delta_q (((m^\epsilon + n^\epsilon) \nabla \phi) * \rho^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 d\tau. \end{aligned} \quad (36)$$

Therefore, multiplying the above inequality by  $2^{2qs}$  and summing over  $q$ , we can obtain

$$\begin{aligned} & \left( \sum_{q \geq 0} 2^{2qs} \|\Delta_q u^\epsilon\|_{L_t^\infty L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} + \left( \sum_{q \geq 0} 2^{2q(s+1)} \|\Delta_q u^\epsilon\|_{L_t^2 L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \\ & \leq \|u_0^\epsilon\|_{H^s(\mathbb{R}^2)} + C\|(u^\epsilon)^2\|_{\tilde{L}_t^2 H^s(\mathbb{R}^2)} + C\|((m^\epsilon + n^\epsilon)\nabla\phi) * \rho^\epsilon\|_{\tilde{L}_t^2 H^{s-1}(\mathbb{R}^2)} \\ & \leq \|u_0^\epsilon\|_{H^s(\mathbb{R}^2)} + C\|u^\epsilon\|_{\tilde{L}_t^\infty H^s(\mathbb{R}^2)}\|u^\epsilon\|_{\tilde{L}_t^2 H^{s+1}(\mathbb{R}^2)} + C_\epsilon\|(m^\epsilon, n^\epsilon)\|_{\tilde{L}_t^2 L^2(\mathbb{R}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} < \infty. \end{aligned} \quad (37)$$

Combining the fact that  $u^\epsilon \in L_t^\infty H^s$ , we obtain  $\sum_{q \geq -1} 2^{2qs} \|\Delta_q u^\epsilon\|_{L_t^\infty L^2(\mathbb{R}^2)}^2 < \infty$  and thus, it follows that

$$\lim_{N \rightarrow \infty} \|S_N u^\epsilon - u^\epsilon\|_{L_t^\infty H^s(\mathbb{R}^2)} = 0. \quad (38)$$

In addition, we can conclude from the fact  $\partial_t u^\epsilon \in L_{loc}^2(\mathbb{R}^+; H^{s-1}(\mathbb{R}^2))$  that

$$u^\epsilon \in C(\mathbb{R}^+; H^{s'}(\mathbb{R}^2))$$

with  $s' < s$ , which means that  $S_N u^\epsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2))$  for a fixed  $N \in \mathbb{Z}^+$ . Consequently, we obtain  $u^\epsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2))$ . In quite a similar way, we can also obtain that  $(m^\epsilon, n^\epsilon, c^\epsilon) \in (C(\mathbb{R}^+; H^s(\mathbb{R}^2)))^3$   $\square$

• **Nonnegativity.** Next, we need to prove the positivity of  $m^\epsilon, n^\epsilon$  and  $c^\epsilon$ . Let us decompose  $n^\epsilon = n_+^\epsilon - n_-^\epsilon$ , where

$$n_-^\epsilon = \max\{0, -n^\epsilon\}, \quad n_+^\epsilon = \max\{0, n^\epsilon\}.$$

We multiply the first equation of (4) by  $n_-^\epsilon$ , noting that  $n^\epsilon = n_+^\epsilon - n_-^\epsilon$ . Recall that the weak derivative of  $n_-^\epsilon$  is  $-\nabla n_-^\epsilon$  if  $n^\epsilon \leq 0$ ; otherwise, it is zero. It holds that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|n_-^\epsilon(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} (\nabla n_-^\epsilon(t))^2 dx \\ & = \int_{\mathbb{R}^2} n_-^\epsilon \nabla(c^\epsilon * \rho^\epsilon) \nabla(n_-^\epsilon) dx + \|n_-^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} (n_-^\epsilon)^2 (-n^\epsilon * \rho^\epsilon - m^\epsilon * \rho^\epsilon) dx \\ & \lesssim \epsilon^{-2} \|n_-^\epsilon\|_{L^2(\mathbb{R}^2)} \|c^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla n_-^\epsilon\|_{L^2(\mathbb{R}^2)} + \|n_-^\epsilon\|_{L^2(\mathbb{R}^d)}^2 + \epsilon^{-1} \|(n^\epsilon, m^\epsilon)\|_{L^2(\mathbb{R}^2)} \|n_-^\epsilon\|_{L^2(\mathbb{R}^d)}^2 \\ & \lesssim \frac{1}{2} \|\nabla n_-^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C_\epsilon \|n_-^\epsilon\|_{L^2(\mathbb{R}^2)}^2 (\|c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + 1 + \|(n^\epsilon, m^\epsilon)\|_{L^2(\mathbb{R}^2)}), \end{aligned}$$

from which, by Grönwall's inequality for any  $t > 0$ , we immediately obtain

$$\|n_-^\epsilon(t)\|_{L^2(\mathbb{R}^2)}^2 \lesssim \|(n_0^\epsilon)_-\|_{L^2(\mathbb{R}^2)}^2 \exp \left\{ C_\epsilon \int_0^t (\|c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + 1 + \|(n^\epsilon, m^\epsilon)\|_{L^2(\mathbb{R}^2)}) dt \right\} = 0.$$

This implies  $n^\epsilon(x, t) \geq 0$ . The terms  $m^\epsilon(x, t) \geq 0$  and  $c^\epsilon(x, t) \geq 0$ , we can estimate in a similar way as the term  $n^\epsilon(x, t) \geq 0$ . Applying the strong maximum principle, we have  $c^\epsilon(x, t) > 0$ . To prove  $n^\epsilon(x, t) > 0$ , we consider  $\tilde{n}^\epsilon(x, t)$ , which satisfies

$$\begin{cases} \partial_t \tilde{n} + (\tilde{u} \cdot \nabla) \tilde{n} = \Delta \tilde{n} - \nabla \cdot (\tilde{n} \nabla \tilde{c}) + \tilde{n}(1 - \tilde{n} - \tilde{m}), \\ \tilde{n}^\epsilon(x, 0) = n_0^\epsilon. \end{cases} \quad (39)$$

Similarly, due to  $n_0^\epsilon \in H^s$  for any  $s > 0$ , using the  $L^2$  estimate on  $\tilde{n}^\epsilon - n^\epsilon$ , we obtain  $\tilde{n}^\epsilon = n^\epsilon$ . If there exists a point  $(x_0, t_0)$  such that  $n^\epsilon(x_0, t_0) = 0$ , which implies  $\tilde{n}^\epsilon(x_0, t_0) = 0$ . By the Harnack inequality, we have  $\tilde{n}^\epsilon(x, t) = 0$ . Due to  $\tilde{n}_0^\epsilon = n_0^\epsilon > 0$ , we have  $\tilde{n}^\epsilon = n^\epsilon > 0$ . By the same method, likewise, we have  $m^\epsilon > 0$ .

• **Uniqueness.** We assume that  $(u_1^\epsilon, m_1^\epsilon, n_1^\epsilon, c_1^\epsilon)$  and  $(u_2^\epsilon, m_2^\epsilon, n_2^\epsilon, c_2^\epsilon)$  are global-in-time solutions of system (4) with the same initial data  $(u_0^\epsilon, m_0^\epsilon, n_0^\epsilon, c_0^\epsilon)$ . We set

$$(\mathbf{U}^\epsilon, \mathbf{M}^\epsilon, \mathbf{N}^\epsilon, \mathbf{C}^\epsilon) = (u_1^\epsilon, m_1^\epsilon, n_1^\epsilon, c_1^\epsilon) - (u_2^\epsilon, m_2^\epsilon, n_2^\epsilon, c_2^\epsilon),$$

then  $(\mathbf{U}^\epsilon, \mathbf{M}^\epsilon, \mathbf{N}^\epsilon, \mathbf{C}^\epsilon)$  solves

$$\begin{cases} \partial_t \mathbf{U}^\epsilon + \mathbf{U}^\epsilon \cdot \nabla u_1^\epsilon + u_2^\epsilon \nabla \mathbf{U}^\epsilon = \Delta \mathbf{U}^\epsilon + \nabla(\mathbb{P}_1^\epsilon - \mathbb{P}_2^\epsilon) + ((\mathbf{M}^\epsilon + \mathbf{N}^\epsilon) \nabla \phi) * \rho^\epsilon, \\ \partial_t \mathbf{N}^\epsilon + \mathbf{U}^\epsilon \cdot \nabla n_1^\epsilon + u_2^\epsilon \nabla \mathbf{N}^\epsilon = \Delta \mathbf{N}^\epsilon - \nabla \cdot (\mathbf{N}^\epsilon \nabla (c_1 * \rho^\epsilon) - \nabla \cdot (n_2 \nabla (\mathbf{C} * \rho^\epsilon))) \\ + (\mathbf{N}^\epsilon - \mathbf{N}^\epsilon (n_1^\epsilon + n_2^\epsilon) * \rho^\epsilon - n_2^\epsilon (\mathbf{N}^\epsilon + \mathbf{M}^\epsilon) * \rho^\epsilon), \\ \partial_t \mathbf{M}^\epsilon + \mathbf{U}^\epsilon \cdot \nabla m_1^\epsilon + u_2^\epsilon \nabla \mathbf{M}^\epsilon = \Delta \mathbf{M}^\epsilon - \nabla \cdot (\mathbf{M}^\epsilon \nabla (c_1^\epsilon * \rho^\epsilon) + m_2^\epsilon \nabla (\mathbf{C}^\epsilon * \rho^\epsilon)) \\ + (\mathbf{M}^\epsilon - \mathbf{M}^\epsilon (m_1^\epsilon + m_2^\epsilon) * \rho^\epsilon - m_2^\epsilon (\mathbf{M}^\epsilon + \mathbf{N}^\epsilon) * \rho^\epsilon), \\ \partial_t \mathbf{C}^\epsilon + \mathbf{U}^\epsilon \cdot \nabla c_1^\epsilon + u_2^\epsilon \nabla \mathbf{C}^\epsilon = \Delta \mathbf{C}^\epsilon - (\mathbf{N}^\epsilon * \rho^\epsilon) c_1^\epsilon - (n_2^\epsilon * \rho^\epsilon) \mathbf{C}^\epsilon - (\mathbf{M}^\epsilon * \rho^\epsilon) c_1^\epsilon - (m_2^\epsilon * \rho^\epsilon) \mathbf{C}^\epsilon. \end{cases} \quad (40)$$

Multiplying the second equation in (40) by  $\mathbf{N}^\epsilon$  and integrating over  $\mathbb{R}^2$ , we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{N}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{N}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq - \int_{\mathbb{R}^2} (\mathbf{U}^\epsilon \cdot \nabla n_1^\epsilon + u_2^\epsilon \nabla \mathbf{N}^\epsilon) \mathbf{N}^\epsilon dx + \int_{\mathbb{R}^2} (\mathbf{N}^\epsilon \nabla (c_1^\epsilon * \rho^\epsilon) + n_2^\epsilon \nabla (\mathbf{C} * \rho^\epsilon)) \nabla \mathbf{N}^\epsilon dx \\ & + \int_{\mathbb{R}^2} (\mathbf{N}^\epsilon - \mathbf{N}^\epsilon (n_1^\epsilon + n_2^\epsilon) * \rho^\epsilon - n_2^\epsilon (\mathbf{N}^\epsilon + \mathbf{M}^\epsilon) * \rho^\epsilon) \mathbf{N}^\epsilon dx = I + II + III. \end{aligned} \quad (41)$$

By means of Hölder's inequality, Young's inequality and the Gagliardo-Nirenberg inequality, since  $\nabla \cdot u_2 = \nabla \cdot \mathbf{U} = 0$ , we have

$$\begin{aligned} I &= \int_{\mathbb{R}^2} n_1^\epsilon \mathbf{U}^\epsilon \cdot \nabla \mathbf{N}^\epsilon dx \\ &\leq C \|n_1^\epsilon\|_{L^2} \|\nabla n_1^\epsilon\|_{L^2} \|\mathbf{U}^\epsilon\|_{L^2} \|\nabla \mathbf{U}^\epsilon\|_{L^2} + \frac{1}{10} \|\nabla \mathbf{N}^\epsilon\|_{L^2}^2 \\ &\leq C \|n_1^\epsilon\|_{L^2}^2 \|\nabla n_1^\epsilon\|_{L^2}^2 \|\mathbf{U}^\epsilon\|_{L^2}^2 + \frac{1}{10} \|\nabla \mathbf{U}^\epsilon\|_{L^2}^2 + \frac{1}{10} \|\nabla \mathbf{N}^\epsilon\|_{L^2}^2 \end{aligned} \quad (42)$$

and

$$II \leq C_\epsilon \|\mathbf{N}^\epsilon\|_{L^2}^2 \|c_1\|_{L^2}^2 + C_\epsilon \|\mathbf{C}^\epsilon\|_{L^2}^2 \|n_2\|_{L^2}^2 + \frac{1}{10} \|\nabla \mathbf{N}^\epsilon\|_{L^2}^2 \quad (43)$$

as well as

$$\begin{aligned} & III + \int_{\mathbb{R}^2} (\mathbf{N}^\epsilon)^2 (n_1^\epsilon + n_2^\epsilon) * \rho^\epsilon dx \\ & \leq C \|\mathbf{N}^\epsilon\|_{L^2}^2 + C_\epsilon \|n_2^\epsilon\|_{L^2} \|\mathbf{N}^\epsilon\|_{L^2} \|\mathbf{N}^\epsilon, \mathbf{M}^\epsilon\|_{L^2} \leq C_\epsilon (\|n_2^\epsilon\|_{L^2} + 1) \|(\mathbf{N}^\epsilon, \mathbf{M}^\epsilon)\|_{L^2}^2. \end{aligned} \quad (44)$$

Thanks to (42), (43) and (44), this entails that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{N}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{4}{5} \|\nabla \mathbf{N}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} (\mathbf{N}^\epsilon)^2 (n_1^\epsilon + n_2^\epsilon) * \rho^\epsilon dx \leq \frac{1}{10} \|\nabla \mathbf{U}^\epsilon\|_{L^2}^2 \\ & + C (\|\nabla n_1^\epsilon\|_{L^2}^2 \|n_1^\epsilon\|_{L^2}^2 + \|(c_1^\epsilon, n_2^\epsilon)\|_{L^2}^2 + \|n_2^\epsilon\|_{L^2} + 1) \|(\mathbf{N}^\epsilon, \mathbf{M}^\epsilon, \mathbf{C}^\epsilon, \mathbf{U}^\epsilon)\|_{L^2}^2, \end{aligned} \quad (45)$$

and similarly,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{M}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{4}{5} \|\nabla \mathbf{M}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} (\mathbf{M}^\epsilon)^2 (m_1^\epsilon + m_2^\epsilon) * \rho^\epsilon dx \leq \frac{1}{10} \|\nabla \mathbf{U}^\epsilon\|_{L^2}^2 \\ & + C (\|\nabla m_1^\epsilon\|_{L^2}^2 \|m_1^\epsilon\|_{L^2}^2 + \|(c_1^\epsilon, m_2^\epsilon)\|_{L^2}^2 + \|m_2^\epsilon\|_{L^2} + 1) \|(\mathbf{N}^\epsilon, \mathbf{M}^\epsilon, \mathbf{C}^\epsilon, \mathbf{U}^\epsilon)\|_{L^2}^2. \end{aligned} \quad (46)$$

We conclude upon adding (45) and (47) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\mathbf{M}^\epsilon, \mathbf{N}^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + \frac{4}{5} \|(\nabla \mathbf{M}^\epsilon, \nabla \mathbf{N}^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq \frac{1}{5} \|\nabla \mathbf{U}^\epsilon\|_{L^2}^2 + C_\epsilon \|(\mathbf{N}^\epsilon, \mathbf{M}^\epsilon, \mathbf{C}^\epsilon, \mathbf{U}^\epsilon)\|_{L^2}^2 \\ & \quad \times (\|(\nabla m_1^\epsilon, \nabla n_1^\epsilon)\|_{L^2}^2 \|m_1^\epsilon, n_1^\epsilon\|_{L^2}^2 + \|(c_1^\epsilon, m_2^\epsilon, n_2^\epsilon)\|_{L^2}^2 + \|(m_2^\epsilon, n_2^\epsilon)\|_{L^2} + 1). \end{aligned} \quad (47)$$

Next, multiplying the fourth equation in (40) by  $\mathbf{C}^\epsilon$  and integrating, we see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{C}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{C}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq \frac{1}{10} \|\nabla \mathbf{U}^\epsilon\|_{L^2}^2 + \frac{1}{10} \|\nabla \mathbf{C}^\epsilon\|_{L^2}^2 \\ & \quad + C_\epsilon (1 + \|c_1^\epsilon\|_{L^2}^2 \|\nabla c_1^\epsilon\|_{L^2}^2 + \|c_1^\epsilon\|_{L^2}^2) \|(\mathbf{U}^\epsilon, \mathbf{N}^\epsilon, \mathbf{M}^\epsilon, \mathbf{C}^\epsilon)\|_{L^2}^2, \end{aligned}$$

from which,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{C}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{9}{10} \|\nabla \mathbf{C}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq \frac{1}{10} \|\nabla \mathbf{U}^\epsilon\|_{L^2}^2 + C_\epsilon (1 + \|c_1^\epsilon\|_{L^2}^2 \|\nabla c_1^\epsilon\|_{L^2}^2 + \|c_1^\epsilon\|_{L^2}^2) \|(\mathbf{U}^\epsilon, \mathbf{N}^\epsilon, \mathbf{M}^\epsilon, \mathbf{C}^\epsilon)\|_{L^2}^2. \end{aligned} \quad (48)$$

Taking the  $L^2$ -inner product of the first equation of (87) with  $\mathbf{U}^\epsilon$  yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
&= \int_{\mathbb{R}^2} (\mathbf{U}^\epsilon \cdot \nabla \mathbf{U}^\epsilon) \cdot u_1 dx + \int_{\mathbb{R}^2} ((\mathbf{M}^\epsilon + \mathbf{N}^\epsilon) \nabla \phi) * \rho^\epsilon \cdot \mathbf{U}^\epsilon dx \\
&\leq C \|\mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)} \|u_1\|_{L^2(\mathbb{R}^2)} \|\nabla u_1\|_{L^2(\mathbb{R}^2)} + \frac{1}{4} \|\nabla \mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + \|\mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)} \|(\mathbf{M}^\epsilon, \mathbf{N}^\epsilon)\|_{L^2(\mathbb{R}^2)} \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)} \\
&\leq C \|\mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 (\|u_1\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2) \\
&\quad + \frac{1}{2} \|\nabla \mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C \|(\mathbf{M}^\epsilon, \mathbf{N}^\epsilon)\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned} \tag{49}$$

which implies that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\nabla \mathbf{U}^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C \|(\mathbf{U}^\epsilon, \mathbf{M}^\epsilon, \mathbf{N}^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 (1 + \|u_1\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2).
\end{aligned} \tag{50}$$

We may combine (47), (48) and (50) to arrive at

$$E(t) + \int_0^t F(s) ds \leq \int_0^t G(s) E(s) ds, \tag{51}$$

where

$$E(t) = \|(\mathbf{U}^\epsilon(t), \mathbf{M}^\epsilon(t), \mathbf{N}^\epsilon(t), \mathbf{C}^\epsilon(t))\|_{L^2(\mathbb{R}^2)}^2,$$

$$F(s) = \|(\nabla \mathbf{U}^\epsilon(s), \nabla \mathbf{M}^\epsilon(s), \nabla \mathbf{N}^\epsilon(s), \nabla \mathbf{C}^\epsilon(s))\|_{L^2(\mathbb{R}^2)}^2$$

and

$$\begin{aligned}
G(s) &= \|(\nabla m_1^\epsilon, \nabla n_1^\epsilon)\|_{L^2}^2 \|(m_1^\epsilon, n_1^\epsilon)\|_{L^2}^2 + \|(c_1^\epsilon, m_2^\epsilon, n_2^\epsilon)\|_{L^2}^2 + \|(m_2^\epsilon, n_2^\epsilon)\|_{L^2}^2 \\
&\quad + 1 + \|c_1^\epsilon\|_{L^2}^2 \|\nabla c_1^\epsilon\|_{L^2}^2 + \|c_1^\epsilon\|_{L^2}^2 + \|u_1\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2.
\end{aligned}$$

From Lemmas 3.1 and 3.2, we have

$$\int_0^t G(s) ds \leq C e^{Ct} + C e^{Ct} + Ct,$$

by which and (51), applying the Grönwall inequality, we have

$$E(t) \leq E(t=0) \exp \left\{ \int_0^t G(s) ds \right\}.$$

We can thus conclude that  $(\mathbf{U}^\epsilon, \mathbf{M}^\epsilon, \mathbf{N}^\epsilon, \mathbf{C}^\epsilon) = 0$ . Therefore, we complete the proof of uniqueness.

**Proposition 3.4.** Let  $\phi \in \dot{W}^{1,\infty}(\mathbb{R}^2)$ ,  $(n_0^\epsilon, m_0^\epsilon, c_0^\epsilon, u_0^\epsilon) \in (H^s(\mathbb{R}^2))^4$  with  $s > 1$  and  $n_0^\epsilon, m_0^\epsilon, c_0^\epsilon > 0$ . Then, system (4) has a unique global smooth solution

$$(n^\epsilon, m^\epsilon, c^\epsilon, u^\epsilon) \in (C(\mathbb{R}^+; H^s(\mathbb{R}^2)))^4 \cap (L_{loc}^2(\mathbb{R}^+; H^{s+1}(\mathbb{R}^2)))^4.$$

Moreover,  $n^\epsilon > 0, m^\epsilon > 0$  for all  $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ .

*Proof.* The statement is evidently implied by Lemmas 3.1, 3.2 and 3.3.  $\square$

## 4 A priori estimates for a regularized system

In this section, we prepare several lemmas to prove our main theorem. We shall prove uniform estimates for smooth solutions  $(u^\epsilon, m^\epsilon, n^\epsilon, c^\epsilon)$  to the regularized problem (4), which are independent of neither  $\epsilon > 0$  nor the mollifier  $\rho^\epsilon$ .

**Lemma 4.1.** Assume that  $(u_0, c_0, m_0, n_0) \in X_0$  and let  $(u^\epsilon, m^\epsilon, n^\epsilon, c^\epsilon)$  be a unique classical solution to the system (4). Then, there exists a positive constant  $C$  such that

$$\|c^\epsilon\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)} \leq \|c_0\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)},$$

$$\|c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\nabla c^\epsilon(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq \|c_0\|_{L^2(\mathbb{R}^2)}^2$$

and

$$\|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} + \int_0^t \|(m^\epsilon(\tau), n^\epsilon(\tau))\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq Ce^{Ct}$$

as well as

$$\|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\nabla u^\epsilon(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq Ce^{Ct}.$$

*Proof.* Integrating the first equation in (4) gives

$$\frac{d}{dt} \|n^\epsilon\|_{L^1(\mathbb{R}^2)} + \|n^\epsilon(n^\epsilon * \rho^\epsilon)\|_{L^1(\mathbb{R}^2)} + \|n^\epsilon(m^\epsilon * \rho^\epsilon)\|_{L^1(\mathbb{R}^2)} = \|n^\epsilon\|_{L^1(\mathbb{R}^2)},$$

from which and Grönwall's inequality, when we let  $\epsilon \rightarrow \infty$ , yields

$$\begin{aligned} & \|n^\epsilon\|_{L^1(\mathbb{R}^2)} + \int_0^t \|n^\epsilon(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau + \int_0^t \|n^\epsilon(\tau)m^\epsilon(\tau)\|_{L^1(\mathbb{R}^2)} d\tau \\ & \leq C\|n_0\|_{L^1(\mathbb{R}^2)}e^{Ct}. \end{aligned} \tag{52}$$

Similarly, by the second equation of (4), we thereby obtain

$$\begin{aligned} & \|m^\epsilon(t)\|_{L^1(\mathbb{R}^2)} + \int_0^t \|m^\epsilon(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau + \int_0^t \|n^\epsilon(\tau)m^\epsilon(\tau)\|_{L^1(\mathbb{R}^2)} d\tau \\ & \leq C\|m_0\|_{L^1(\mathbb{R}^2)}e^{Ct}. \end{aligned} \tag{53}$$

We can make use of the third equation of (4) to find that

$$\|c^\epsilon(t)\|_{L^1(\mathbb{R}^2)} + \int_0^t \|n^\epsilon(\tau)c^\epsilon(\tau)\|_{L^1(\mathbb{R}^2)}d\tau + \int_0^t \|c^\epsilon(\tau)m^\epsilon(\tau)\|_{L^1(\mathbb{R}^2)}d\tau \leq \|c_0\|_{L^1(\mathbb{R}^2)},$$

and therefore, the weak maximum principle gives rise to

$$\|c^\epsilon\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)} \leq \|c_0\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)}. \quad (54)$$

Testing the third equation in (4) by  $c^\epsilon$  and integrating it over  $\mathbb{R}^2$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \leq 0,$$

from which,

$$\|c^\epsilon(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla c^\epsilon(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq \|c_0\|_{L^2(\mathbb{R}^2)}^2. \quad (55)$$

On the other hand, multiplying the fourth equation in (4) by  $u^\epsilon$  and integrating by parts over  $\mathbb{R}^2$ , we see that

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \leq C \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2 \|(m, n)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|u\|_{L^2(\mathbb{R}^2)}^2,$$

from which, by using (52) and (53) and Grönwall's inequality, we further have

$$\begin{aligned} & \|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 d\tau \\ & \leq \left( \|u_0\|_{L^2(\mathbb{R}^2)}^2 + C \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2 \int_0^t \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 d\tau \right) + \frac{1}{2} \|u\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq C(\|u_0\|_{L^2(\mathbb{R}^2)}, \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}, \|(m_0, n_0)\|_{L^1(\mathbb{R}^2)}) e^{Ct}. \end{aligned}$$

Combining this with (52)–(55) directly results in Lemma 4.1.  $\square$

**Lemma 4.2.** *Suppose  $m_0, n_0, c_0 > 0$  and that  $(u_0, c_0, m_0, n_0) \in X_0$  and  $\nabla \phi \in L^\infty(\mathbb{R}^2)$ . Let  $(n^\epsilon, m^\epsilon, c^\epsilon, u^\epsilon)$  be a smooth solution of system (4). Then, there exists a constant  $C > 0$  independent of  $\epsilon$  such that*

$$\begin{aligned} & \int_0^t \|(c^\epsilon)^{-1} |\nabla \sqrt{c^\epsilon}|^4\|_{L^1(\mathbb{R}^2)} ds + \int_{\mathbb{R}^2} n^\epsilon |\ln n^\epsilon| dx + \int_{\mathbb{R}^2} m^\epsilon |\ln m^\epsilon| dx \\ & + \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|(\Delta \sqrt{c^\epsilon}, \nabla u^\epsilon, \nabla \sqrt{m^\epsilon}, \nabla \sqrt{n^\epsilon}, m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 ds \\ & + \int_0^t \int_{\mathbb{R}^2} \left( (1 + 2|x|) n^\epsilon m^\epsilon + |x| (n^\epsilon)^2 + |x| (m^\epsilon)^2 + (n^\epsilon + m^\epsilon) |\nabla \sqrt{c^\epsilon}|^2 \right) dx ds \\ & \leq C \exp(Ct). \end{aligned} \quad (56)$$

*Proof.* Because  $s \ln s \geq -\frac{1}{e}$  for all  $s > 0$ , we first invoke the inequality

$$-\int_{\mathbb{R}^2} n^\epsilon m^\epsilon \ln n^\epsilon dx \leq \frac{1}{e} \|m^\epsilon\|_{L^1(\mathbb{R}^2)}. \quad (57)$$

Therefore, multiplying the first equation of (4) by  $1 + \ln n^\epsilon$  and integrating the desired equation, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx + 4 \int_{\mathbb{R}^2} |\nabla \sqrt{n^\epsilon}|^2 dx + \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} n^\epsilon m^\epsilon dx \\ &= \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon) dx + \|n^\epsilon\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx \\ & \quad - \int_{\mathbb{R}^2} n^\epsilon m^\epsilon \ln n^\epsilon dx - \int_{\mathbb{R}^2} (n^\epsilon)^2 \ln n^\epsilon dx. \end{aligned} \quad (58)$$

In treating the above term, we make use of the pointwise identity

$$\Delta c^\epsilon = 2|\nabla \sqrt{c^\epsilon}|^2 + 2\sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon} \quad (59)$$

and the following fact

$$\begin{aligned} & - \int_{\mathbb{R}^2} (n^\epsilon)^2 \ln n^\epsilon dx \\ &= - \int_{n^\epsilon \geq 1} (n^\epsilon)^2 \ln n^\epsilon dx - \int_{0 < n^\epsilon < e^{-|x|}} (n^\epsilon)^2 \ln n^\epsilon dx - \int_{e^{-|x|} < n^\epsilon < 1} (n^\epsilon)^2 \ln n^\epsilon dx \\ &\leq - \int_{0 < n^\epsilon < e^{-|x|}} (n^\epsilon)^2 \ln n^\epsilon dx - \int_{e^{-|x|} < n^\epsilon < 1} (n^\epsilon)^2 \ln n^\epsilon dx \leq \int_{\mathbb{R}^2} (n^\epsilon)^2 |x| dx + C. \end{aligned} \quad (60)$$

Therefore, going back to (58) and in view of (57), (59) and (60), we readily have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx + 4 \int_{\mathbb{R}^2} |\nabla \sqrt{n^\epsilon}|^2 dx + \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} n^\epsilon m^\epsilon dx \\ &\leq 2\|c\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} (\|\Delta \sqrt{c}\|_{L^2(\mathbb{R}^2)} + \|\sqrt{c}^{-1} |\nabla \sqrt{c}|^2\|_{L^2(\mathbb{R}^2)}) + \|n^\epsilon\|_{L^1(\mathbb{R}^2)} \\ & \quad + \int_{\mathbb{R}^2} (n^\epsilon)^2 |x| dx + C + e^{-1} \|m^\epsilon\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx \\ &\leq C \|n\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \|\Delta \sqrt{c}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \int_{\mathbb{R}^2} |\nabla \sqrt{c}|^4 c^{-1} dx \\ & \quad + C \|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} + C + \int_{\mathbb{R}^2} (n^\epsilon)^2 |x| dx + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx. \end{aligned} \quad (61)$$

By multiplying the first equation of (4) by  $|x|$ , this shows that

$$\begin{aligned} & \partial_t (|x| n^\epsilon) + u^\epsilon \cdot \nabla (|x| n^\epsilon) \\ &= u \cdot x \frac{n^\epsilon}{|x|} + |x| \Delta n^\epsilon - |x| \nabla \cdot (n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) + |x| n^\epsilon (1 - n^\epsilon - m^\epsilon), \end{aligned} \quad (62)$$

from which, then integrating it over  $\mathbb{R}^2$ , by Young's inequality, we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} |x| n^\epsilon dx + \int_{\mathbb{R}^2} |x| (n^\epsilon)^2 dx + \int_{\mathbb{R}^2} |x| m^\epsilon n^\epsilon dx \\
& \leq \int_{\mathbb{R}^2} |x| n^\epsilon dx + \|u^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} + \|\nabla c^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} + 2\|\sqrt{n^\epsilon}\|_{L^2(\mathbb{R}^2)} \|\nabla \sqrt{n^\epsilon}\|_{L^2(\mathbb{R}^2)} \quad (63) \\
& \leq \int_{\mathbb{R}^2} |x| n^\epsilon dx + 2\|n^\epsilon\|_{L^1(\mathbb{R}^2)} + \|\nabla \sqrt{n^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + C\|(u^\epsilon, \nabla c^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4}\|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}$$

Multiplying the above inequality by 2 and using (61), one obtains

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + 2|x|n^\epsilon) dx + 2 \int_{\mathbb{R}^2} |\nabla \sqrt{n^\epsilon}|^2 dx + \frac{1}{2} \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
& + \int_{\mathbb{R}^2} (1 + 2|x|) n^\epsilon m^\epsilon dx + \int_{\mathbb{R}^2} |x| (n^\epsilon)^2 dx \\
& \leq C\|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \int_{\mathbb{R}^2} |\nabla \sqrt{c^\epsilon}|^4 (c^\epsilon)^{-1} dx + C \\
& + C\|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + 2|x|n^\epsilon) dx + C\|(u^\epsilon, \nabla c^\epsilon)\|_{L^2(\mathbb{R}^2)}^2. \quad (64)
\end{aligned}$$

Similar to (64), by the third equation of system (4), we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} (m^\epsilon \ln m^\epsilon + 2|x|m^\epsilon) dx + 2 \int_{\mathbb{R}^2} |\nabla \sqrt{m^\epsilon}|^2 dx + \frac{1}{2} \|m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
& + \int_{\mathbb{R}^2} (1 + 2|x|) m^\epsilon n^\epsilon dx + \int_{\mathbb{R}^2} |x| (m^\epsilon)^2 dx \\
& \leq C\|m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \int_{\mathbb{R}^2} |\nabla \sqrt{c^\epsilon}|^4 (c^\epsilon)^{-1} dx + C \\
& + C\|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} (m^\epsilon \ln m^\epsilon + 2|x|m^\epsilon) dx + C\|(u^\epsilon, \nabla c^\epsilon)\|_{L^2(\mathbb{R}^2)}^2. \quad (65)
\end{aligned}$$

By the pointwise identity (59), the third equation of (4) thereupon turns into the relation

$$\partial_t \sqrt{c^\epsilon} - \Delta \sqrt{c^\epsilon} - (\sqrt{c^\epsilon})^{-1} |\nabla \sqrt{c^\epsilon}|^2 + u^\epsilon \cdot \nabla \sqrt{c^\epsilon} = -\frac{1}{2} \sqrt{c^\epsilon} (n^\epsilon * \rho^\epsilon) - \frac{1}{2} \sqrt{c^\epsilon} (m^\epsilon * \rho^\epsilon) \quad (66)$$

Multiplying the above equation by  $-\Delta \sqrt{c^\epsilon}$ , by direct differentiation, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} u^\epsilon \cdot \nabla \sqrt{c^\epsilon} \cdot \Delta c^\epsilon dx - \int_{\mathbb{R}^2} (c^\epsilon)^{-\frac{1}{2}} |\nabla \sqrt{c^\epsilon}|^2 \Delta \sqrt{c^\epsilon} dx \\
& + \int_{\mathbb{R}^2} \frac{1}{2} (n^\epsilon * \rho^\epsilon) \sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon} + \frac{1}{2} (m^\epsilon * \rho^\epsilon) \sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon} dx = I_1 + I_2 + I_3. \quad (67)
\end{aligned}$$

For  $I_1$ , by Young's inequality, we integrate by parts to obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2} u^\epsilon \cdot \nabla \sqrt{c^\epsilon} \cdot \Delta \sqrt{c^\epsilon} dx \\
&= - \int_{\mathbb{R}^2} \sqrt{c^\epsilon} (\nabla \sqrt{c^\epsilon} \cdot \nabla) u^\epsilon \cdot (\nabla \sqrt{c^\epsilon}) \sqrt{c^{\epsilon-1}} dx \\
&\leq 3 \|c^\epsilon\|_{L^\infty(\mathbb{R}^2)} \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{12} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c^\epsilon}|^4}{c^\epsilon} dx \\
&\leq C \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{12} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c^\epsilon}|^4}{c^\epsilon} dx.
\end{aligned} \tag{68}$$

We proceed further to rewrite  $I_3$  as

$$\begin{aligned}
& \int_{\mathbb{R}^2} \frac{1}{2} (n^\epsilon * \rho^\epsilon) \sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon} + \frac{1}{2} (m^\epsilon * \rho^\epsilon) \sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon} dx \\
&= - \int_{\mathbb{R}^2} \left( \frac{1}{4} \nabla n^\epsilon + \frac{1}{4} \nabla m^\epsilon \right) \cdot \nabla (c^\epsilon * \rho^\epsilon) + \left( \frac{1}{2} n^\epsilon + \frac{1}{2} m^\epsilon \right) * \rho^\epsilon |\nabla \sqrt{c^\epsilon}|^2 dx.
\end{aligned} \tag{69}$$

For  $I_2$ , we have that

$$\begin{aligned}
I_2 &= - \int_{\mathbb{R}^2} (c^\epsilon)^{-\frac{1}{2}} |\nabla \sqrt{c^\epsilon}|^2 \Delta \sqrt{c^\epsilon} dx = - \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} (\partial_j \sqrt{c^\epsilon})^2 (\partial_{i,i} \sqrt{c^\epsilon})^2 dx \\
&= - \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c^\epsilon}|^2 |\partial_j \sqrt{c^\epsilon}|^2 dx + 2 \sum_{i=j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} \partial_i \sqrt{c^\epsilon} \partial_{i,j} \sqrt{c^\epsilon} \partial_j \sqrt{c^\epsilon} dx \\
&\quad + 2 \sum_{i \neq j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} \partial_i \sqrt{c^\epsilon} \partial_{i,j} \sqrt{c^\epsilon} \partial_j \sqrt{c^\epsilon} dx \\
&= - \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c^\epsilon}|^2 |\partial_j \sqrt{c^\epsilon}|^2 dx - 2I_2 - 2 \sum_{i \neq j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} (\partial_j \sqrt{c^\epsilon})^2 \partial_{i,i} \sqrt{c^\epsilon} dx \\
&\quad + 2 \sum_{i \neq j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} \partial_i \sqrt{c^\epsilon} \partial_{i,j} \sqrt{c^\epsilon} \partial_j \sqrt{c^\epsilon} dx,
\end{aligned} \tag{70}$$

from which, by Young's inequality, it follows that

$$\begin{aligned}
I_2 &= -\frac{1}{3} \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c}|^2 |\partial_j \sqrt{c^\epsilon}|^2 dx - \frac{2}{3} \sum_{i \neq j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} (\partial_j \sqrt{c^\epsilon})^2 \partial_{i,i} \sqrt{c^\epsilon} dx \\
&\quad + \frac{2}{3} \sum_{i \neq j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-1} \partial_i \sqrt{c^\epsilon} \partial_{i,j} \sqrt{c^\epsilon} \partial_j \sqrt{c^\epsilon} dx, \\
&\leq -\frac{1}{3} \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c^\epsilon}|^2 |\partial_j \sqrt{c^\epsilon}|^2 dx \\
&\quad + \frac{1}{6} \sum_i \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c^\epsilon}|^4 dx + \frac{2}{3} \sum_i \int_{\mathbb{R}^2} (\partial_{i,i} \sqrt{c^\epsilon})^2 dx \\
&\quad + \frac{1}{6} \sum_{i \neq j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c^\epsilon}|^2 |\partial_j \sqrt{c}|^2 dx + \frac{2}{3} \sum_{i \neq j} \int_{\mathbb{R}^2} (\partial_{i,j} \sqrt{c^\epsilon})^2 dx \\
&\leq -\frac{1}{6} \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c^\epsilon}|^2 |\partial_j \sqrt{c^\epsilon}|^2 dx + \frac{2}{3} \sum_{i,j} \int_{\mathbb{R}^2} (\partial_{i,j} \sqrt{c^\epsilon})^2 dx,
\end{aligned} \tag{71}$$

Plugging estimates (68), (69) and (71) into (67),

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \left( \frac{1}{2} n^\epsilon + \frac{1}{2} m^\epsilon \right) * \rho^\epsilon |\nabla \sqrt{c^\epsilon}|^2 dx \\
&\quad + \frac{1}{6} \sum_{i,j} \int_{\mathbb{R}^2} (\sqrt{c^\epsilon})^{-2} |\partial_i \sqrt{c}|^2 |\partial_j \sqrt{c^\epsilon}|^2 dx \\
&\leq C \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{12} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c^\epsilon}|^4}{c^\epsilon} dx \\
&\quad - \int_{\mathbb{R}^2} \left( \frac{1}{4} \nabla n^\epsilon + \frac{1}{4} \nabla m^\epsilon \right) \cdot \nabla (c^\epsilon * \rho^\epsilon) dx + \frac{2}{3} \sum_{i,j} \int_{\mathbb{R}^2} (\partial_{i,j} \sqrt{c^\epsilon})^2 dx.
\end{aligned} \tag{72}$$

Due to  $\|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)} = \|\nabla^2 \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}$ , we have by (72) that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \left( \frac{1}{2} n^\epsilon + \frac{1}{2} m^\epsilon \right) * \rho^\epsilon |\nabla \sqrt{c}|^2 dx \\
&\quad + \frac{1}{3} \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{12} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c^\epsilon}|^4}{c^\epsilon} dx \\
&\leq C \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 - \int_{\mathbb{R}^2} \left( \frac{1}{4} \nabla n^\epsilon + \frac{1}{4} \nabla m^\epsilon \right) \cdot \nabla (c^\epsilon * \rho^\epsilon) dx.
\end{aligned} \tag{73}$$

By integrating by parts and Young's inequality, we arrive at

$$\begin{aligned}
& - \int_{\mathbb{R}^2} \left( \frac{1}{4} \nabla n^\epsilon + \frac{1}{4} \nabla m^\epsilon \right) \cdot \nabla (c^\epsilon * \rho^\epsilon) dx \\
& = \int_{\mathbb{R}^2} \Delta c^\epsilon \left( \frac{1}{4} n^\epsilon + \frac{1}{4} m^\epsilon \right) * \rho^\epsilon dx \\
& = \int_{\mathbb{R}^2} (|\nabla \sqrt{c^\epsilon}|^2 + \sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon}) \left( \frac{1}{2} n^\epsilon + \frac{1}{2} m^\epsilon \right) * \rho^\epsilon dx \\
& \leq \|c^\epsilon\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)} (\|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)} + \|\Delta \sqrt{c^\epsilon}|^2 \sqrt{c^\epsilon}^{-1}\|_{L^2(\mathbb{R}^2)}) \\
& \leq C \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{24} \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{24} \int_{\mathbb{R}^2} |\Delta \sqrt{c^\epsilon}|^4 (c^\epsilon)^{-1} dx.
\end{aligned} \tag{74}$$

Inserting (74) into (73), we can find that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \left( \frac{1}{2} n^\epsilon + \frac{1}{2} m^\epsilon \right) * \rho^\epsilon |\nabla \sqrt{c^\epsilon}|^2 dx \\
& + \frac{7}{24} \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{24} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c^\epsilon}|^4}{c^\epsilon} dx \\
& \leq C \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{75}$$

Summing up (64), (65) and (75), we have

$$\begin{aligned}
& \frac{d}{dt} \left( \frac{1}{2} \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + 2|x|n^\epsilon) dx \right) + 2 \int_{\mathbb{R}^2} |\nabla \sqrt{c^\epsilon}|^2 dx + \frac{1}{2} \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
& + \int_{\mathbb{R}^2} (1 + 2|x|) n^\epsilon m^\epsilon dx + \int_{\mathbb{R}^2} |x| (n^\epsilon)^2 dx \\
& + \frac{13}{48} \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{48} \int_{\mathbb{R}^2} |\nabla \sqrt{c^\epsilon}|^4 (c^\epsilon)^{-1} dx + \int_{\mathbb{R}^2} \left( \frac{1}{2} n^\epsilon + \frac{1}{2} m^\epsilon \right) * \rho^\epsilon |\nabla \sqrt{c^\epsilon}|^2 dx \\
& \leq C \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + C \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + C \|(u^\epsilon, \nabla c^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + C \|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} + C \\
& + \int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + 2|x|n^\epsilon) dx.
\end{aligned} \tag{76}$$

On the other hand, we multiply  $u^\epsilon$  with the fourth equation of (4) and integrate over  $\mathbb{R}^2$  to obtain

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \leq C \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2. \tag{77}$$

Multiplying the above equation by  $12\|c_0\|_{L^\infty(\mathbb{R}^2)}$  and adding the resulting inequality to (64), (65) and (75), we see that

$$\begin{aligned}
& \frac{d}{dt} \left( \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + m^\epsilon \ln m^\epsilon + 2|x|m + 2|x|n)dx + 6\|c_0\|_{L^\infty(\mathbb{R}^2)}\|u\|_{L^2(\mathbb{R}^2)}^2 \right) \\
& + \int_{\mathbb{R}^2} 2|\nabla \sqrt{n^\epsilon}|^2 + 2|\nabla \sqrt{m^\epsilon}|^2 dx + \frac{1}{2}\|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2}\|m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \\
& + \int_{\mathbb{R}^2} (1 + 2|x|)n^\epsilon m^\epsilon dx + \int_{\mathbb{R}^2} (1 + 2|x|)n^\epsilon m^\epsilon dx \\
& + \int_{\mathbb{R}^2} |x|(n^\epsilon)^2 dx + \int_{\mathbb{R}^2} |x|(m^\epsilon)^2 dx + 6\|c_0\|_{L^\infty(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \\
& + \frac{13}{24}\|\Delta \sqrt{c}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{12} \int_{\mathbb{R}^2} |\nabla \sqrt{c}|^4 c^{-1} dx + \int_{\mathbb{R}^2} (\alpha n + \beta m) * \rho^\epsilon |\nabla \sqrt{c}|^2 dx \\
& \leq C + C\|(m, n)\|_{L^2(\mathbb{R}^2)}^2 + C\|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} + C\|(u, \nabla c^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 \\
& + \int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + 2|x|n)dx + \int_{\mathbb{R}^2} (m^\epsilon \ln m^\epsilon + 2|x|m)dx,
\end{aligned}$$

from which and Grönwall's inequality, we have

$$\begin{aligned}
& \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|u\|_{L^2(\mathbb{R}^2)}^2 + \|(n^\epsilon \ln n^\epsilon, m^\epsilon \ln m^\epsilon, 2|x|m^\epsilon, 2|x|n^\epsilon)\|_{L^1(\mathbb{R}^2)} \\
& + \int_0^t \left( \|(c^\epsilon)^{-1} \nabla \sqrt{c}|^4\|_{L^1(\mathbb{R}^2)} + \|(\Delta \sqrt{c^\epsilon}, \nabla u^\epsilon, \nabla \sqrt{m^\epsilon}, \nabla \sqrt{n^\epsilon}, m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 \right) ds \\
& + \int_0^t \int_{\mathbb{R}^2} \left( (1 + 2|x|)n^\epsilon m^\epsilon + |x|(n^\epsilon)^2 + |x|(m^\epsilon)^2 dx ds + (n^\epsilon + m^\epsilon) * \rho^\epsilon |\nabla \sqrt{c}|^2 \right) dx ds \quad (78) \\
& \leq \left( C \int_0^t \|(m^\epsilon, n^\epsilon, \nabla c^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 ds + C \int_0^t \|(m^\epsilon, n^\epsilon)\|_{L^1(\mathbb{R}^2)} ds + Ct + U_0 \right) \exp(Ct) \\
& \leq C \exp(Ct),
\end{aligned}$$

where

$$U_0 = \|\nabla \sqrt{c_0}\|_{L^2(\mathbb{R}^2)}^2 + \|(n_0 \ln n_0, m_0 \ln m_0, 2|x|m_0, 2|x|n_0)\|_{L^1(\mathbb{R}^2)} + \|u_0\|_{L^2(\mathbb{R}^2)}^2.$$

Thanks to the fact that

$$\begin{aligned}
\int_{\mathbb{R}^2} n^\epsilon |\ln n^\epsilon| dx &= \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx + 2 \int_{0 < n^\epsilon < e^{-x}} n^\epsilon \ln \frac{1}{n^\epsilon} dx + 2 \int_{e^{-x} \leq n^\epsilon \leq 1} n^\epsilon \ln \frac{1}{n^\epsilon} dx \\
&\leq \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx + C + 2 \int_{\mathbb{R}^2} n^\epsilon |x| dx.
\end{aligned} \quad (79)$$

It follows from (78) and (79) that

$$\begin{aligned}
& \int_0^t \|(c^\epsilon)^{-1} |\nabla \sqrt{c^\epsilon}|^4\|_{L^1(\mathbb{R}^2)} ds + \int_{\mathbb{R}^2} n^\epsilon |\ln n^\epsilon| dx + \int_{\mathbb{R}^2} m^\epsilon |\ln m^\epsilon| dx \\
& + \|\nabla \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|(\Delta \sqrt{c^\epsilon}, \nabla u^\epsilon, \nabla \sqrt{m^\epsilon}, \nabla \sqrt{n^\epsilon}, m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 ds \\
& + \int_0^t \int_{\mathbb{R}^2} \left( (1 + 2|x|)n^\epsilon m^\epsilon + |x|(n^\epsilon)^2 + |x|(m^\epsilon)^2 + (n^\epsilon + m^\epsilon) |\nabla \sqrt{c^\epsilon}|^2 \right) dx ds \\
& \leq C \exp(Ct),
\end{aligned}$$

where  $C$  depends on

$$U_0 = \|\nabla \sqrt{c_0}\|_{L^2(\mathbb{R}^2)}^2 + \|(n_0 \ln n_0, m_0 \ln m_0, 2|x|m_0, 2|x|n_0)\|_{L^1(\mathbb{R}^2)} + \|u_0\|_{L^2(\mathbb{R}^2)}^2.$$

The proof of Lemma 4.2 is completed.  $\square$

Based on Lemma 4.2, we can further obtain a uniform estimate for the high regularity of the smooth solution  $(n^\epsilon, m^\epsilon, c^\epsilon, u^\epsilon)$ .

**Lemma 4.3.** *Assume that  $m_0, n_0, c_0 > 0$ ,  $(u_0, c_0, m_0, n_0) \in X_0$  and  $\nabla \phi \in L^\infty(\mathbb{R}^2)$ . Let  $(n^\epsilon, m^\epsilon, c^\epsilon, u^\epsilon)$  be a smooth solution of system (4). Then, there exists a constant  $C > 0$  independent of  $\epsilon$  such that*

$$\|c^\epsilon\|_{L_t^\infty H^1} + \|c^\epsilon\|_{L_t^2 H^2} \leq C e^{Ct},$$

$$\|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left( \|\nabla n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|n^\epsilon\|_{L^3(\mathbb{R}^2)}^3 + \|n^2 m\|_{L^1(\mathbb{R}^2)} \right) ds \leq C e^{Ct}$$

and

$$\|m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left( \|\nabla m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|m^\epsilon\|_{L^3(\mathbb{R}^2)}^3 + \|m^2 n\|_{L^1(\mathbb{R}^2)} \right) ds \leq C e^{Ct}$$

as well as

$$\|\nabla u^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\Delta u^\epsilon\|_{L^2(\mathbb{R}^2)} ds \leq C e^{Ct}.$$

*Proof.* Employing Hölder's inequality, Lemmas 4.1 and 4.2, we have

$$\begin{aligned}
& \|c^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|\nabla c^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} = \|c^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + 2\|\sqrt{c^\epsilon} \nabla \sqrt{c^\epsilon}\|_{L_t^\infty(L^2(\mathbb{R}^2))} \\
& \leq \|c^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + 2\|c^\epsilon\|_{L_t^\infty(L^\infty(\mathbb{R}^2))}^{\frac{1}{2}} \|\nabla \sqrt{c^\epsilon}\|_{L_t^\infty(L^2(\mathbb{R}^2))} \\
& \leq \|c_0\|_{L^2(\mathbb{R}^2)} + C\|c_0\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} e^{Ct} \leq C e^{Ct}.
\end{aligned} \tag{80}$$

By the pointwise identity  $\Delta c^\epsilon = 2\sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon} + 2|\nabla \sqrt{c^\epsilon}|^2$ , it follows from Lemmas 4.1 and 4.2 that

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^2} |\Delta c^\epsilon|^2 dx d\tau \leq 8 \int_0^t \int_{\mathbb{R}^2} c^\epsilon |\Delta \sqrt{c^\epsilon}|^2 + |\Delta \sqrt{c^\epsilon}|^2 dx d\tau \\
& \leq 8\|c_0\|_{L^\infty(\mathbb{R}^2)} \int_0^t \|\Delta \sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2 + \|(c^\epsilon)^{-1} |\nabla \sqrt{c^\epsilon}|^4\|_{L^1(\mathbb{R}^2)} ds \leq C e^{Ct}.
\end{aligned} \tag{81}$$

Using Lemma 4.1, we have

$$\int_0^t \|c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 d\tau + \int_0^t \|\nabla c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq t\|c_0\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2}\|c_0\|_{L^2(\mathbb{R}^2)}^2. \quad (82)$$

Then, we deduce from (80), (81) and (82) that

$$\|c^\epsilon\|_{L_t^\infty H^1} + \|c^\epsilon\|_{L_t^2 H^2} \leq Ce^{Ct}. \quad (83)$$

Testing the first equation in (4) against  $n^\epsilon$  and using Young's inequality yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|n^\epsilon\|_{L^3(\mathbb{R}^2)}^3 + \int_{\mathbb{R}^2} (n^\epsilon)^2 m dx \\ &= \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{2} \int_{\mathbb{R}^2} \Delta(c * \rho^\epsilon) (n^\epsilon)^2 dx \\ &\leq \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\Delta c^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^4(\mathbb{R}^2)}^2 \\ &\leq \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\Delta c^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla n^\epsilon\|_{L^2(\mathbb{R}^2)} \\ &\leq \left(1 + \frac{1}{8} \|\Delta c^\epsilon\|_{L^2(\mathbb{R}^2)}^2\right) \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\nabla n^\epsilon\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

from which, by Grönwall's inequality and (81), we obtain

$$\begin{aligned} & \|n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left( \|\nabla n^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + 2\|n^\epsilon\|_{L^3(\mathbb{R}^2)}^3 + 2\|(n^\epsilon)^2 m^\epsilon\|_{L^1(\mathbb{R}^2)} \right) ds \\ &\leq \|n_0\|_{L^2(\mathbb{R}^2)}^2 \exp \left( \int_0^t \left( 2 + \frac{1}{4} \|\Delta c^\epsilon\|_{L^2(\mathbb{R}^2)}^2 \right) ds \right) \\ &\leq \|n_0\|_{L^2(\mathbb{R}^2)}^2 \exp \left( 2t + \frac{1}{4} Ce^{Ct} \right) \leq Ce^{Ce^{Ct}}. \end{aligned} \quad (84)$$

Similarly, taking the  $L^2$ -inner product of the second equation of (4) with  $n^\epsilon$  gives

$$\|m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left( \|\nabla m^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + 2\|m^\epsilon\|_{L^3(\mathbb{R}^2)}^3 + 2\|m^2 n\|_{L^1(\mathbb{R}^2)} \right) ds \leq Ce^{Ce^{Ct}}. \quad (85)$$

The integrability of the second derivative of  $u^\epsilon$  remains to be investigated. We first let  $\omega = \nabla^\perp u$  be the vorticity of  $u^\epsilon$  and then consider the vorticity equation

$$\omega_t^\epsilon + u^\epsilon \nabla \omega^\epsilon - \Delta \omega^\epsilon = \nabla^\perp((m^\epsilon + n^\epsilon) \nabla \phi) * \rho^\epsilon.$$

A direct energy method gives that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \omega^\epsilon\|_{L^2(\mathbb{R}^2)}^2 &= - \int_{\mathbb{R}^2} \nabla^\perp \omega \cdot ((m^\epsilon + n^\epsilon) \nabla \phi) * \rho^\epsilon dx \\ &\leq C \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\nabla \omega^\epsilon\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

which leads to the vorticity estimate

$$\|\omega^\epsilon\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\nabla \omega^\epsilon\|_{L^2(\mathbb{R}^2)}^2 ds \leq C \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2 \int_0^t \|(m^\epsilon, n^\epsilon)\|_{L^2(\mathbb{R}^2)}^2 ds \leq C e^{Ct}. \quad (86)$$

Collecting terms from (83), (84), (85) and (86), we can thereby complete the proof of Lemma 4.3.  $\square$

## 5 Proof of Theorem 1.1

This section mainly deals with the proof of Theorem 1.1.

• **Existence.** By Lemmas 4.1, 4.2 and 4.3, we can achieve that

$$\|(m^\epsilon, n^\epsilon)\|_{L_t^2(H^1(\mathbb{R}^2))} + \|(m^\epsilon, n^\epsilon)\|_{L_t^3(L^3(\mathbb{R}^2))} + \|(m^\epsilon, n^\epsilon)\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|(m^\epsilon, n^\epsilon)\|_{L_t^\infty(L^1(\mathbb{R}^2))} \leq C_t$$

and

$$\|c^\epsilon\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)} \leq \|c_0^\epsilon\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)}, \quad \|c\|_{L_t^\infty(H^1(\mathbb{R}^2))} + \|c\|_{L_t^2(H^2(\mathbb{R}^2))} \leq C_t$$

as well as

$$\|u^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \leq C_t,$$

where  $C_t$  is independent of  $\epsilon$ . We thus have the following bounds uniformly with  $\epsilon$ ,

$$(m^\epsilon, n^\epsilon) \in (L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)))^2,$$

$$c^\epsilon \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L_{loc}^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^2(\mathbb{R}^2)),$$

$$u^\epsilon \in L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)).$$

By the Aubin-Lions compactness Lemma 2.3, we will prove the uniform boundedness for  $\partial_t m^\epsilon, \partial_t n^\epsilon, \partial_t c^\epsilon$  and  $\partial_t u^\epsilon$ . Making use of (4) and the Gagliardo-Nirenberg inequality  $\|f\|_{L_t^4(L^4(\mathbb{R}^2))}^2 \leq \|f\|_{L_t^\infty(L^2(\mathbb{R}^2))} \|\nabla f\|_{L_t^2(L^2(\mathbb{R}^2))}$  and  $H^{s+\epsilon}(\mathbb{R}^2) \hookrightarrow H^s(\mathbb{R}^2)$  for any  $\epsilon > 0$ , one can readily obtain that

$$\begin{aligned} \|n_t^\epsilon\|_{L_t^2 H^{-1}} &\lesssim \|u^\epsilon \cdot n^\epsilon\|_{L_t^2(L^2(\mathbb{R}^2))} + \|n^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \\ &\quad + \|n^\epsilon \nabla(c^\epsilon * \rho^\epsilon)\|_{L_t^2(L^2(\mathbb{R}^2))} + \|n^\epsilon(1 - n^\epsilon - m^\epsilon)\|_{L_t^2(L^2(\mathbb{R}^2))} \\ &\lesssim \|u^\epsilon\|_{L_t^4(L^4(\mathbb{R}^2))}^2 + \|n^\epsilon\|_{L_t^4(L^4(\mathbb{R}^2))}^2 + \|n^\epsilon\|_{L_t^2 H^1} + \|n^\epsilon\|_{L_t^2 L^2} + \|\nabla(c^\epsilon * \rho^\epsilon)\|_{L_t^4(L^4(\mathbb{R}^2))}^2 \\ &\lesssim \|u^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} \|u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} \|n^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \\ &\quad + t^{\frac{1}{2}} \|n^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|\rho^\epsilon\|_{L^1(\mathbb{R}^2)}^2 \|c^\epsilon\|_{L_t^\infty(H^1(\mathbb{R}^2))} \|c^\epsilon\|_{L_t^2(H^2(\mathbb{R}^2))} \leq C_t \end{aligned}$$

and

$$\begin{aligned}
\|c_t^\epsilon\|_{L_t^2(L^2(\mathbb{R}^2))} &\lesssim \|\nabla c^\epsilon\|_{L_t^4(L^4(\mathbb{R}^2))}^2 + \|u^\epsilon\|_{L_t^4(L^4(\mathbb{R}^2))}^2 + \|c^\epsilon\|_{L_t^2(H^2(\mathbb{R}^2))} \\
&\quad + \|(m^\epsilon * \rho^\epsilon, n^\epsilon * \rho^\epsilon)\|_{L_t^4(L^4(\mathbb{R}^2))}^2 + \|c^\epsilon\|_{L_t^4(L^4(\mathbb{R}^2))}^2 \\
&\lesssim \|c^\epsilon\|_{L_t^\infty(H^1(\mathbb{R}^2))} \|c^\epsilon\|_{L_t^2(H^2(\mathbb{R}^2))} + \|c^\epsilon\|_{L_t^2(H^2(\mathbb{R}^2))} \\
&\quad + \|u^\epsilon\|_{L_t^\infty L^2} \|u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} \|n^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \\
&\quad + \|n^\epsilon\|_{L_t^\infty L^2} \|n^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + \|c^\epsilon\|_{L_t^\infty L^2} \|c^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \leq C_t
\end{aligned}$$

as well as

$$\begin{aligned}
\|u_t^\epsilon\|_{L_t^2(H^{-1}(\mathbb{R}^2))} &\lesssim \|u^\epsilon\|_{L_t^4(L^4(\mathbb{R}^2))}^2 + \|u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + \|(m^\epsilon, n^\epsilon)\|_{L_t^2(L^2(\mathbb{R}^2))} \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)} \\
&\lesssim \|u^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} \|\nabla u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + \|u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \\
&\quad + \|(m^\epsilon, n^\epsilon)\|_{L_t^2(L^2(\mathbb{R}^2))} \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)} \leq C_t,
\end{aligned}$$

which proves that  $\partial_t n^\epsilon, \partial_t u^\epsilon$  are bounded in  $L_{loc}^2(\mathbb{R}^+; H^{-1}(\mathbb{R}^2))$  and  $\partial_t c^\epsilon$  is bounded in  $L_{loc}^2(\mathbb{R}^+; L^2(\mathbb{R}^2))$ . Similarly, we can also obtain that  $\partial_t m^\epsilon$  is bounded in  $L_{loc}^2(\mathbb{R}^+; H^{-1}(\mathbb{R}^2))$ . Since  $L^2(\mathbb{R}^2)$  is locally compactly embedded  $H^s(\mathbb{R}^2)$  and  $H^s(\mathbb{R}^2)$  continuously embedded in  $H^{-1}(\mathbb{R}^2)$  with  $s \in (-1, 0)$ , by a compactness argument, we thus deduce that there exists some function such that for all  $\tilde{\phi} \in \mathcal{D}$ , the sequence  $(\tilde{\phi} m^\epsilon, \tilde{\phi} n^\epsilon, \tilde{\phi} c^\epsilon, \tilde{\phi} u^\epsilon)$  converges (up to a subsequence independent of  $\tilde{\phi}$ ) to  $(\tilde{\phi} m, \tilde{\phi} n, \tilde{\phi} c, \tilde{\phi} u)$  in  $C(\mathbb{R}^+; H^s(\mathbb{R}^2))$ . Therefore,  $(m^\epsilon, n^\epsilon, c^\epsilon, u^\epsilon)$  converges to  $(m, n, c, u)$  in the sense of distribution. Taking advantage of Fatou's Lemma, we have

$$\begin{aligned}
(m, n) &\in (L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)))^2, \\
c &\in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L_{loc}^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^2(\mathbb{R}^2)), \\
u &\in L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L_{loc}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)).
\end{aligned}$$

Hence, we readily obtain that  $(m, n, c, u)$  is a global-in-time weak solution of system (1).

• **Uniqueness.** Let us consider the two solutions  $(m_1, n_1, c_1, u_1)$  and  $(m_2, n_2, c_2, u_2)$  of system (1) associated with the same initial data  $(m_0, n_0, c_0, u_0)$ . Let  $\mathbf{M} = m_1 - m_2$ ,  $\mathbf{N} = n_1 - n_2$ ,  $\mathbf{C} = c_1 - c_2$  and  $\mathbf{U} = u_1 - u_2$ , which solves

$$\left\{ \begin{array}{l} \mathbf{U}_t + \mathbf{U} \cdot \nabla u_1 + u_2 \nabla \mathbf{U} = \Delta \mathbf{U} + \nabla(\mathbb{P}_1 - \mathbb{P}_2) + (\mathbf{M} + \mathbf{N}) \nabla \phi, \quad \nabla \cdot \mathbf{U} = 0, \\ \mathbf{N}_t + \mathbf{U} \cdot \nabla n_1 + u_2 \nabla \mathbf{N} = \Delta \mathbf{N} - \nabla \cdot (\mathbf{N} \nabla c_1 + n_2 \nabla \mathbf{C}) \\ \quad + (\mathbf{N} - \mathbf{N}(n_1 + n_2) - \mathbf{M} n_1 - \mathbf{N} m_2), \\ \mathbf{M}_t + \mathbf{U} \cdot \nabla m_1 + u_2 \nabla \mathbf{M} = \Delta \mathbf{M} - \nabla \cdot (\mathbf{M} \nabla c_1 + m_2 \nabla \mathbf{C}) \\ \quad + (\mathbf{M} - \mathbf{M}(m_1 + m_2) - \mathbf{N} m_1 - \mathbf{M} n_2), \\ \mathbf{C}_t + \mathbf{U} \cdot \nabla c_1 + u_2 \nabla \mathbf{C} = \Delta \mathbf{C} - \mathbf{N} c_1 - n_2 \mathbf{C} - \mathbf{M} c_1 - m_2 \mathbf{C}. \end{array} \right. \quad (87)$$

Taking the  $L^2$ -inner product of the second equation of (87) with  $\mathbf{N}$  yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq - \int_{\mathbb{R}^2} (\mathbf{U} \cdot \nabla n_1 + u_2 \nabla \mathbf{N}) \mathbf{N} dx + \int_{\mathbb{R}^2} (\mathbf{N} \nabla c_1 + n_2 \nabla \mathbf{C}) \nabla \mathbf{N} dx \\
& \quad + \int_{\mathbb{R}^2} (\mathbf{N} - \mathbf{N}(n_1 + n_2) - M n_1 - \mathbf{N} m_2) \mathbf{N} dx = I + II + III.
\end{aligned} \tag{88}$$

Using Hölder's inequality, Young's inequality and the Gagliardo-Nirenberg interpolation inequality, by virtue of  $\nabla \cdot u_2 = \nabla \cdot \mathbf{U} = 0$ , we deduce

$$\begin{aligned}
I &= \int_{\mathbb{R}^2} n_1 \mathbf{U} \cdot \nabla \mathbf{N} dx \\
&\leq C \|n_1\|_{L^2(\mathbb{R}^2)} \|\nabla n_1\|_{L^2(\mathbb{R}^2)} \|\mathbf{U}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)} + \frac{1}{10} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C \|n_1\|_{L^2(\mathbb{R}^2)}^2 \|\nabla n_1\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{10} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{10} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
II &\leq C \|\mathbf{N}\|_{L^4(\mathbb{R}^2)}^2 \|\nabla c_1\|_{L^4(\mathbb{R}^2)}^2 + C \|n_2\|_{L^4(\mathbb{R}^2)}^2 \|\nabla \mathbf{C}\|_{L^4(\mathbb{R}^2)}^2 + \frac{1}{20} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C \|\mathbf{N}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)} \|\nabla c_1\|_{L^2(\mathbb{R}^2)} \|\Delta c_1\|_{L^2(\mathbb{R}^2)} \\
&\quad + \|n_2\|_{L^2(\mathbb{R}^2)} \|\nabla n_2\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)} + \frac{1}{20} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 + \|n_2\|_{L^2(\mathbb{R}^2)}^2 \|\nabla n_2\|_{L^2(\mathbb{R}^2)}^2 \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + \frac{1}{10} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{10} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2
\end{aligned} \tag{90}$$

as well as

$$\begin{aligned}
III &\leq C \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + C \|(n_1, n_2, m_2)\|_{L^2(\mathbb{R}^2)} \|\mathbf{N}\|_{L^4(\mathbb{R}^2)}^2 + C \|n_1\|_{L^2(\mathbb{R}^2)} \|\mathbf{N}\|_{L^4(\mathbb{R}^2)} \|\mathbf{M}\|_{L^4(\mathbb{R}^2)} \\
&\leq C \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + C \|(n_1, n_2, m_2)\|_{L^2(\mathbb{R}^2)} \|\mathbf{N}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)} \\
&\quad + C \|n_1\|_{L^2(\mathbb{R}^2)} \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\mathbf{M}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla \mathbf{M}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\leq C \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + C \|(n_1, n_2, m_2)\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{10} \|(\nabla \mathbf{N}, \nabla \mathbf{M})\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + C \|n_1\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + C \|n_1\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{M}\|_{L^2(\mathbb{R}^2)}^2 \\
&\leq C (\|(n_1, n_2, m_2)\|_{L^2(\mathbb{R}^2)}^2 + 1) \|(\mathbf{N}, \mathbf{M})\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{10} \|(\nabla \mathbf{N}, \nabla \mathbf{M})\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{91}$$

Substituting (89), (90) and (91) into (88), one has

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{N}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq \frac{1}{10} \|(\nabla \mathbf{U}, \Delta \mathbf{C})\|_{L^2}^2 + \frac{3}{10} \|(\nabla \mathbf{N}, \nabla \mathbf{M})\|_{L^2(\mathbb{R}^2)}^2 \\
& + C(\|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 + \|(n_1, n_2, m_2)\|_{L^2(\mathbb{R}^2)}^2 + 1) \|(\mathbf{N}, \mathbf{M})\|_{L^2(\mathbb{R}^2)}^2 \\
& + C\|(n_1, n_2)\|_{L^2(\mathbb{R}^2)}^2 \|(\nabla n_1, \nabla n_2)\|_{L^2(\mathbb{R}^2)}^2 \|(\mathbf{U}, \nabla \mathbf{C})\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{92}$$

Similarly, taking the  $L^2(\mathbb{R}^2)$ -inner product of the third equation of (87) with  $\mathbf{M}$  yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{M}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{M}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq \frac{1}{10} \|(\nabla \mathbf{U}, \Delta \mathbf{C})\|_{L^2(\mathbb{R}^2)}^2 + \frac{3}{10} \|(\nabla \mathbf{M}, \nabla \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 \\
& + C(\|\nabla c_1\|_{L^2}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 + \|(m_1, m_2, n_2)\|_{L^2}^2 + 1) \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 \\
& + C\|(m_1, m_2)\|_{L^2(\mathbb{R}^2)}^2 \|(\nabla m_1, \nabla m_2)\|_{L^2(\mathbb{R}^2)}^2 \|(\mathbf{U}, \nabla \mathbf{C})\|_{L^2(\mathbb{R}^2)}^2
\end{aligned} \tag{93}$$

We derive from (92) and (93) that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 + \frac{2}{5} \|(\nabla \mathbf{M}, \nabla \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq \frac{1}{5} \|(\nabla \mathbf{U}, \Delta \mathbf{C})\|_{L^2(\mathbb{R}^2)}^2 + C(\|(n_1, n_2, m_1, m_2)\|_{L^2}^2 \|(\nabla n_1, \nabla n_2, \nabla m_1, \nabla m_2)\|_{L^2(\mathbb{R}^2)}^2 \\
& + \|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 + \|(m_1, m_2, n_1, n_2)\|_{L^2(\mathbb{R}^2)}^2 + 1) \|(\mathbf{M}, \mathbf{N}, \mathbf{U}, \nabla \mathbf{C})\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{94}$$

Taking the  $L^2$ -inner product of the fourth equation of (87) with  $\mathbf{C}$  yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq \|\mathbf{U}\|_{L^2(\mathbb{R}^2)} \|\mathbf{C}\|_{L^4(\mathbb{R}^2)} \|\nabla c_1\|_{L^4(\mathbb{R}^2)} + \|(\mathbf{N}, \mathbf{M})\|_{L^2(\mathbb{R}^2)} \|c_1\|_{L^\infty(\mathbb{R}^2)} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)} \\
& + \|(n_2, m_2)\|_{L^2(\mathbb{R}^2)} \|\mathbf{C}\|_{L^4(\mathbb{R}^2)}^2 \\
& \leq \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla c_1\|_{L^2(\mathbb{R}^2)} \|\Delta c_1\|_{L^2(\mathbb{R}^2)} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)} \\
& + \|(\mathbf{N}, \mathbf{M})\|_{L^2(\mathbb{R}^2)} \|c_1\|_{L^\infty(\mathbb{R}^2)} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)} \\
& + \|(n_2, m_2)\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)} \\
& \leq \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + C\|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + C\|(\mathbf{N}, \mathbf{M})\|_{L^2(\mathbb{R}^2)}^2 + \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\
& + C\|(n_2, m_2)\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned} \tag{95}$$

from which, we have

$$\begin{aligned} \frac{d}{dt} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 &\leq \|(\mathbf{U}, \mathbf{N}, \mathbf{M}, \mathbf{C})\|_{L^2(\mathbb{R}^2)}^2 \\ &+ C \left( \|(n_2, m_2)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 \right) \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (96)$$

By the fourth equation of (87), we see that

$$\begin{aligned} \partial_t(\partial_i \mathbf{C}) + \partial_i(\mathbf{U} \cdot \nabla c_1) + u_2 \nabla \partial_i \mathbf{C} + \partial_i u_2 \nabla \mathbf{C} \\ = \Delta \partial_i \mathbf{C} - \partial_i(\mathbf{N} c_1 + n_2 \mathbf{C} + \mathbf{M} c_1 + m_2 \mathbf{C}). \end{aligned} \quad (97)$$

Multiplying (97) by  $\partial_i \mathbf{C}$  and integrating over  $\mathbb{R}^2$ , we infer that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\ = \int_{\mathbb{R}^2} (\mathbf{U} \cdot \nabla c_1) \Delta \mathbf{C} dx - \int_{\mathbb{R}^2} (\nabla \mathbf{C} \cdot \nabla) u_2 \cdot \nabla \mathbf{C} dx \\ + \int_{\mathbb{R}^2} (\mathbf{N} c_1 + n_2 \mathbf{C} + \mathbf{M} c_1 + m_2 \mathbf{C}) \Delta \mathbf{C} dx = II_1 + II_2 + II_3. \end{aligned} \quad (98)$$

Using Hölder's inequality, Young's inequality and the Gagliardo-Nirenberg interpolation inequality, one has

$$\begin{aligned} II_1 &\leq \frac{1}{8} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + C \|\mathbf{U}\|_{L^4(\mathbb{R}^2)}^2 \|\nabla c_1\|_{L^4(\mathbb{R}^2)}^2 \\ &\leq \frac{1}{8} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + C \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 \end{aligned} \quad (99)$$

and

$$\begin{aligned} II_2 &\leq \|\nabla \mathbf{C}\|_{L^4(\mathbb{R}^2)}^2 \|\nabla u_2\|_{L^2(\mathbb{R}^2)} \\ &\leq \frac{1}{8} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + C \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_2\|_{L^2(\mathbb{R}^2)}^2 \end{aligned} \quad (100)$$

as well as

$$\begin{aligned} II_3 &\leq \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)} \|\Delta c\|_{L^2(\mathbb{R}^2)} \|c_1\|_{L^\infty(\mathbb{R}^2)} \\ &+ \|(n_2, m_2)\|_{L^4(\mathbb{R}^2)} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)} \|\mathbf{C}\|_{L^4(\mathbb{R}^2)} \\ &\leq C \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 \|c_1\|_{L^\infty(\mathbb{R}^2)}^2 + \frac{1}{8} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\ &+ \|(\nabla n_2, \nabla m_2)\|_{L^2(\mathbb{R}^2)}^2 \|(n_2, m_2)\|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (101)$$

Substituting (99), (100) and (101) into (98), we arrive at

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{5}{8} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C \|(\nabla n_2, \nabla m_2)\|_{L^2(\mathbb{R}^2)}^2 \| (n_2, m_2) \|_{L^2(\mathbb{R}^2)}^2 \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\
& \quad + C \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_2\|_{L^2(\mathbb{R}^2)}^2 + C \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 \\
& \quad + C \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 \|c_1\|_{L^\infty(\mathbb{R}^2)}^2 + \frac{1}{8} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{102}$$

Combining (96) and (102), we have

$$\begin{aligned}
& \frac{d}{dt} \|\mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{5}{8} \|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 + \frac{7}{8} \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2 \\
& \leq C (\|\nabla c_1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta c_1\|_{L^2(\mathbb{R}^2)}^2 + \|(\nabla n_2, \nabla m_2)\|_{L^2(\mathbb{R}^2)}^2 \| (n_2, m_2) \|_{L^2(\mathbb{R}^2)}^2) \\
& \quad + \| (n_2, m_2, \nabla u_2) \|_{L^2(\mathbb{R}^2)}^2 + 1) \|(\mathbf{C}, \mathbf{U}, \mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{103}$$

Taking the  $L^2$ -inner product of the first equation of (87) with  $\mathbf{U}$  yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 \\
& = \int_{\mathbb{R}^2} (\mathbf{U} \cdot \nabla \mathbf{U}) \cdot u_1 dx + \int_{\mathbb{R}^2} (\mathbf{M} + \mathbf{N}) \nabla \phi \cdot \mathbf{U} dx \\
& \leq C \|\mathbf{U}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)} \|u_1\|_{L^2(\mathbb{R}^2)} \|\nabla u_1\|_{L^2(\mathbb{R}^2)} + \frac{1}{4} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 \\
& \quad + \|\mathbf{U}\|_{L^2(\mathbb{R}^2)} \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)} \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)} \\
& \leq C \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 (\|u_1\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2) \\
& \quad + \frac{1}{2} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + C \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned} \tag{104}$$

which implies that

$$\begin{aligned}
& \frac{d}{dt} \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 \leq C \|(\mathbf{M}, \mathbf{N})\|_{L^2(\mathbb{R}^2)}^2 \\
& \quad + C \|\mathbf{U}\|_{L^2(\mathbb{R}^2)}^2 (\|u_1\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \phi\|_{L^\infty(\mathbb{R}^2)}^2).
\end{aligned} \tag{105}$$

Collecting (94), (103) and (105) leads

$$\tilde{E}(t) + \int_0^t \tilde{F}(s) ds \leq \int_0^t \tilde{G}(s) \tilde{E}(s) ds, \tag{106}$$

with

$$\begin{aligned}
\tilde{E}(t) &= \|(\mathbf{U}, \mathbf{M}, \mathbf{N}, \mathbf{C})(t)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{C}\|_{L^2(\mathbb{R}^2)}^2, \\
\tilde{F}(s) &= \|(\nabla \mathbf{U}, \nabla \mathbf{M}, \nabla \mathbf{N}, \nabla \mathbf{C})(s)\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta \mathbf{C}(s)\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned}$$

and

$$\begin{aligned}\tilde{G}(s) = & C(\|\nabla u_1\|_{L^2}^2 \|u_1\|_{L^2}^2 + \|(n_1, n_2, m_1, m_2)\|_{L^2}^2 \|(\nabla n_1, \nabla n_2, \nabla m_1, \nabla m_2)\|_{L^2}^2 \\ & + \|\nabla c_1\|_{L^2}^2 \|\Delta c_1\|_{L^2}^2 + \|(\nabla u_2, m_1, m_2, n_1, n_2)\|_{L^2}^2 + 1)\end{aligned}$$

Applying Lemmas 4.1, 4.2 and (4.3), we have

$$\int_0^t \tilde{G}(s) ds \leq C e^{Ce^{Ct}} + C e^{Ct} + Ct,$$

from which and (106), applying Grönwall's inequality, we have

$$\tilde{E}(t) \leq \tilde{E}(t=0) \exp \left\{ \int_0^t \tilde{G}(s) ds \right\},$$

from which, we conclude that  $(\mathbf{U}, \mathbf{M}, \mathbf{N}, \mathbf{C}) = 0$ , and we thus complete the proof of uniqueness.

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## Declarations

**Conflict of interest.** We declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

**Ethics approval.** Not applicable.

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