Supplementary Information

A House Divided: Cooperation, Polarization, and the Power of Reputation

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This supplementary information contains further technical details. These include previous and new work on reputation systems (Section 1), as well as a quantitative proof that GANDHI is an effective discriminator (Section 2),

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1 Reputation Systems

Formally, a *reputation system* determines a label "good" or "bad" for each individual, based on the history of interactions the individual was involved in. It is important to note that we allow for the possibility of including the reputation of former opponents as well, i. e., players have access to higher-order information. For example, the reputation system may rate defection against good or bad players differently.

In the base model, we initially assume that an individual's reputation is globally agreed upon and based on public information; the later discussion of polarization and tribalism hinges on divergence from this appraisal. To model the equivalence in the parallel interaction with all neighbors, we update reputation only *after* all eight neighbor duels of one propagation round have taken place. This also accounts for a delay in the exchange of information between neighbors until more tangible outcomes are visible; more responsive update rules only enhance the advantage of discriminator systems.

1.1 Previous Related Work

A discriminating strategy is only successful if its underlying reputation system provides useful guidance. To this end, a considerable variety of functions have been proposed. In some settings, simple reputation systems may suffice[1,2], but often they are not successful under all circumstances[3,4]. This observation led to the study of more advanced reputation systems that are often able to overcome shortcomings of simpler ones[4–7], frequently at the expense of using a larger amount of input data. In the following, we review the most important previously proposed reputation systems.

1.1.1 Image Scoring

The seminal work of Nowak and Sigmund[1,2] popularized investigation of indirect reciprocity in an evolutionary setting. The reputation system studied seems intuitive: Image Scoring rewards cooperative actions and penalizes defection. It keeps a *score* per player from the fixed discrete interval $\{b, \dots, g\}$ ($b, g \in \mathbb{Z}$ with $b < g$). Each game yields a score change of $+1$ or -1 if the player in question cooperates or defects, respectively — ignoring updates that leave the range $\{b, \dots, g\}$. Players with positive score have good reputation, all others are bad. The binary case $b = 0$, $g = 1$ was investigated in great detail[2], as its simplicity facilitates the use of analytical tools. In out experiments, We also considered the case $b = -1$ and $g = 1$. Both variants yield identical results.

Using Image Scoring, a player can extrapolate past actions to future behavior. However, using this knowledge comes at a price: If the other player is likely to defect, the best response is to defect as well, which in turn causes a loss in the player's own reputation. This makes punishment of uncooperative behavior costly.

1.1.2 Good Standing

Leimar and Hammerstein investigate the deficiency of Image Scoring and propose that Standing — as initially suggested by Sugden[8] — should be used instead[3]. All players are initially good. After each duel, reputation is updated as shown in Table S1. The row shows the *opponent*'s reputation, columns indicate the chosen action. The entries in the table show the new reputation resulting from the player's action. Several update rules for actions against bad players are referred to as "Standing" reputation systems in

	cooperate	defect
vs. good	good	bad
vs. bad	good *or* no change	no change

Table S1: Reputation update rules for STANDING and STRICT STANDING.

	cooperate	defect
vs. good	good	bad
vs. bad	bad	good

Table S2: Reputation update rules for the KANDORI system with $T = 1$.

the literature. We refer to the variant with good in the lower left cell as STANDING and with "no change" as STRICT STANDING. The latter is stricter in the sense that it requires more to regain good standing.

All of them allow non-costly or even beneficial punishment of defectors. However, this is not enough to maintain stable cooperation over time. Tolerated ALLC's can soften up the population and subsequently allow defectors to take over[1,9,10]. This makes it desirable to also discriminate between "justified" and "blind" cooperation.

1.1.3 The "Kandori" Reputation

Kandori[11] showed that mutual cooperation can always be established as a *sequential equilibrium* by choosing a suitable reputation system in a repeated random matching game. One such system is proposed by the author, which we refer to as KANDORI reputation.

For some fixed $T \in \mathbb{N}$, a player's *score* is a number in $\{-T, \ldots, -1, 0\}$, with only score 0 being regarded as good. After each duel, we increment the score by 1 if the player acted "compliant", i.e., cooperated with a good player or defected against a bad one. If she acted differently, her score is *reset* to $-T$. This means a dissenter will undergo T rounds of punishment during which she is considered bad and receives defecting treatment by the community. If T is chosen large enough, any immediate incentive to deviate is overcompensated by imminent outcasting. This mechanism has the drawback that excessively long "rehabilitation phases" make the system vulnerable to errors: If there is some small probability that players are wrongfully perceived as acting non-compliant, accidental punishment is amplified excessively. Keeping track of all scores also poses a memory demand on involved players (KANDORI requires $\lceil \log_2(T+1) \rceil$ bits of memory per player).

We considered KANDORI for values $T \in \{1, 2, 3, 8, 9\}$. The simplest choice $T = 1$ gives the reputation system shown in Table S2.

1.1.4 The Leading Eight

The reputation systems described above were mostly motivated by common sense. Ohtsuki and Iwasa took a more comprehensive approach. In[5], they studied the PD with reputation and considered *all* possible reputation systems that compute the new reputation deterministically from only the old reputation of both players and the chosen action. To judge their quality, the authors combined reputation systems with all possible (pure) strategies; for each of those pairs they determined whether it is an *evolutionary stable strategy* in the sense of classical evolutionary game theory; see[12]. Of the considerable number

Leading 2:

	cooperate	defect
vs. good	good	bad
vs. bad	flip	no change

Leading 3:

	cooperate	defect
vs. good	good	bad
vs. bad	good	good

Leading 4:

	cooperate	defect
vs. good	good	bad
vs. bad	no change	good

Leading 5:

	cooperate	defect
vs. good	good	bad
vs. bad	flip	good

Leading 8:

	cooperate	defect
vs. good	good	bad
vs. bad	bad	no change

Table S3: Reputation update rules for Leading 2, 3, 4, 5 and 8.

of 4096 pairs, many were evolutionary stable, but only eight pairs yielded consistently highest payoffs under varying cost/benefit ratios. A main contribution of Ohtsuki and Iwasa was the identification of common features shared among all those stable pairs: Both (a) cooperating with good players and (b) defecting against bad players must be considered as good. Of these eight pairs, only two different strategies for selecting the next action are used. In fact, all but the first two pairs use the Disc strategy (with varying reputation mechanisms): Cooperate if and only if your duel partner is good according to the corresponding reputation system.

In the following, "Leading *i*" refers to the pair in the *i*th line of Table 4 of Ohtsuki and Iwasa[5]. For Leading 1 and Leading 2, the so-called Or strategy is used instead, which cooperates if the opponent is good *or* if we ourselves are seen as bad. This behavior can be seen as the plausible incentive of a bad player to cooperate with just *anybody* to escape ostracism as fast as possible. Or is thus a less strictly discriminating strategy. It turns out that three of the leading eight pairs in fact use reputation strategies we already described under a different name:

- Leading 1 uses Standing reputation system,
- Leading 6 is equivalent to Kandori with $T = 1$ and
- Leading 7 employs Strict Standing.

For the remaining pairs, the reputation system is given in Table S3, where the row indicates the reputation of the opponent, the column gives the action the current player used against this opponent and the cell entry then describes the action corresponding to this player's reputation: An entry good or bad simply requires the reputation to be set to good or bad, "no change" leaves the reputation unchanged,

last action: vs. bad \ vs. good	defected	cooperated
defected	bad	good
cooperated	bad	bad

Table S4: Computation of reputation in Gandhi.

and "flip" inverts the reputation: If the player is good at the moment, it becomes bad, and if it was bad, it now becomes good.

1.2 The Gandhi Reputation

All reputation systems described so far only make use of a single piece of information per player. Observing defection, this makes it hard to distinguish between unconditional defection and rightful punishment. Similarly, a single cooperative action cannot be used to tell AllC and Disc apart.

We address this issue by introducing a new reputation system, called Gandhi, which only uses *two bits* of memory; the name is based on a well-known quote by Mahatma Gandhi: "Non-cooperation with evil is as much a duty as is cooperation with good."[13]. To the best of our knowledge, Gandhi has not been described in the study of reputation systems for indirect reciprocity. Other reputation systems, e. g. Kandori with large *T*, achieve similar discrimination, but they only do it implicitly over time (leading to inferior replicator dynamics), and by utilizing more memory.

Gandhi remembers for each individual *two* actions played: *the last action against a bad opponent* and *the last action against a good opponent*. A player is only regarded as good if its last action against a good opponent was cooperate *and* the last action against a bad opponent was defect. The other (three) combinations yield the label bad.

This self-referential aspect of the Gandhi reputation system hinges on updating preexisting values, hence on an initial state. There are different possibilities for initialization; for our base model, we initially consider all players to have cooperated with everybody. Accordingly, all players start out bad, and they can only turn good by defecting against a bad player.

1.3 Hidden Information and Perfect Group Cohesion: The Mafia Reputation

When studying cooperation within a group, a perfect mechanism for group cooperation would make group members always cooperate with other group members and defect against all outsiders, without giving away group membership to outsiders. This kind of strategy can be based on a hidden "membership bit" that is passed on to newly acquired group members, invisible to outsiders. We refer to this strategy as Mafia. Such a mechanism is not available in systems based on only public information, making Mafia a strong benchmark for measuring the ability of a discriminating strategy, i.e., by evaluating how close systems can get to Mafia's performance when only publicly available information can be used.

2 Quantitative Proof: Gandhi Is an Effective Discriminator

The two-dimensional spatial model used in the experiments exhibits rich and complex dynamics that are hard to predict analytically. In spatial settings without reputation, *pair approximation*[14] is usually

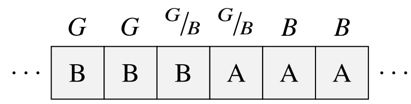

Figure S1: The setup: `good` (G) Gandhi (B) (respectively Mafia) on the left side, `bad` (B) AllD (A) players on the right.

the method of choice to predict the model's long-term behavior or to study the evolution of certain parameters[15]. However, pair approximation is not well suited for analyzing our spatial setting augmented with reputation, for the following reasons.

- The number of differential equations needed to describe the state of the system grows large (it doubles, even if one considers only binary reputations).
- The bookkeeping when deriving the equations by hand gets quite complicated.
- Pair approximation is in general not very precise and only predicts the *qualitative* behavior well.

As we are interested in a specific parameter (i.e., *invasion speed*), we have developed a Markov chain model that turns out to be both surprisingly accurate and relatively simple to analyze. We carry out our theoretical analysis for a *one-dimensional* spatial setting, as opposed to our two-dimensional model. We could empirically validate that the analytical invasion speeds derived in 1D fit the 2D experimental results very accurately. These observations suggest a general method to analyze suitable parameters (like invasion speed) of evolutionary spatial 2D models via a simpler 1D model.

2.1 One-Dimensional Approximation and Results

We consider a simple 1D model in which two homogeneous populations with different strategies face each other at a boundary (cf. Figure S1). The region left of the boundary is occupied by the Gandhi or Mafia players, the right region by AllD players. We then analyze how the boundary moves over time. Here we only discuss the situation of Gandhi (resp. Mafia) versus AllD, because the analysis versus AllC is considerably simpler.

The process of duel selection, strategy and reputation updating is analogous to the 2D setting. Note, however, that we consider an infinite one-dimensional chain of players, so we ignore any boundary effects.

It is not hard to see that there can never be a case in which a player is isolated from the other players of its strategy population. Hence, the strategy configuration can be fully described by the position of the boundary $n \in \mathbb{Z}$. In addition, one can easily show (for both Gandhi and Mafia) that the reputation configuration is fully determined by the reputation of the two players left and right of the boundary, which we denote by XY, where $X, Y \in \{G, B\}$. Because a defector will never be considered `good` after a game it participated in (as *focal* or as *opponent*), the combination BG can never occur. Therefore, the tuple (n, XY), where $XY \in \{GG, GB, BB\}$, describes the current state of the process. Also, it fully encodes the information that is relevant for performing the next round — the dynamics of (n, XY) over time is therefore a Markov process, and completely described by the transition probabilities between two such tuples.

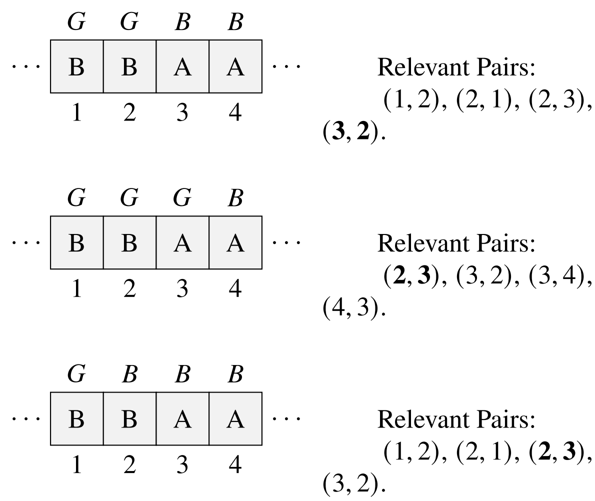

Figure S2: The three possible configurations of the boundary along with the relevant (*focal*, *opponent*) pairs. In the bottom two cases, a pair is defined as " relevant" if its choice can lead to a change in the configuration; in the first case, we arbitrarily define the above four pairs as "relevant" in order to generate an equal number of pairs as in the other two cases (only $(3, 2)$ is crucial here, the other pairs could be chosen differently). Of these, only the pairs in boldface can lead to a change in strategies — all other pairs will lead to the "standard" reputation configuration GB.

Analyzing the speed of the (reputation-free) setting MAFIA VS. ALLD is relatively straighforward, as the boundary can only move to the right. The mathematical analysis of Section 2.2 yields

$$\psi_+ \;=\; \frac{1}{8}\frac{1-u}{1+u}$$

for the drift speed of the boundary.

The invasion speed of GANDHI VS. ALLD can be analyzed by solving a recurrence equation for the expected time to go from state $(BB, n-1)$ to (BB, n) (see below for details). We obtain

$$T^{-1} \;=\; \frac{-5u^2 - 3u + 8}{41u^2 + 111u + 72}$$

for the drift speed. This can be well approximated by $8/9$ of that of the (optimal) MAFIA VS. ALLD drift (cf. Extended Data Fig. 2a), as follows.

$$T^{-1} \;\approx\; \frac{1}{9}\frac{1-u}{1+u}.$$

There is a simple intuitive explanation for this factor: After defeating an opponent and extending the boundary of GANDHI, one in eight cases requires an additional round at the boundary for updating the reputation; this is not necessary for MAFIA, as it uses hidden information that is updated instantaneously.

2.2 Mathematical Analysis

For easier exposition, we only consider situations from which a change in strategies or reputations can result. It suffices to consider four pairs of players for that (see Figure S2); when we report probabilities

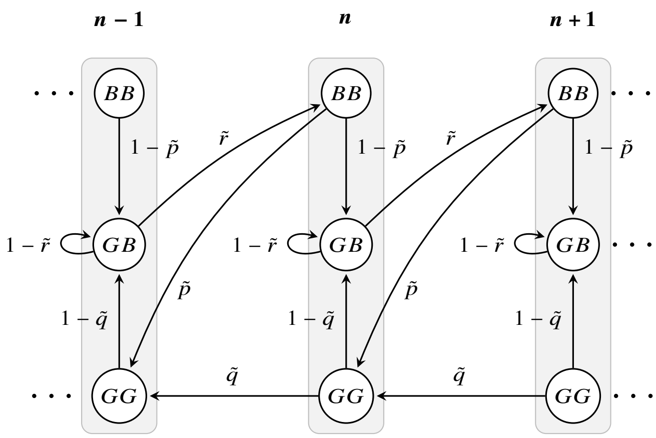

Figure S3: Transition probabilities for Gandhi vs. AllD (PD), where $\tilde{p} = p/4$, $\tilde{q} = q/4$ and $\tilde{r} = r/4$.

in this section, they are always meant *conditional* to one of these four pairs being chosen. (Note that the pairs change over time, but there are always four of them.)

There are the following different *focal*/*opponent* combinations in which the boundary can move with the given probabilities, conditional to the pair being selected in this particular *focal* ← *opponent* order:

$$\overset{B}{\textsc{Gandhi}} \leftarrow \overset{B}{\textsc{AllD}} \qquad \text{with prob. } p = \frac{u}{2(1+u)}$$

$$\overset{G}{\textsc{Gandhi}} \leftarrow \overset{G}{\textsc{AllD}} \qquad \text{with prob. } q = \frac{2u}{2(1+u)}$$

$$\overset{G}{\textsc{Gandhi}} \rightarrow \overset{B}{\textsc{AllD}} \qquad \text{with prob. } r = \frac{1-u}{2(1+u)}$$

$$\textsc{Mafia} \rightarrow \textsc{AllD} \qquad \text{with prob. } r = \frac{1-u}{2(1+u)}$$

For Mafia playing Prisoner's Dilemma against AllD, reputation does not play a role. The only possible transition is

$$\textsc{Mafia} \;\rightarrow\; \textsc{AllD},$$

which has (conditional) probability r. Hence the probability of the boundary moving from $n-1$ to n is

$$\psi_+ \;=\; \frac{1}{4}r \;=\; \frac{1}{8}\frac{1-u}{1+u},$$

where we have re-scaled to match our time normalization.

For Gandhi playing Prisoner's Dilemma against AllD, the situation is more involved, because it is now possible that the boundary moves left as well as right. The transition probabilities are now as shown in Figure S3. While these probabilities define the Markov chain completely, we proceed to provide a more analytic measure for the overall drift of the boundary, i.e., the probability of the boundary moving left or right. We analyze the infinite 1-dimensional Markov chain by a recursion that yields the *expected time* for reaching n from $n-1$. The inverse of this time serves as an estimate of the drift speed.

Note that moving the boundary from $n-1$ to n requires going through the state (BB, n). This motivates the following definition: Denote by $T_n(XY, k)$ the expected time to reach (BB, n) from (XY, k), where $XY \in \{GG, GB, BB\}$. Conversely, every transition from n to $n-1$ has to go through $(GG, n-1)$, so we are particularly interested in $t_n = T_n(GG, n-1)$.

First, we find that

$$\begin{aligned} t_n &= T_n(GG, n-1) \\ &= \frac{q}{4}\big(1 + T_n(GG, n-2)\big) + \Big(1 - \frac{q}{4}\Big)\big(1 + T_n(GB, n-1)\big). \end{aligned} \tag{1}$$

The only transition away from $(GB, n-1)$ leads to (BB, n) and is taken with probability $r/4$. So we will eventually reach (BB, n) and the number of steps needed is geometrically distributed with parameter $r/4$. The expectation is then given by the inverse of this parameter:

$$T_n(GB, n-1) = 4/r. \tag{2}$$

Also, we have that

$$\begin{aligned} T_n(GG, n-2) &= T_{n-1}(GG, n-2) + T_n(BB, n-1) \\ &= t_{n-1} + \frac{p}{4}\big(1 + T_n(GG, n-2)\big) + \Big(1 - \frac{p}{4}\Big)\big(1 + \underbrace{T_n(GB, n-1)}_{=4/r \text{ by } (2)}\big) \\ &= t_{n-1} + 1 + \frac{p}{4} T_n(GG, n-2) + \Big(1 - \frac{p}{4}\Big)\frac{4}{r}. \end{aligned}$$

Rearranging the terms yields

$$T_n(GG, n-2) = \frac{t_{n-1} + 1}{1 - p/4} + \frac{4}{r}. \tag{3}$$

Substituting back into (1) results in

$$t_n = \frac{q}{4}\Big(1 + \frac{t_{n-1} + 1}{1 - p/4} + \frac{4}{r}\Big) + \Big(1 - \frac{q}{4}\Big)\Big(1 + \frac{4}{r}\Big).$$

Because there is no boundary, we have $t_n = t_{n-1} =: t$ by symmetry. Hence,

$$t = \frac{4 - p + 4\frac{4-p}{r} + q}{4 - p - q}.$$

Then, the expected time it takes from $(BB, n-1)$ to (BB, n) is

$$\begin{aligned} T &= T_n(BB, n-1) \\ &= \frac{p}{4}\big(1 + \underbrace{T_n(GG, n-2)}_{=\frac{t+1}{1-p/4} + 4/r \text{ by } (3)}\big) + \Big(1 - \frac{p}{4}\Big)\big(1 + \underbrace{T_n(GB, n-1)}_{=4/r \text{ by } (2)}\big) \\ &= 1 + p\frac{t+1}{4-p} + \frac{4}{r} \\ &= \Big(1 + \frac{4}{r}\Big)(1 + p) + p\frac{q+1}{4-p}. \end{aligned}$$

By substituting u in, we obtain

$$T^{-1} = \frac{-5u^2 - 3u + 8}{41u^2 + 111u + 72}$$

as an estimate for the drift speed.

Our analysis suggests that the drift of the 1-dimensional Markov chain for GANDHI VS. ALLD can be approximated by $\frac{8}{9}$ of that of the (optimal) MAFIA VS. ALLD drift. Hence,

$$T^{-1} \approx \frac{1}{9}\frac{1-u}{1+u}$$

appears to be a good estimate for the invasion speed.

As we show in the following Section 2.2.1, data from two-dimensional simulation and from this one-dimensional analysis match extremely well. This is no coincidence: a strongly invading population will tend to form a large shape that is close to being convex, resulting in a well-focused, uniform average degree with respect to the own subpopulation for individuals along the boundary. (In a grid setting, this average degree very rapidly converges to five for large clusters.) Thus, we get the same fundamental behavior as in the one-dimensional case, subject to some scaling to account for different normalization.

2.2.1 Validation of the 1D Model on 2D Data

If we compare the experimental invasion speed plots from Figure 2b to the functions from above, we see that their qualitative agreement is excellent. More precisely, if $M(u)$, $u \in \{0.1, 0.2, \ldots, 0.9\}$, is the vector of experimental MAFIA speeds and $f(u)$ is the analytically derived speed, we see that $\frac{M(u)}{f(u)}$ is (almost) constant. Because we count time steps differently in our 1D analysis and in order to adjust for geometric effects of dimensionality, we scale the functions by this constant (which we call c), which turns out to be $c = 4.8167$.

Because we only use MAFIA in the PD to determine c, we can use the exact same scaling for the GANDHI speeds in order to validate our claim that c only captures the effects of dimensionality and the different counting of timesteps. A plot of these scaled functions together with the respective data and figures from the main paper shows that the 1D analysis matches the 2D data remarkably well; see Extended Data Fig. 2b.

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