Supplementary Information

In the present Supplementary Information, we give a simple description on the design of the dispersion section using the transfer-matrix method. Here, we leave (y, y') out for simplicity, i.e. (x, x', z, δ) is used in the following analysis. The transfer matrix for a thin-lens magnet dipole and the chicane can be described as:

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{chicane} = \begin{pmatrix} 1 & 3L_c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2b_c^2 L_c \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S1}$$

where b is the bending angle of the dipole, b_c and L_c is the bending angle and the distance between each dipole in the chicane. The total transfer matrix for cascading a dipole and a chicane can be expressed as:

$$M_{dipole-chicane} = M_{chicane} M_{drift} M_{dipole} = \begin{pmatrix} 1 & 3L_c + L_{dr} & 0 & -b(3L_c + L_{dr}) \\ 0 & 1 & 0 & -b \\ b & 0 & 1 & 2b_c^2 L_c \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S2}$$

where L_{dr} is the drift length between the dipole and the chicane. The total transfer matrix has the required coupling coefficients $R_{51} = b$ and $R_{56} = 2b_c^2 L_c$, which can be separately adjusted and applied in the phase merging scheme with normal dispersion.

For the phase merging with angular dispersion scheme, the coefficient R_{52} is needed and can be provided by a dogleg with the corresponding transfer matrix of

$$M_{dogleg} = \begin{pmatrix} 1 & L_d & 0 & -b_d L_d \\ 0 & 1 & 0 & 0 \\ 0 & -b_d L_d & 1 & b_d^2 L_d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{S3}$$

where b_d and L_d are the bending angle and distance between the dipoles in the dogleg. Despite the coupling coefficients R_{52} and R_{56} can be maintained simultaneously with a vanishing R_{51}, these two coefficients are coupled with each other and causes poor scalability for various parameters. Cascading a dogleg and a chicane can provide an additional term in R_{56}, which is independent with the dogleg and is more feasible for various situations. The corresponding transfer matrix can be expressed as

$$M_{dogleg-chicane} = M_{chicane} M_{drift} M_{dogleg} = \begin{pmatrix} 1 & 5L_D & 0 & -b_d L_D \\ 0 & 1 & 0 & 0 \\ 0 & -b_d L_D & 1 & b_d^2 L_D + 2b_c^2 L_D \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S4}$$

where the distance between each two dipoles is assumed to be L_D in the dogleg-chicane cascading configuration.

For the phase merging with angular modulation and dispersion scheme, the coefficients R_{51} can be induced with a dipole, as indicated by Eq. (S1). Along with an appropriate drift section, the dipole can provide the coupling coefficient R_{52} to achieve the phase merging. An inserted quadruple makes the DS-II more feasible and the corresponding transfer matrix for DS-II can be expressed as:

$$M_{quadruple-dipole} = M_{dipole} M_{drift} M_{quadruple} = \begin{pmatrix} 1-kL_q L_{dr} & L_q + L_{dr} & 0 & 0 \\ -kL_q & 1 & 0 & -b \\ b(1-kL_q L_{dr}) & b(L+L_{dr}) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{(S5)}$$

where k and L_q are the focusing strength and the length of the quadruple.

Equations (S2), (S4) and (S5) give the design principles on DS-II of the phase merging schemes with normal dispersion, angular dispersion and the angular modulation and dispersion, respectively. The more specific parameters and designs are discussed in the start-to-end simulations part in the manuscript.