Information scrambling and the growth of errors in noisy tomography - a quantum signature of chaos: Supplementary information

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I. DERIVATION OF ERROR SCRAMBLING EXPRESSION

Let the operators at time t be $\mathcal{O}(t) = U^\dagger(t)\mathcal{O}U(t)$ and $\mathcal{O}'(t) = U'^\dagger(t)\mathcal{O}U'(t)$, where $U(t)$ is the unitary for the unperturbed dynamics and $U'(t)$ is the unitary for perturbed dynamics. Thus, the operator incompatibility is

$$\mathcal{I}_\mathcal{O}(t) = \frac{1}{2j^4}\mathrm{Tr}\,(|[\mathcal{O}(t), \mathcal{O}'(t)]|^2). \tag{1}$$

Now one can simplify this expression using the Hermiticity property of the physical observables.

$$\begin{aligned}\mathrm{Tr}\,(|[\mathcal{O}(t), \mathcal{O}'(t)]|^2) &= \mathrm{Tr}\,([\mathcal{O}(t), \mathcal{O}'(t)]^\dagger[\mathcal{O}(t), \mathcal{O}'(t)]) \\ &= \mathrm{Tr}\left(\big(\mathcal{O}(t)\mathcal{O}'(t) - \mathcal{O}'(t)\mathcal{O}(t)\big)^\dagger\big(\mathcal{O}(t)\mathcal{O}'(t) - \mathcal{O}'(t)\mathcal{O}(t)\big)\right) \\ &= \mathrm{Tr}\left(\big(\mathcal{O}'(t)\mathcal{O}(t) - \mathcal{O}(t)\mathcal{O}'(t)\big)\big(\mathcal{O}(t)\mathcal{O}'(t) - \mathcal{O}'(t)\mathcal{O}(t)\big)\right) \\ &= \mathrm{Tr}\left(\mathcal{O}'(t)\mathcal{O}(t)\mathcal{O}(t)\mathcal{O}'(t) - \mathcal{O}'(t)\mathcal{O}(t)\mathcal{O}'(t)\mathcal{O}(t) - \mathcal{O}(t)\mathcal{O}'(t)\mathcal{O}(t)\mathcal{O}'(t) + \mathcal{O}(t)\mathcal{O}'(t)\mathcal{O}'(t)\mathcal{O}(t)\right)\end{aligned} \tag{2}$$

We can further simplify the above expression by exploiting the cyclic property of trace operation and the property of a unitary operator $U^\dagger U = UU^\dagger = \mathbb{1}$. Let us consider the first term in the above expression

$$\begin{aligned}\mathrm{Tr}\left(\mathcal{O}'(t)\mathcal{O}(t)\mathcal{O}(t)\mathcal{O}'(t)\right) &= \mathrm{Tr}\left(U'^\dagger(t)\mathcal{O}U'(t)U^\dagger(t)\mathcal{O}U(t)U^\dagger(t)\mathcal{O}U(t)U'^\dagger(t)\mathcal{O}U'(t)\right) \\ &= \mathrm{Tr}\left(U^\dagger(t)U(t)U'^\dagger(t)\mathcal{O}U'(t)U^\dagger(t)\mathcal{O}\mathcal{O}U(t)U'^\dagger(t)\mathcal{O}U'(t)\right) \\ &= \mathrm{Tr}\left(U(t)U'^\dagger(t)\mathcal{O}U'(t)U^\dagger(t)\mathcal{O}\mathcal{O}U(t)U'^\dagger(t)\mathcal{O}U'(t)U^\dagger(t)\right) \\ &= \mathrm{Tr}\left(\mathcal{U}^\dagger(t)\mathcal{O}\mathcal{U}(t)\mathcal{O}\mathcal{O}\mathcal{U}^\dagger(t)\mathcal{O}\mathcal{U}(t)\right),\end{aligned} \tag{3}$$

where we define the error unitary $\mathcal{U}(t) = U'(t)U^\dagger(t)$. Similarly we can modify the remaining terms of the RHS in Eq. (2) and rewrite Eq. (1) as

$$\mathcal{I}_\mathcal{O}(t) = \frac{1}{2j^4}\mathrm{Tr}\,(|[\mathcal{O}, \mathcal{U}^\dagger(t)\mathcal{O}\mathcal{U}(t)]|^2). \tag{4}$$

II. UNDERSTANDING INFORMATION GAIN IN TOMOGRAPHY WITH ERRORS

Here, we describe information gain and the general behavior of reconstruction fidelity as shown in the main text. Later in this section, we will demonstrate the effect of the magnitude of perturbation on the fidelity obtained in tomography. We observe that independent of the degree of chaos in the dynamics, the fidelity initially rises despite errors and then starts to decline after attaining a peak. As described in the main text, we consider the density matrix of Hilbert-space dimension d that can be

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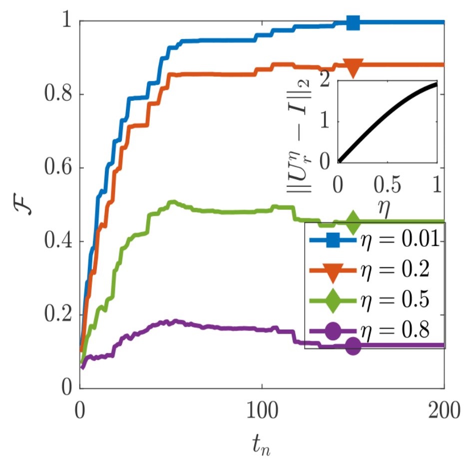

FIG. 1. Reconstruction fidelity as a function of time in the limit of vanishing shot noise for an increase in the perturbation of $\{E_\alpha\}$. The measurement operators are the perturbed ordered set $\{E'_1, E'_2, ..., E'_k\}$ with ordered Bloch vector components of the initial state ρ_0 (i.e. that corresponds to the Bloch vector components, $r_\alpha = \mathrm{Tr}\,[\rho_0 E_\alpha]$ in a particular order of their magnitudes). The perturbed operators $\{E'_\alpha\}$ are generated by applying a fractional power η of a random unitary U_r. The inset figure shows the Euclidean norm of the difference between U_r^η and Identity I, which increases with an increase in the value η.

realized as a generalized Bloch vector $\mathbf{r}$ by expanding $\rho_0 = I/d + \Sigma_{\alpha=1}^{d^2-1}\, r_\alpha E_\alpha$ in an orthonormal basis of traceless Hermitian operators $\{E_\alpha\}$.

The probability of reconstructing a state ρ_0 is [1]

$$p(\rho_\mathbf{0}|\mathbf{M}, \mathcal{L}, \mathcal{M}) = A\, p(\mathbf{M}|\rho_\mathbf{0}, \mathcal{L}, \mathcal{M})\; p(\rho_0|\mathcal{L}, \mathcal{M})\; p(\mathcal{L}, \mathcal{M}), \tag{5}$$

where A is a normalization constant. The first term $p(\mathbf{M}|\rho_\mathbf{0}, \mathcal{L}, \mathcal{M})$, is the probability of acquiring a measurement record $\mathbf{M}$, given an initial state ρ_0, the dynamics $\mathcal{L}$ (choice of unitaries), and the measurement process $\mathcal{M}$ (choice of operators $\mathcal{O}$ for generating measurement record). This term contains the errors due to shot noise and helps one to quantify the signal-to-noise ratio in various directions in the operator space independent of the state to be estimated. Thus, in the limit of negligible backaction $p(\mathbf{M}|\rho_\mathbf{0}, \mathcal{L}, \mathcal{M})$ is identical to the probability distribution corresponding to the measurement history $\mathbf{M}$ for a given Bloch vector $\mathbf{r}$ [2–6],

$$\begin{aligned} p(\mathbf{M}|\mathbf{r}) &\propto \exp\Big\{-\frac{N^2}{2\sigma^2}\sum_i [M_i - \sum_\alpha \tilde{\mathcal{O}}_{i\alpha} r_\alpha]^2\Big\} \\ &\propto \exp\Big\{-\frac{N^2}{2\sigma^2}\sum_{\alpha,\beta} (\mathbf{r}-\mathbf{r_{ML}})_\alpha\; C^{-1}_{\alpha\beta}\; (\mathbf{r}-\mathbf{r_{ML}})_\beta\Big\}. \end{aligned} \tag{6}$$

Therefore, this term estimates the information gained, given a density matrix, in different directions in the operator space. Now we have the second term $p(\rho_0|\mathcal{L}, \mathcal{M})$, which is the posterior probability distribution relating the knowledge of the dynamics and the measurement operators. On that account, in the limit of vanishing shot noise and with complete knowledge of the system dynamics for given measurement observables $\{E_\alpha\}$, this conditional probability is continuously updated and ultimately becomes a product of Dirac-delta functions. Once we obtain an informationally complete measurement record, each Dirac-delta function identifies a particular Bloch vector component. The term $p(\mathcal{L}, \mathcal{M})$ in Eq. (5) can be absorbed in the constant as it gives the prior information about the choice of dynamics and measurement operators. Thus, Eq. (5) separates the probability of quantum state estimation into a product of two terms (up to a constant) [1].

$$\begin{aligned} p(\rho_\mathbf{0}|\mathbf{M}, \mathcal{L}, \mathcal{M}) &\propto \exp\Big\{-\frac{N^2}{2\sigma^2}\sum_i [M_i - \sum_\alpha \mathcal{O}_{i\alpha} r_\alpha]^2\Big\}\; p(\rho_0|\mathcal{L}, \mathcal{M}) \\ &\propto \exp\Big\{-\frac{N^2}{2\sigma^2}\sum_{\alpha,\beta} (\mathbf{r}-\mathbf{r_{ML}})_\alpha\; C^{-1}_{\alpha\beta}\; (\mathbf{r}-\mathbf{r_{ML}})_\beta\Big\}\; p(\rho_0|\mathcal{L}, \mathcal{M}) \end{aligned} \tag{7}$$

In the limit of zero shot-noise, the errors due to the first term are zero, and we may purely focus on the conditional probability distribution, $p(\rho_0|\mathcal{L}, \mathcal{M})$. In terms of the observables in continuous measurement tomography, one can express $p(\rho_0|\mathcal{L}, \mathcal{M}) = p(\mathbf{r}|\mathcal{O}_\mathbf{1}, \mathcal{O}_\mathbf{2}, ..., \mathcal{O}_\mathbf{n})$, giving the conditional probability of the density matrix parameters $\mathbf{r}$ till the time step n. For example, consider the measurement operator at the first k time steps are the ordered set $\{E_1, E_2, ..., E_k\}$, giving precise information about Bloch vector components $\{r_1, r_2, ..., r_k\}$. The conditional probability distribution at time k is,

$$p(\mathbf{r}|E_1, E_2, ..., E_k) = \delta(r_1 - \mathrm{Tr}\,[E_1\rho_0])\; \delta(r_2 - \mathrm{Tr}\,[E_2\rho_0])\; ...\; \delta(r_k - \mathrm{Tr}\,[E_k\rho_0])\; \delta\big(\sum_{\alpha\neq 1,2,...k}^{d^2-1} r_\alpha^2 = 1 - 1/d - r_1^2 - r_2^2 ... - r_k^2\big). \tag{8}$$

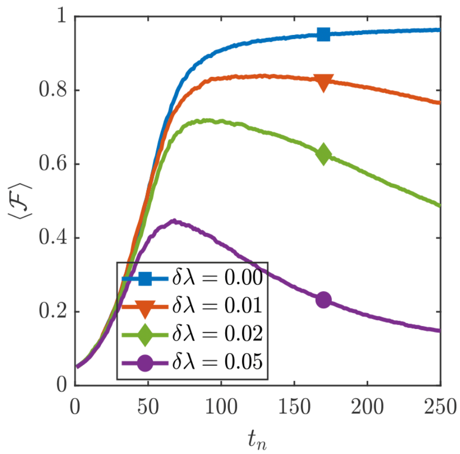

FIG. 2. Reconstruction fidelity as a function of time for an increase in perturbation strength. The measurement record is generated for spin $j = 10$. Here we consider rotation angle $\alpha = 1.4$ and kicking strength $\lambda = 7.0$ for the quantum kicked top.

Each noiseless measurement above gives us complete information in one of the orthogonal directions. For example, after the first measurement,

$$p(\mathbf{r}|E_1) = \delta(r_1 - \mathrm{Tr}\,[E_1\rho_0])\; \delta\big(\sum_{\alpha\neq 1}^{d^2-1} r_\alpha^2 = 1 - 1/d - r_1^2\big). \tag{9}$$

Therefore, once r_1 is determined, the rest of the $d^2 - 2$ Bloch vector components are constrained to reside on a surface given by the equation $\sum_{\alpha\neq 1}^{d^2-1} r_\alpha^2 = 1 - 1/d - r_1^2$. The state estimation procedure shall select a state based on incomplete information consistent with r_1 as determined precisely by the first measurement and the remaining Bloch vector components from a point on this surface. Therefore, qualitatively, the average fidelity of the estimated state is proportional to the area of this surface. After k time steps, the error is proportional to the area of the surface consistent with the equation $\sum_{\alpha\neq 1}^{d^2-1} r_\alpha^2 = 1 - 1/d - r_i^2 - r_j^2 - ... - r_k^2$. This area, quantifying the average error, decreases with each subsequent measurement.

To see it in another way, consider the fidelity between the actual and reconstructed state. The fidelity $\mathcal{F} = \langle\psi_0|\,\bar\rho\,|\psi_0\rangle$,

$$\mathcal{F} = 1/d + \Sigma_{\alpha=1}^{d^2-1}\; \bar r_\alpha r_\alpha \tag{10}$$

Here r_α and $\bar r_\alpha$ are the Bloch vectors for ρ_0 and $\bar\rho$ respectively. As one makes measurements, E_1, E_2, ..., E_k and gets information about the corresponding Bloch vector components (with absolute certainty in the case of zero noise for example), one can express the fidelity as

$$\mathcal{F} = 1/d + \Sigma_{i=1}^{k}\; r_i^2 + \Sigma_{\alpha\neq 1,2,...k}^{d^2-1}\; \bar r_\alpha r_\alpha \tag{11}$$

The term $\Sigma_{i=1}^{k}\; r_i^2$ puts a lower bound on the fidelity obtained after k measurements. Here $\bar r_\alpha$ for $\alpha \neq 1, 2, ..., k$ represent the state estimator's guess for the unmeasured Bloch vector components consistent with the constraint $\big(\sum_{\alpha\neq 1,2,...,k}^{d^2-1} r_\alpha^2 = 1 - 1/d - r_1^2 - r_2^2 ... - r_k^2\big)$. It is this guess that picks a point from the surface with an area consistent with the above constraint.

Consider the same scenario but now with perturbations to the system dynamics. The estimate of the density matrix gets modified as

$$p(\mathbf{r}|E_1', E_2', ..., E_k') = \delta(r_1' - \mathrm{Tr}\,[E_1'\rho_0])\; \delta(r_2' - \mathrm{Tr}\,[E_2'\rho_0])\; ...\; \delta(r_k' - \mathrm{Tr}\,[E_k'\rho_0])\; \delta\big(\sum_{\alpha\neq 1,2,...k}^{d^2-1} {r'}_\alpha^2 = 1 - 1/d - {r'}_1^2 - {r'}_2^2 ... - {r'}_k^2\big). \tag{12}$$

Here, $E_1', E_2', ..., E_k'$ are the perturbed operators leading to a slightly inaccurate estimate of the Bloch vector components $r_1', r_2', ..., r_k'$ respectively. The operators $\{E_\alpha'\}$ are obtained by rotating $\{E_\alpha\}$ by a unitary U_r^η, where U_r is a random unitary and η is a fractional power which makes U_r^η close to identity. The Euclidean norm of the operator $U_r^\eta - I$ is less when η is small. Thus, η serves as the strength of perturbation in this analysis of Bloch vector components.

Despite the perturbation, the uncertainty of the Bloch vector components r_α for $\alpha \neq 1, 2, ..., k$ reduces to the area of the surface consistent with the equation

$\sum_{\alpha\neq 1,2,...k}^{d^2-1} {r'}_{\alpha}^{2} = 1 - 1/d - {r'}_{1}^{2} - {r'}_{2}^{2}... - {r'}_{k}^{2}$. The fidelity between the original and the estimated state now reads as

$$\mathcal{F} = 1/d + \Sigma_{i=1}^{k}\; r'_i r_i + \Sigma_{\alpha\neq 1,2,...k}^{d^2-1}\; \bar{r}_\alpha r_\alpha, \qquad (13)$$

that we can see in Fig. 1. We know that the overlap between two Bloch vectors is maximum only when they are exactly aligned in the same direction, and the overlap decreases when they move far from each other. Comparing the second terms of Eq. (11) and Eq. (13) it is now clear why with an increase in perturbation, the initial rise in fidelity is less. Therefore, the drop in fidelity is more, and the fidelity saturates at a lower value if the perturbation is more, as illustrated in Fig. 2. For relatively weaker perturbations, the fidelity will continue to increase when there is an information gain despite such errors to the measurement operators. The partially inaccurate information about the jth Bloch vector owing to perturbations to the dynamics still offsets the estimator's guess of the Bloch components of the unmeasured jth direction in the operator space determined by E_j.

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