Appendix : Details for the prediction variance

The prediction variance of x in each fidelity model are decried as follow:

$$\begin{aligned}
v_1(x) &= v_1^p(x) - c_1^\top C_1^{-1} c_1 && (1)\\
v_2(x) &= v_2^p(x) - c_2^\top C_2^{-1} c_2 && (2)\\
&\cdots && (3)\\
& && (4)
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \operatorname{cov}\left[f_1(x), Y_1\right] = \sigma_1^2 \cdot r_1\left(x, X_1\right) && (5)\\
c_2 &= \operatorname{cov}\left[f_2(x), \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\right] = \begin{bmatrix} \rho_1 \sigma_1^2 r_1\left(x, X_1\right) \\ \rho_1^2 \cdot {\sigma_1}^2 r_1\left(x, X_2\right) + r_2^2 r_2\left(x, X_2\right) \end{bmatrix} && (6)\\
C_1 &= \operatorname{cov}\left[Y_1, Y_1\right] = \sigma_1^2 \cdot r_1\left(X_1, X_1\right) && (7)\\
C_2 &= \operatorname{cov}\left[\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\right] = \begin{bmatrix} \sigma_1^2 \cdot r_1\left(X_1, X_1\right) & \rho_1 \sigma_1^2 r_1\left(X_1, X_2\right) \\ \rho_1 \cdot \sigma_1^2 \cdot r_1\left(X_2, X_1\right) & \rho_1^2 \cdot \sigma_1^2 r_1\left(X_2, X_2\right) + \sigma_2^2 r_2\left(X_2, X_2\right) \end{bmatrix} && (8)\\
& && (9)
\end{aligned}$$

Rewrite C_2 as $\begin{bmatrix} C_{2(1)} & C_{2(2)} \\ C_{2(3)} & C_{2(4)} \end{bmatrix}$, where

$$\begin{aligned}
C_{2(1)} =& \sigma_1^2 \cdot r_1\left(X_1, X_1\right) && (10)\\
C_{2(2)} =& \rho_1 \sigma_1^2 r_1\left(X_1, X_2\right) && (11)\\
C_{2(3)} =& \rho_1 \cdot \sigma_1^2 \cdot r_1\left(X_2, X_1\right) = C_{2(2)}^\top && (12)\\
C_{2(4)} =& \rho_1^2 \cdot \sigma_1^2 r_1\left(X_2, X_2\right) + \sigma_2^2 r_2\left(X_2, X_2\right) && (13)\\
& && (14)
\end{aligned}$$

Then,

$$\begin{aligned}
C_2^{-1} &= \begin{bmatrix} C_{2(1)} & C_{2(2)} \\ C_{2(3)} & C_{2(4)} \end{bmatrix}^{-1} && (15)\\
&= \begin{bmatrix} C_{2(1)}^{-1} + C_{2(1)}^{-1} C_{2(2)} \left(C_{2(4)} - C_{2(3)} C_{2(1)}^{-1} C_{2(2)}\right)^{-1} C_{2(3)} C_{2(1)}^{-1} & -C_{2(1)}^{-1} C_{2(2)} \left(C_{2(4)} - C_{2(3)} C_{2(1)}^{-1} C_{2(2)}\right)^{-1} \\ -\left(C_{2(4)} - C_{2(3)} C_{2(1)}^{-1} C_{2(2)}\right)^{-1} C_{2(3)} C_{2(1)}^{-1} & \left(C_{2(4)} - C_{2(3)} C_{2(1)}^{-1} C_{2(2)}\right)^{-1} \end{bmatrix} && (16)\\
&= \begin{bmatrix} C_{2(1)}^{-1} + C_{2(1)}^{-1} C_{2(2)} \left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} C_{2(2)}^\top C_{2(1)}^{-1} & -C_{2(1)}^{-1} C_{2(2)} \left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} \\ -\left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} C_{2(2)}^\top C_{2(1)}^{-1} & \left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} \end{bmatrix} && (17)
\end{aligned}$$

Rewrite C_2^{-1} as $\begin{bmatrix} C_1^{-1} + W_{2(1)} & W_{2(2)} \\ (W_{2(2)}^\top) & W_{2(3)} \end{bmatrix}$, where

$$\begin{aligned}
W_{2(1)} &= C_{2(1)}^{-1} C_{2(2)} \left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} C_{2(2)}^\top C_{2(1)}^{-1} && (18)\\
W_{2(2)} &= -C_{2(1)}^{-1} C_{2(2)} \left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} && (19)\\
W_{2(3)} &= \left(C_{2(4)} - C_{2(2)}^\top C_{2(1)}^{-1} C_{2(2)}\right)^{-1} && (20)
\end{aligned}$$

Therefore, the prediction variance in each fidelity models are evaluated as follow:

$$
\begin{aligned}
v_1(x) &= v_1^p - c_1^T C_1^{-1} c_1 && (21)\\
&= \sigma_1^2 - \sigma_1^4 r_1^\top(x, X_1) \cdot C_1^{-1} \cdot r_1(x, X_1) && (22)\\
v_2(x) &= v_2^p - c_2^T C_2^{-1} c_2 && (23)\\
&= \underbrace{\rho_1^2\sigma_1^2 - \rho_1^2\sigma_1^4 r_1^\top(x, X_1) \cdot C_1^{-1} \cdot r_1^\top(x, X_1) - \rho_1^2\sigma_1^4 r_1^\top(x, X_1) \cdot W_{2(1)} \cdot r_1^\top(x, X_1)}_{\text{Term controlled only by LF samples}} && (24)\\
&\quad \underbrace{+2\rho_1^3\sigma_1^4 r_1^\top(x, X_1) \cdot W_{2(2)} \cdot r_1(x, X_2) + 2\rho_1\sigma_1^2\sigma_2^2 r_1^\top(x, X_1) \cdot W_{2(2)} \cdot r_1(x, X_2)}_{\text{Term controlled by LF and HF samples}} && (25)\\
&\quad \underbrace{+\sigma_2^2 - \sigma_2^4 r_2^\top(x, X_2) \cdot W_{2(3)} \cdot r_2(x, X_2) - \rho^4\sigma_1^4 r_1^\top(x, X_2) \cdot W_{2(3)} \cdot r_1(x, X_2)}_{\text{Term only controlled by HF samples}} && (26)\\
&\quad \underbrace{-2\rho^2\sigma_1\sigma_2 r_1^\top(X_2, X_2) \cdot W_{2(3)} \cdot r_2(x, X_2)}_{\text{Term only controlled by HF samples}} && (27)\\
&\cdots && (28)\\
& && (29)
\end{aligned}
$$