The waveform comparison of three common-used fractional viscous acoustic wave equations

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The waveform comparison of three common-used fractional viscous acoustic wave equations

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Abstract
The forward simulation of the viscous acoustic wave equation is an essential part of geophysics and energy resources exploration research. The viscous acoustic seismic wave equations are diverse, even if we limit the study scope to the fractional viscous wave equations. In the present study, we consider three commonly used fractional-order viscous wave equations: the fractional viscous acoustic wave (FV AW) equation, dispersion-dominated wave (DDW) equation, and attenuation-dominated wave (ADW) equation. The acoustic wave (AW) equation, as a special fractional wave equation, is used to compare with the three viscous acoustic equations. The asymptotic local finite difference (ALFD) method is adopted to solve the three fractional wave equations, while the Lax-Wendroff Correction (LWC) scheme is used to solve the integer wave equation. The analysis shows that the stability of the ADW equation is the most rigorous, and that of the DDW equation is the most flexible. When the numerical wave number \( \vartheta = \pi \), the maximum phase velocity errors of the FV AW equation, DDW equation, ADW equation, and AW equation are 27.78\%, 28.02\%, 2.25\%, and 3.04\%, respectively. Numerical experiments show that the waveforms simulated by the four equations with the same parameters are distinct. Specifically, the FV AW equation, DDW equation, and quality factor \( Q \) are sensitive to the arrival time, while the FV AW equation, ADW equation, and quality factor \( Q \) are sensitive to the amplitude. Furthermore, the change of amplitude is more apparent than that of the arrival time, giving the results that the arrival time is more robust than the amplitude.

Keywords:
Fractional viscous wave equation, Arrival time and amplitude, Dispersion and attenuation, Finite difference method.

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All authors have contributed to the creation of this manuscript for important intellectual content and read and approved the final manuscript. We further confirm that the order of author listed in the manuscript has been approved by all of us. We declare there is no conflict of interest.

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1. Introduction

The wave propagation simulation in complex media is important to geophysics and energy resources exploration research. Usually, wave propagation medium is regarded as perfectly elastic media. However, many studies show that there are unignorable differences in both the amplitude and phase between the seismic record generated with perfectly elastic media hypothesis and the collected seismic record (Aki and Richards 1980). In fact, the underground medium is the media between elasticity and viscosity. Especially, in exploration geophysics, since the presence of fluid pockets, the target domain shows high seismic attenuation (Dvorkin and Mavko, 2006). Therefore, many studies have conducted viscoelastic medium models to describe underground media (Aki and Richards 1980; Stokes 1845; Madja et al. 1985).

Currently, commonly used constitutive models for viscoelastic media include Kelvin model (Chen 2008), Maxwell model (Madja et al. 1985), standard linear solid (SLS) model (Aki and Richards 1980) and Generalized Standard Linear Solid (GSLS) model (Emmerich and Korn 1987). The viscous acoustic seismic wave equations are also diverse, and can be mathematically divided into two categories: integer-order and fractional-order. The fractional-order operator is essentially differential and integral operator. The integration term in definition can fully reflect the historical dependence of functions, making them a mathematical tool for modeling memory processes. Therefore, fractional-order operator has become the useful mathematical tool for describing the viscosity characteristics of underground medium in previous geophysics studies. For example, Caputo (1969) and Mainardi (1971) applied fractional calculus methods to complex viscoelastic and rheological media, developing dissipative linear models in elastic solids. Kjartansson (1979) proposed a frequency-independent three-parameter (reference velocity, attenuation factor, and reference frequency) constant quality factor $Q$ linear attenuation model. Carcione et al. (2002, 2009, 2010) derived time-domain fractional-order viscous wave equations, using Riemann-Liouville type time-fractional derivatives to characterize the time accumulation effect of viscous properties. Carcione (2013) incorporated the Standard Linear Solid (SLS) relaxation mechanism into the time-fractional constant $Q$ model to describe the propagation of viscous and viscoelastic waves. Since time-fractional derivatives require high storage, scholars have proposed converting time-fractional constant $Q$ wave equations to spatial-fractional constant $Q$ wave equations (Chen and Holm 2004; Treeby and Cox 2010; Zhu and Carcione 2014; Zhang et al. 2022).

There is diversity in the spatial-fractional viscosity seismic wave equations commonly used in geophysics. Carcione (2010) considered a constant $Q$ acoustic wave equation that emphasizes lossless acoustic dispersion phenomena, in which the losses and dispersion of the geological medium are considered together. Treeby and Cox (2010) studied the viscous acoustic wave equation proposed by Chen et al. (2004) based on the fractional Laplacian operator, which emphasizes the attenuation of viscous waves during propagation. Zhu et al. (2014a, b), Sun et al. (2016), and Yao et al. (2017) studied a fractional Laplacian wave equation containing both dispersion and attenuation operators. Wang et al. (2018) developed an almost equivalent constant fractional viscosity wave equation based on a spatially variable order fractional viscoelastic wave equation. Li et al. (2019) derived a viscous acoustic wave equation with amplitude decay controlled by a fractional Laplacian operator and a viscous acoustic wave equation with phase dispersion. These studies show that the form of the fractional control equation is not unique. Thus, the difference of these fractional viscous acoustic wave equation is one of our research focuses.

The common numerical forward methods include Finite Difference (FD) method (Altman and Karal 1968; Madariaga 1976; Dablain 1986; Yang et al. 2014), spectral element method (Priolo and Seriani 1991), pseudospectral method (Gazdag 1981; Dan and Baysal 1982; Fornberg 1987; Carcione 2010; Mu et al. 2021). Among them, the FD method is one of the most widely used forward modeling methods in geophysics for its relatively low requirements on calculation, high computational efficiency and simplicity of the implementation. However, the traditional FD method of fractional differential equations uses a global difference operator to approximate the fractional derivative, resulting in low computational efficiency of forward modelling. It is worth mentioning that Song et al. (2020) give
an Asymptotic Local Finite Difference (ALFD) discretization method with a reasonable truncated stencil of
traditional fractional derivatives to improve the computational efficiency. The computational speed of ALFD is much
faster than that of the original FD method, making it suitable for seismic simulating of fractional wave equation.

In present study, we first consider four seismic wave equations. Three of them are fractional-order wave equation
and the remaining one is a classical integer wave equation. Secondly, we introduce the numerical discrete schemes
used in this paper. We use the Lax-Wendroff Correction (LWC) discretization method (Lax and Wendroff 1964;
Dablain 1986) and ALFD discretization method (Song et al. 2020) to deal with the integer- and fractional- order
partial derivative, respectively. Next, we give the theoretical properties of the four equations and numerical
experiments to study the features of seismic wave with different governing equation and $Q$. At last, we give the
conclusion of this paper.

2. Fractional viscous acoustic wave equations

The generally fractional viscous acoustic wave equation (FV AW) (Zhu and Harris 2014; Zhu 2014; Sun et al.
2016; Yao et al. 2017) is expressed as

$$\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \eta \kappa_1 P + \tau \frac{\partial}{\partial t} \kappa_2 P \quad (1)$$

where $\kappa_1 = (\partial^{2+2}/\partial|x|^{2+2} + \partial^{2+2}/\partial|z|^{2+2})$ represents the dispersion operator, $\kappa_2 = (\partial^{2+1}/\partial|x|^{2+1} + \partial^{2+1}/\partial|z|^{2+1})$ denotes the attenuation operator, $\partial^{2+2}/\partial|x|^{2+2}$ (or $\partial^{2+2}/\partial|z|^{2+2}$) is the Riesz fractional partial derivative of $2\gamma + 2$ order. $\eta = \omega_0 c_0^2 \cos(\pi\gamma)$ is the dispersion coefficient, $\tau = \omega_0 c_0^2 \sin(\pi\gamma)$ is the attenuation factor, $c = c_0 \cos(\pi\gamma / 2)$ is the P-wave propagation velocity and $\gamma = 1/\pi \arctan(1/Q)$. The parameter $c_0$ is the phase
velocity, $\omega_0$ is the reference frequency, $Q$ is the quality factor. $P$ is the pressure.

The second viscous wave equation (Carcione 2010; Treeby and Cox 2010; Zhu and Harris 2014) is

$$\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \eta \kappa_1 P. \quad (2)$$

We name equation (2) as the dispersion-dominated wave (DDW) equation to distinguish it from the other viscous
wave equations. The third viscous wave equation (Treeby and Cox 2010; Zhu and Harris 2014) is

$$\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2}\right) + \tau \frac{\partial}{\partial t} \kappa_2 P. \quad (3)$$

We call equation (3) the attenuation-dominated wave (ADW) equation. When $Q$ approaches to infinity, three
fractional viscous equation approaches to the traditional acoustic wave (AW) equation

$$\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2}. \quad (4)$$

3. Numerical solution to fractional viscous wave equations

3.1 The scheme of the temporal derivative

The temporal partial derivative is discretized by the first-order backward difference (5) and second-order central
difference (6).

$$\frac{\partial P_{i,j}^n}{\partial t} = \frac{P_{i,j}^n - P_{i,j}^{n-1}}{\Delta t} \quad (5)$$

$$\frac{\partial^2 P_{i,j}^n}{\partial t^2} = \frac{P_{i,j}^{n+1} - 2P_{i,j}^n + P_{i,j}^{n-1}}{\Delta t^2} \quad (6)$$

3.2 The scheme of the integer-order spatial derivative

The integer-order spatial derivative in x direction is discretized by the fourth-order LWC method.

$$\frac{\partial^2 P_{i,j}^n}{\partial x^2} = \frac{1}{\Delta x^2} \left[ -\frac{5}{2} P_{i,j}^n + \frac{4}{3} \left( P_{i+1,j}^n + P_{i-1,j}^n \right) - \frac{1}{12} \left( P_{i+2,j}^n + P_{i-2,j}^n \right) \right] \quad (7)$$

Similarly, we can obtain the fourth-order LWC scheme in z direction.
3.3 The scheme of the fractional-order spatial derivative

Since the traditional FD method of fractional spatial derivative need to calculate global difference, resulting in low computational efficiency of forward modelling. Song et al. (2020) give an asymptotic local finite difference (ALFD) method with a reasonable truncated stencil of traditional fractional derivatives to improve the computational efficiency. In present study, the fractional-order spatial derivative in \( x \) direction is discretized by the fourth-order ALFD scheme (Song et al. 2020)

\[
\frac{\partial^\alpha P^n_{i,j}}{\partial|x|^\alpha} = -\frac{\alpha}{24\Delta x^\alpha} \sum_{k=-N}^{N} g_k^{(a)} P^{n}_{i-1,k,j} - \frac{\alpha}{24\Delta x^\alpha} \sum_{k=-N}^{N} g_k^{(a)} P^{n}_{i+1,k,j} - \left(1 - \frac{\alpha}{12}\right) \frac{1}{\Delta x^\alpha} \sum_{k=-N}^{N} g_k^{(a)} P^{n}_{i,k-j} \quad (8)
\]

where

\[
g_k^{(a)} = \frac{(-1)^k}{\Gamma(\alpha + 1)} \Gamma(\alpha / 2 - k + 1) \Gamma(\alpha / 2 + k + 1), \quad (9)
\]

\[
\alpha = 2 + 2\gamma \quad \text{(or) } \alpha = 1 + 2\gamma. \quad (10)
\]

Song et al. (2020) gave the minimum differential stencil required under different truncation error conditions. We choose \( N = 7 \) to approach the truncation error \( \varepsilon = 10^{-5} \) (Song et al. 2020). Similarly, we can obtain the fractional partial derivative difference scheme in \( z \)-direction.

3.4 Stability conditions

In this section, we use the Fourier method to analyze the stability of four viscous wave equations. Let the harmonic solution \( w^n_{i,j} = [p^n_{i,j}, v^n_{i,j}]^T = V^n e^{i(h(x, z))} \), where \( h = \Delta x = \Delta z \) is the spatial interval. \( I = \sqrt{-1} \) is an imaginary unit. \( k_x \) and \( k_z \) are the wave numbers in the \( x \) and \( z \) directions, respectively.

We take the FVAW equation as an example to show the procedure of stability analysis. We bring the harmonic solution into (5), (6), (8) and rewrite the FD scheme into a matrix form

\[
V^{n+1} = GV^n, \quad (11)
\]

where \( G \) is the update matrix

\[
G = \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} \quad (12)
\]

\[
a_1 = 2 - R_M M_1 - R_M M_2 \quad (13)
\]

\[
a_2 = -1 + R_M M_2 \quad (14)
\]

\[
R_1 = \frac{\Delta \Gamma^2 c \eta}{h^{N(y+1)}} \quad (15)
\]

\[
R_2 = \frac{\Delta \Gamma c^2 \tau}{h^{N(z+0.5)}} \quad (16)
\]

\[
M_1 = \sum_{k_x = -N_1}^{N_1} g_{k_x}^{(2,2)} e^{-i h k_x} \left[ \frac{(1 + \gamma)}{6} \cos(h k_x) + \frac{(5 - \gamma)}{6} \right] \\
+ \sum_{k_x = -N_1}^{N_1} g_{k_x}^{(2,2)} e^{-i h k_x} \left[ \frac{(1 + \gamma)}{6} \cos(h k_x) + \frac{(5 - \gamma)}{6} \right] \quad (17)
\]

\[
M_2 = \sum_{k_x = -N_1}^{N_1} g_{k_x}^{(2,2)} e^{-i h k_x} \left[ \frac{(0.5 + \gamma)}{6} \cos(h k_x) + \frac{(5.5 - \gamma)}{6} \right] \\
+ \sum_{k_x = -N_1}^{N_1} g_{k_x}^{(2,2)} e^{-i h k_x} \left[ \frac{(0.5 + \gamma)}{6} \cos(h k_x) + \frac{(5.5 - \gamma)}{6} \right] \quad (18)
\]

Subsequently, we can get the two eigenvalues \( \lambda_{1,2} \) of the update matrix \( G \).

\[
\lambda_{1,2} = \frac{1}{2} \left[ 2 - R_M M_1 - R_M M_2 \pm \sqrt{(R_M M_1 + R_M M_2)^2 - 4R_M M_1} \right] \quad (19)
\]

According to Fourier method, the stability condition should satisfy \( \max \{|\lambda_1|, |\lambda_2|\} \leq 1 \).

Unlike the integer-order FD scheme, the stability of the FVAW equation’s numerical method depends on the...
truncation error $\epsilon$ and the fractional-order $\gamma$. We choose $\gamma = 0.0317, \epsilon = 10^{-5}$ and give the stability conditions of the FVAW equation as

$$\Delta t_{\text{FVAW}} \leq 0.693 \frac{h^{1.0317} \epsilon^{0.0317}}{c_0^{1.0317}}.$$  \hfill (20)

Similarly, we can respectively give the stability condition of the DDW equation, the ADW equation, the AW equation as

$$\Delta t_{\text{DDW}} \leq 0.879 \frac{h^{1.0317} \epsilon^{0.0317}}{c_0^{1.0317}},$$  \hfill (21)

$$\Delta t_{\text{ADW}} \leq \frac{0.524 h}{c},$$  \hfill (22)

$$\Delta t_{\text{AW}} \leq \frac{0.729 h}{c}. \hfill (23)$$

It is obvious that the stability of the above four viscous wave equations are different. The stability of the ADW equation is the most rigorous and that of the DDW equation is the most flexible.

3.5 Numerical dispersion analysis

In this section, we perform numerical dispersion analysis by using Fourier method. We substitute the harmonic solution $P_n = \exp[I(\Omega t + jhk_x - lc'n\Delta t)]$ into the difference scheme of above the four equations. After the mathematical derivation, we can give the numerical phase velocity expressions corresponding to the four equations as:

$$c'_{\text{FVAW}} = \frac{1}{l\Delta t} \arccos \left( 1 - \frac{R_1 W_1}{2 - R_1 W_2} \right),$$  \hfill (24)

$$c'_{\text{DDW}} = \frac{1}{l\Delta t} \arccos \left( 1 - \frac{1}{2} R_1 W_1 \right),$$  \hfill (25)

$$c'_{\text{ADW}} = \frac{1}{l\Delta t} \arccos \left( 1 + \frac{c^2 \Delta^2 S}{2 - R_1 W_2} \right).$$  \hfill (26)

$$c'_{\text{AW}} = \frac{1}{l\Delta t} \arccos \left( 1 + \frac{c^2 \Delta^2 S}{2h^2} \right),$$  \hfill (27)

where

$$W_1 = \sum_{i=1}^{N} \left( \frac{1 + \gamma}{6} \cos X_i + \frac{5 - \gamma}{6} \right) \sum_{i=N}^{N} S_i^{1+22} e^{-hX_i},$$  \hfill (28)

$$W_2 = \sum_{i=1}^{N} \left( \frac{0.5 + \gamma}{6} \cos X_i + \frac{5.5 - \gamma}{6} \right) \sum_{i=N}^{N} S_i^{1+22} e^{-hX_i},$$  \hfill (29)

$$S = -\frac{8}{3} \left( \cos X_1 + \cos X_2 \right) - \frac{1}{6} \left( \cos (2X_1) + \cos (2X_2) \right).$$  \hfill (30)

$$X_1 = h k_x = h k \cos \theta,$$  \hfill (31)

$$X_2 = h k_x = h k \sin \theta.$$  \hfill (32)

Figure 1 gives the dispersion relationship of the four viscous wave equations. The horizontal axis represents numerical wavenumber $\vartheta = k h$. The vertical axis represents the ratio $c'/c_0$, where $c'$ represents the numerical phase velocity.

From the Figure 1 we can see that the algorithm of four equations can effectively capture the low frequency components of the wave field. But for high frequency components, the numerical phase velocity has a certain deviation from the real phase velocity which is particularly evident in diagonal direction. When the numerical wave number $\vartheta = \pi$, the maximum phase velocity errors of the FVAW equation, DDW equation, ADW equation and AW equation are 27.78%, 28.02%, 2.25% and 3.04%, respectively.
4. Numerical experiment

In this section, we give the waveform of FVAW, DDW and ADW, calculating by the scheme in Section 3. For subsequent comparison, we regard AW as the reference wave.

4.1 Uniform velocity model

To compare the dispersion and attenuation of seismic waves controlled by the four equations before mentioned, our first model is a sample uniform velocity model. We choose a computational domain \(2km \times 2km\), the phase velocity \(c_0 = 2.1km/s\), \(Q=10\). The source located at the center of the computational domain, \((1km, 1km)\), is a Ricker wavelet

\[ s(t) = -5.76f_0^2 \left[1 - 16(0.6f_0 \Delta t - 1)^2 \right] e^{-(0.6f_0 \Delta t)^2}, \]  

where the frequency \(f_0 = 30Hz\). The temporal step \(\Delta t = 0.001s\), and the spatial step \(\Delta x = \Delta z = 0.005km\).

Figure 2 is a stitching snapshot of the wave field at 0.4s controlled by the four equations spliced with black dashed lines. From Figure 2, compared with the snapshot of AW, the snapshot of DDW shows obvious phase dispersion and no amplitude attenuation. The snapshot of ADW shows apparent amplitude attenuation but no phase dispersion. The snapshot of FVAW shows both apparent phase dispersion and amplitude attenuation.

In order to observe the attenuative and dispersive phenomena clearly, Figure 3 give single-point waveform
recorded at (1km, 0.9km) from 60ms to 300ms. In order to clearly observe the difference at the troughs, we amplify the troughs as shown in the black rectangle at the bottom right of Figure 3.

The simulation parameters in Figure 3 are the same as Figure 2. Compared with the record of AW, we observe that the record of FV AW has phase shift and amplitude attenuation. The record of DDW shows a waveform shift without amplitude attenuation. The record of ADW shows amplitude attenuation without waveform shift.

Figure 3. Waveform comparison of four viscous equations received at (1km, 0.9km).

Figure 4 gives the comparison of the record controlled by equation (1), (2) and (3) at (1km, 0.9km) when the quality factor $Q$ takes $\infty$, 50, 20, 10, 8, 5, respectively. In order to observe the changes of the waveform clearly, we highlight the trough of each record with black solid line.

When the quality factor $Q$ decreases, Figure 4(a) shows that the amplitude attenuation and shifted arrival time of the FV AW equation are noticeable. Figure 4(b) shows that the arrival time of the DDW is shifted and the records have small amplitude attenuation but not significant. From Figure 4(c), we can see a evident amplitude attenuation of the ADW and the arrival time are almost identical. Based on the numerical experiment above, the change of amplitude is more obvious than that of the arrival time, giving the results that the arrival time is more robust than the amplitude.
Fig. 4. Waveform records comparison of (a) the FV AW equation, (b) the DDW equation and (c) the ADW equation at (1 km, 0.9 km). Quality factor $Q$: $\infty$, 50, 20, 10, 8, 5.

4.2 Hess/Salt model

In order to compare the dispersion and attenuation of seismic waves controlled by the above four equations in a complex model, our second model is a sampled Hess/Salt model shown in Figure 5. The computational domain is $1.875 \times 2.265 km$. The range of wave velocity is 1.524 km/s - 4.512 km/s, quality factor $Q=10$. The source function is the Ricker wavelet described by formula (33) placed at (1.1325 km, 0.05 km) which is shown as “☆” in the Figure 5 with a frequency $f_0=25$ Hz. The temporal step $\Delta t=0.6$ ms, and the space step $\Delta x=\Delta z=0.005$ km. Convolution Perfect Matching Layer (CPML) boundary conditions (Komatitsch and Martin 2007; Martin et al. 2008; Ma et al. 2018, 2019) are used to eliminate reflected waves near artificial boundaries.

Figure 6 shows the single-point waveform record, from 240ms to 720ms, at (1.2 km, 0.0685 km). In order to distinguish easily, we let (a), (b), (c), and (d) represent the record of FV AW, DDW, ADW, AW, respectively. From Figure 6, we can see that, compared with the wave simulated with the AW equation, the record of the DDW shows waveform shift without amplitude attenuation. The record of the ADW show amplitude attenuation without time shift. The record of the FVAW shows both times shift and amplitude attenuation. In complex...
media, even if the physical parameters are identical, the waveform records simulated by different control equations are still different.

Fig. 6. Waveform comparison of four viscous equations at the receiving point (1.2 km, 0.0685 km) with the Hess/Salt model.

5. Conclusion

In this study, we select three commonly used fractional viscous acoustic wave equations and one classical acoustic wave equation to study the difference between the seismic wave propagation. In forward modelling, we use LWC and ALFD to discretize the integer- and fractional-order partial derivatives of the equation. Analysis shows that the stability of the four equations is different. The stability of the ADW equation is the most rigorous and that of the DDW equation is the most flexible. In fact, with the same parameters, the stability conditions of the FV AW equation, the DDW equation, the ADW equation, and the AW equation are

\[
\begin{align*}
\Delta_{\text{FV AW}} &\leq 0.693 \frac{h}{c_0} \frac{1}{\Omega_0} \frac{1}{h^0} \frac{1}{c_0^0} \\
\Delta_{\text{DDW}} &\leq 0.879 \frac{h}{c_0} \frac{1}{\Omega_0} \frac{1}{h^0} \frac{1}{c_0^0} \\
\Delta_{\text{ADW}} &\leq 0.524 \frac{h}{c_0} \\
\Delta_{\text{AW}} &\leq 0.729 \frac{h}{c_0}
\end{align*}
\]

respectively. Numerical dispersion analysis shows that the FD method of the four equations can effectively capture the low frequency components of the wave field. But for high frequency components, the numerical phase velocity has a certain deviation from the actual phase velocity. When the numerical wave number \( \vartheta = \pi \), the maximum phase velocity errors of the FV AW equation, DDW equation, ADW equation and AW equation are 27.78%, 28.02%, 2.25%, and 3.04%, respectively. At last, numerical experiments show that the FV AW equation, DDW equation, and quality factor \( Q \) are sensitive to the arrival time, while the FV AW equation, ADW equation, and quality factor \( Q \) are sensitive to the amplitude. Relatively speaking, the change of amplitude is more apparent than that of the arrival time, giving the results that the arrival time is more robust than the amplitude.

6. Discussion

As exploration progresses and more information is gathered about the block's viscous properties, a more reliable and efficient viscous model can be established to better fit the real waveform. The waveforms generated by different governing equations with different \( Q \) values are diverse. However, for a given study area or medium, there is only one most suitable control equation and \( Q \) value. In the present study, our results show that, under the identical parameter conditions, the equation itself and the quality factor \( Q \) show different sensitivities to changes in waveform amplitude and phase. Therefore, our next work is to study how to match the best governing equation and quality factor \( Q \) based on the target medium conditions to numerically simulate the most suitable waveform recording.

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