I. SUPLEMENTAL INFORMATION

A. Criterion for a Quantum Time Crystal

Here, we present the proper mathematical criterion for a quantum time crystal. A time crystal (TC) is a phase of matter which was proposed in 2012 by Wilczek [1] and has been the object of much interest since then.

Like many other phases of matter, TCs arise when the system spontaneously breaks one of the symmetries of its Hamiltonian. When a system spontaneously breaks, for example, translational symmetry, it becomes a crystal. If it breaks spin rotational symmetry, it becomes a magnet. TCs, on the other hand, appear when a system breaks the time translational symmetry (TTS).

In 2015, Watanabe and Oshikawa provided a mathematical criterion for verifying whether a system is in a quantum time crystal phase [2]. They followed steps based on the mathematical definition of a spatial crystal, in which, for the system to be considered a crystal, it has to show a non-trivial two-point correlator in an order parameter $\hat{O}$ in space far apart $\left(\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty}\langle\hat{O}(\mathbf{r})\hat{O}(\mathbf{r}')\rangle = f(\mathbf{r}-\mathbf{r}')\right)$. Analogously, for a system to be a TC, it has to show some non-trivial long-range correlations in time for some order parameter $\hat{O}$ at two different and far-apart times, namely,

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty}\ \lim_{t-t'\to\infty}\left\langle \hat{O}(\mathbf{r},t)\hat{O}(\mathbf{r}',t')\right\rangle = c(t), \tag{1}$$

for some non-trivial, or, in other words, non-stationary, function $c(t)$.

It is important to note that, in order for the correlator proposed in Eq. (1) to be evaluated, one needs to know the proper time evolution of the density matrix of the studied system. This is not always an easy task. In this paper, we instead studied the mean-field evolution of the condensates. In the mean-field approach, one cannot analyze the criterion exactly as proposed in Ref. [2]. Because of that, we will present the semiclassical modification of Eq. (1), which will serve as the criterion for semiclassical time crystals. The modified criterion, unlike the one in Eq. (1), can be used to identify TTS breaking in the mean-field approach.

B. Criterion for Semiclassical Time Crystals

Here, we present the criterion which is appropriate in the study of non-trivial time correlations in the mean-field dynamics and can be used to verify whether a system behaves as a semiclassical time crystal.

In Ref. [2], Watanabe and Oshikawa were focusing their studies entirely on strictly quantum time crystals. Because of that, we see in Eq. (1) the expected value of a product of order parameters which are quantum observables at different times and positions, represented by $\hat{O}$. In order to calculate the expected value in Eq. (1) one would, therefore, need to know the full density matrix of the system at both times, which might not always be an easy task. By replacing the system's density matrix by the mean-field wave function, one ends up losing all information on strictly quantum correlations. Therefore, it is impossible for us to compute precise expected values of a product of two quantum observables, as proposed by Watanabe and Oshikawa. However, one can still extract the time evolution of some order parameters from the mean-field dynamics. This is the case, for example, for the condensate mean-field density, which is the order parameter usually considered in the study of BECs.

We modify the correlator proposed by Watanabe and Oshikawa by replacing the strictly quantum order parameter $\hat{O}(\mathbf{r},t)$ by a semiclassical order parameter $\rho(\mathbf{r},t)$, which must depend on the mean-field wave function $\varphi(\mathbf{r},t)$. Namely, we replace $\left\langle \hat{O}(\mathbf{r},t)\hat{O}(\mathbf{r}',t')\right\rangle \to \langle\rho(\mathbf{r},t)\rho(\mathbf{r}',t')\rangle$ in Eq. (1). In this case, the quantum expectation value is replaced by the average over all space, while keeping $\mathbf{r}-\mathbf{r}' \equiv \Delta\mathbf{r}$ constant. In other words, the semiclassical analogue to the criterion shown in Eq. (1) is

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty}\ \lim_{|t-t'|\to\infty}\langle\rho(\mathbf{r},t)\rho(\mathbf{r}',t')\rangle = \lim_{|\Delta\mathbf{r}|\to\infty}\ \lim_{|t-t'|\to\infty}\frac{1}{V}\int\rho(\mathbf{r},t)\rho(\mathbf{r}+\Delta\mathbf{r},t')d\mathbf{r} = c(t), \tag{2}$$

where V is the system's volume, and $t' \ll t$ is an arbitrary time. The order parameter ρ could be, for example, the average local magnetization for a magnetic system, or the mean-field condensate density. A system that obeys the criterion in Eq. (2) shows time-dependent two-point correlations that spontaneously break TTS and is, therefore, a time crystal, albeit not necessarily a quantum time crystal since correlations caused by strictly quantum phenomena will be completely missed by this analysis. Following the nomenclature of Refs. [3, 4], both of which have conducted studies of TCs within the mean-field approximation, we refer to TCs which obey the criterion in Eq. (2) as Semiclassical Time Crystals.

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[2] Watanabe, H. & Oshikawa, M. Absence of Quantum Time Crystals, Phys. Rev. Lett. **114**, 251603 https://doi.org/10.1103/PhysRevLett.114.251603 (2015).

[3] Thies, M. Semiclassical time crystal in the chiral Gross- Neveu model, arXiv:1411.4236 (2014).

[4] Nalitov, A. V. *et al.*, Optically trapped polariton condensate as a semiclassical time crystal Phys. Rev. A **99**, 033830 https://doi.org/10.1103/PhysRevA.99.033830 (2019).