

An operational matrix method based on scaling function of Daubechies wavelet to solve stochastic differential equations

Abstract

In our joint work, by introducing Quasi Block-Pulse (QBP) functions, we provide integration and stochastic integration operational matrices based on scaling function of Daubechies wavelet to solve linear and non-linear stochastic differential equations. Due to interesting properties of the Daubechies wavelet such as orthogonality, compactly support and vanishing moments, these operational matrices have good properties. The convergence of the proposed method is presented and in order to verify the accuracy of the proposed method some numerical examples are provided.

Keywords: Stochastic differential equation, Scaling function, Daubechies wavelet, Quasi Block-Pulse functions, Integration operational matrix, Stochastic integration operational matrix.

1 Introduction

Many scientists and researchers in various fields such as finance and biology [3], control [9], social science [15], boundary value problems [12, 14, 44], statistic analysis [19], image processing [30], and signal analysis [35] need to have enough knowledge about stochastic calculus in order to model and solve their problems. Many real problems in applied mathematics, physics and engineering lead to stochastic models. So, being able to solve stochastic equations is a necessity.

In order to model stochastic processes, it is unavoidable to deal with stochastic differential or integral equations. It is not always possible to obtain the exact solution of these equations easily. So, in these cases we rely on the suitable numerical methods such as explicit and implicit approximation schemes [2, 20–22, 29, 39], Runge-Kutta method [31–33, 40, 42], Monte Carlo method [8], weak Simpson method [1], and wavelets [18, 36]. In recent years, some researchers used the operational matrices of wavelets to solve stochastic equations [11, 23–25, 34, 37, 38]. The Daubechies wavelet has vanishing moments of order $N \in \mathbb{N}$ which becomes smoother when moment's

degree increases. Ingrid Daubechies made it in 1988 and used it in order to approximate functions.

In this paper, we are going to provide operational matrix and stochastic operational matrix for scaling function of the Daubechies wavelet and use them to solve stochastic differential equations in the form of

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X_0 = x_0, \quad t \in [0, T],$$

or equivalently stochastic Itô-Volterra integral equations in the following form [2, 10, 16, 17, 29]:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s, \quad 0 \leq t \leq T, \quad (1)$$

such that W_t is the *standard Brownian motion* and the *random variable* $X = X(\omega)$ is a real-valued function defined on *outcome space* Ω [2, 7, 10, 16, 17, 29]. The first and the second integrals in Eq. (1) are Riemann and Itô stochastic integrals, respectively. Numerical solution for Eq. (1) is a stochastic process $X^{(n)} = (X_t^{(n)}, t \in [0, T])$ which demonstrates an approximation for the exact solution $X = (X_t, t \in [0, T])$. In order to have such a numerical solution, we need a discretization of the time interval

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

where grid points are not equidistant necessarily. This paper is planned as follows: Section 2 provides some required information about stochastic calculus and wavelet theory that we need in the following. Then in Section 3 quasi Block-Pulse functions are introduced and some of their important properties are declared. Next, in Section 4, integration and stochastic integration operational matrices of the scaling function for the Daubechies wavelet are provided. Section 5 explains the numerical method by using the introduced matrices in the previous section to solve differential or integral stochastic equations. In Section 6 the convergence of the numerical method is argued. Finally, several examples are presented in Section 7. In conclusion, a discussion is provided in Section 8.

2 Preliminaries

2.1 Stochastic calculus

Definition 1 [16] A stochastic process $W = (W_t, t \in [0, +\infty))$ is called *standard Brownian motion* or a *Wiener process* if following conditions are satisfied:

1. $W_0 = 0$.
2. W_t has stationary and independent increments.
3. For all $t > 0$, W_t has $N(0, t)$ which N means the normal distribution.
4. W_t has continuous sample paths.

Definition 2 [16] The *Itô stochastic integral* for a simple process C on $[0, T]$ is defined in the following form:

$$\int_0^T C_s dW_s := \sum_{i=1}^n C_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}).$$

The Itô stochastic integral has the *isometry property*:

$$\mathbb{E} \left(\int_0^t C_s dW_s \right)^2 = \int_0^t \mathbb{E} C_s^2 ds, \quad t \in [0, T]. \quad (2)$$

Assume C is adapted to Brownian motion on $[0, T]$, in the other words, C_t is dependent on $W_s, s \leq t$. Also assume $\int_0^T \mathbb{E} C_s^2 ds$ is finite, there exists a sequence $(C^{(n)})$ of simple processes such that

$$\int_0^T \mathbb{E} [C_s - C_s^{(n)}]^2 ds \rightarrow 0.$$

Theorem 1 Let $\mathbb{E}(X_0^2) < \infty$ and X_0 is independent of the standard Brownian motion. If for all $t \in [0, T]$ and $x, y \in \mathbb{R}$, the functions $a(t, x)$ and $b(t, x)$ are continuous and Lipschitz condition with respect to the second variable is hold for them, then Eq. (1) has a unique strong solution on $[0, T]$.

Proof The proof is available in [26]. □

2.2 Wavelet theory

Definition 3 [13, 27, 28] A multiresolution analysis of $L^2(\mathbb{R})$ is a sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ that the following properties are hold:

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$.
2. $f(t) \in V_0 \Leftrightarrow f(2^j t) \in V_j$.
3. $f(2^j t) \in V_j \Leftrightarrow f(2^j t + 1) \in V_j$.
4. $\bigcap_j V_j = \{0\}, (\bigcup_j V_j) = L^2(\mathbb{R})$.
5. There exists a function $\phi(t) \in V_0$ with nonzero integral which $\{\phi(t - n)\}$ is an orthonormal base for V_0 . The function ϕ is called scaling function.

Definition 4 [43] The function $f(t)$ has vanishing moments of order N if

$$\int_{\mathbb{R}} t^m f(t) dt = 0, \quad 0 \leq m \leq N - 1,$$

but

$$\int_{\mathbb{R}} t^N f(t) dt \neq 0.$$

Ingrid Daubechies made a basis from smooth wavelet with vanishing moments in its support [5]. Also, if Daubechies wavelet has vanishing moments of order $N \in \mathbb{N}$, the support of scaling function is $[0, 2N - 1]$. So, dilation relation for the scaling function $\phi(t)$ can be have the following form

$$\phi(t) = \sqrt{2} \sum_{k=0}^{D-1} h_k \phi(2t - k), \quad (3)$$

where $D = 2N$ and $\{h_k\}_{k \in \mathbb{Z}}$ are named filter coefficients. Their values can be obtained by using some useful relations which are related to properties of the Daubechies wavelet. Of course, Daubechies calculated and set them in tables for various values of N . They are available in some books [6, 27].

From definition of the multiresolution analysis, it is clear that

$$\{\phi_{j,k}(t) := 2^{j/2} \phi(2^j t - k) | j, k \in \mathbb{Z}\},$$

is an orthonormal base for V_j . So we can write an orthogonal projection of function $f(t)$ belongs $L^2(\mathbb{R})$ in subspace V_j by using scaling function [28, 41]:

$$\mathcal{P}_j f(t) = \sum_{k=-\infty}^{+\infty} c_{j,k} \phi_{j,k}(t), \quad (4)$$

where

$$c_{j,k} = \int_{-\infty}^{+\infty} f(t) \phi_{j,k}(t) dt.$$

Assume $f \in V_j$ and $t \in [a, b]$ such that $a, b \in \mathbb{Z}$, due to compactly support of the scaling function of the Daubechies wavelet, (4) can be written in the following form

$$\mathcal{P}_j f(t) = \sum_{k=2^j a+1-L}^{2^j b-1} c_{j,k} \phi_{j,k}(t).$$

From the forth property of multiresolution analysis, it can be found

$$\lim_{j \rightarrow \infty} \|f - \mathcal{P}_j f\| = 0.$$

Theorem 2 *If we use the Daubechies wavelet with vanishing moments of order N to approximate the smooth function $f \in \mathcal{C}^N$ in space V_j , then*

$$\exists C > 0, \quad \|f - \mathcal{P}_j f\|_{L^2} \leq C 2^{-jN}.$$

Proof The proof can be found in [4]. □

2.2.1 Scaling function in integer and dyadic points

Dyadic points are defined in the following form [43]

$$t = \frac{k}{2^j}, \quad j, k \in \mathbb{Z}.$$

For $j \geq 0$ we have the dyadic points in $[0, D - 1]$ as follows

$$t = \frac{n}{2^j}, \quad n = 0, 1, 2, \dots, 2^j(D - 1).$$

As we can see, when $j = 0$ the dyadic points are the same integer points.

Since the scaling function of the Daubechies wavelet does not have an explicit form, first we can obtain the values of the scaling function in integer points and then in dyadic points. Substituting integer points of support interval in relation (3) results in a homogenous system

$$(A - I)\Phi = 0,$$

such that

$$A_{D \times D} = \sqrt{2} \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 & 0 & 0 \\ h_4 & h_3 & h_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_{D-3} & h_{D-4} & h_{D-5} \\ 0 & 0 & 0 & \cdots & h_{D-1} & h_{D-2} & h_{D-3} \\ 0 & 0 & 0 & \cdots & 0 & 0 & h_{D-1} \end{pmatrix},$$

and

$$\Phi_{D \times 1} = \begin{pmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \vdots \\ \phi(D-3) \\ \phi(D-2) \\ \phi(D-1) \end{pmatrix},$$

and I is the identity matrix by the same order of matrix A .

In order to have a unique solution, we need a normalization condition which is [27]

$$\sum_{k=0}^{D-1} \phi(x - k) = 1.$$

Now, by solving the linear equations system, we can obtain values of scaling function in integer points. Then, we can determine values of scaling function

in dyadic points via a recursive process named cascade algorithm which is as follows

$$\phi\left(\frac{n}{2^j}\right) = \sqrt{2} \sum_{k=0}^{D-1} h_k \phi\left(\frac{n}{2^{j-1}} - k\right), \quad n = 1, 3, 5, \dots, 2^j(D-1) - 1.$$

3 Quasi Block-Pulse Functions

Definition 5 For $m \in \mathbb{N}$ we define an m element partition of quasi Block-Pulse functions on $[0, T)$:

$$b_i(t) = \begin{cases} 1, & t_i \leq t < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

where $i = 1, \dots, m$, and

$$0 = t_1 < t_2 < \dots < t_m < t_{m+1} = T,$$

these grid points are not equidistant necessarily.

The functions in (5) have disjointness and orthogonality properties, in the other words, for $i, j = 1, \dots, m$,

$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$\int_0^T b_i(t)b_j(t)dt = \begin{cases} \Delta_i t, & i = j, \\ 0, & i \neq j, \end{cases}$$

where $\Delta_i t = t_{i+1} - t_i$.

A real and bounded function $f(t) \in L^2([0, T))$ can be expanded by quasi Block-Pulse functions

$$f(t) = \sum_{i=1}^{\infty} f_i b_i(t),$$

where

$$f_i = \frac{1}{\Delta_i t} \int_0^T f(t)b_i(t)dt.$$

For any function $f \in L^2([0, T))$, we have

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|b_i(t)\|^2.$$

Let vector of the quasi Block-Pulse functions in the following form

$$\mathbf{B}_m(t) = [b_1(t), b_2(t), \dots, b_m(t)]^T,$$

then

$$\mathbf{B}_m(t) \left(\mathbf{B}_m(t) \right)^T = \begin{bmatrix} b_1(t) & & & \mathbf{0} \\ & b_2(t) & & \\ & & \ddots & \\ \mathbf{0} & & & b_m(t) \end{bmatrix},$$

and

$$\left(\mathbf{B}_m(t) \right)^T \mathbf{B}_m(t) = 1, \quad \forall t \in [0, T].$$

Also for a given vector $V = [v_1, v_2, \dots, v_m]^T$, we can write

$$\mathbf{B}_m(t) \left(\mathbf{B}_m(t) \right)^T V = \begin{bmatrix} v_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & v_m \end{bmatrix} \begin{bmatrix} b_1(t) \\ \vdots \\ b_m(t) \end{bmatrix} = \text{diag}(V) \mathbf{B}_m(t).$$

In order to approximate functions, we use truncated series of quasi Block-Pulse functions with m terms:

$$f(t) \approx \sum_{i=1}^m f_i b_i(t) = F^T \mathbf{B}_m(t) = \mathbf{B}_m(t)^T F,$$

where $F = [f_1, f_2, \dots, f_m]^T$.

3.1 Integration Operational Matrix for the quasi Block-Pulse Functions

Integrating on $[0, t]$ from functions in (5) results in

$$\int_0^t b_i(t) dt = \begin{cases} 0, & t < t_i, \\ \int_0^{t_i} b_i(t) dt + \int_{t_i}^t b_i(t) dt, & t_i \leq t < t_{i+1}, \\ \int_0^{t_i} b_i(t) dt + \int_{t_i}^{t_{i+1}} b_i(t) dt + \int_{t_{i+1}}^t b_i(t) dt, & t_{i+1} \leq t, \end{cases}$$

according the definition of the quasi Block-Pulse functions we have

$$\int_0^t b_i(t) dt = \begin{cases} 0, & t < t_i, \\ t - t_i, & t_i \leq t < t_{i+1}, \\ \Delta_i t, & t_{i+1} \leq t. \end{cases}$$

Since $t - t_i$ in the middle point of the $[t_i, t_{i+1})$ is $\frac{\Delta_i t}{2}$, we approximate the latter integral on $[t_i, t_{i+1})$ with $\frac{\Delta_i t}{2}$, therefore

$$\int_0^t b_i(t)dt \approx [0, \dots, 0, \frac{\Delta_i t}{2}, \Delta_i t, \dots, \Delta_i t] \begin{bmatrix} b_1(t) \\ \vdots \\ b_{i-1}(t) \\ b_i(t) \\ b_{i+1}(t) \\ \vdots \\ b_m(t) \end{bmatrix}.$$

Hence

$$\int_0^t B_m(s)ds \approx \mathbf{P}_m B_m(t), \quad (6)$$

where

$$\mathbf{P}_m = \begin{bmatrix} \frac{\Delta_1 t}{2} & \Delta_1 t & \Delta_1 t & \dots & \Delta_1 t \\ & \frac{\Delta_2 t}{2} & \Delta_2 t & \dots & \Delta_2 t \\ & & \ddots & & \\ & & & \ddots & \\ \mathbf{0} & & & & \frac{\Delta_m t}{2} \end{bmatrix}.$$

3.2 Stochastic Integration Operational Matrix for the quasi Block-Pulse Functions

By stochastic integrating on $[0, t]$ from functions in (5) we have

$$\int_0^t b_i(t)dW_t = \begin{cases} 0, & t < t_i, \\ \int_0^{t_i} b_i(t)dW_t + \int_{t_i}^t b_i(t)dW_t, & t_i \leq t < t_{i+1}, \\ \int_0^{t_i} b_i(t)dW_t + \int_{t_i}^{t_{i+1}} b_i(t)dW_t + \int_{t_{i+1}}^t b_i(t)dW_t, & t_{i+1} \leq t, \end{cases}$$

definition of the quasi Block-Pulse functions leads to

$$\int_0^t b_i(t)dW_t = \begin{cases} 0, & t < t_i, \\ W(t) - W(t_i), & t_i \leq t < t_{i+1}, \\ W(t_{i+1}) - W(t_i), & t_{i+1} \leq t. \end{cases}$$

Since $W(t) - W(t_i)$ in the middle point of $[t_i, t_{i+1})$ is $W\left(t_i + \frac{\Delta_i t}{2}\right) - W(t_i)$, we approximate the latter integral on $[t_i, t_{i+1})$ with $W\left(t_i + \frac{\Delta_i t}{2}\right) - W(t_i)$, which equals to $W\left(\frac{t_i + t_{i+1}}{2}\right)$. Therefore

$$\int_0^t \mathbf{B}_m(s)dW_s \approx \mathbf{R}_s \mathbf{B}_m(t), \quad (7)$$

where

$$\mathbf{R}_s = \begin{bmatrix} W\left(\frac{t_1+t_2}{2}\right) - W(t_1) & W(t_2) - W(t_1) & W(t_2) - W(t_1) & \dots & W(t_2) - W(t_1) \\ & W\left(\frac{t_2+t_3}{2}\right) - W(t_2) & W(t_3) - W(t_2) & \dots & W(t_3) - W(t_2) \\ & & \ddots & \vdots & \vdots \\ & & & \ddots & \vdots \\ & 0 & & & W\left(\frac{t_m+t_{m+1}}{2}\right) - W(t_m) \end{bmatrix}$$

4 Operational Matrix of the Scaling function of the Daubechies Wavelet

Consider vector of the scaling functions for the Daubechies wavelet

$$\phi_m(t) = [\phi_{j,k_1}(t), \phi_{j,k_1+1}(t), \dots, \phi_{j,k_2}(t)]^T.$$

Let grid points in the following form

$$\mathbf{T} = \left\{ t_i | i = 1, \dots, m \right\}, \quad (8)$$

we introduce matrix of the scaling functions of the Daubechies wavelet in the following form

$$\Phi_m = [\phi_m(t_1), \phi_m(t_2), \dots, \phi_m(t_m)],$$

such that Φ_m is a sparse matrix due to compactly support of the scaling function of the Daubechies wavelet. For grid points we have

$$\phi_m(t) = \Phi_m \mathbf{B}_m(t), \quad (9)$$

and hence

$$\mathbf{B}_m(t) = \left(\Phi_m \right)^{-1} \phi_m(t). \quad (10)$$

4.1 Integration Operational Matrix for the Scaling function of the Daubechies Wavelet

Consider the integration operational matrix for the scaling functions of the Daubechies wavelet in the following form

$$\int_0^t \phi_m(s) ds \approx \mathbf{P}_D \phi_m(t). \quad (11)$$

In the grid points, from (9) we have

$$\int_0^t \phi_m(s) ds = \int_0^t \Phi_m \mathbf{B}_m(s) ds = \Phi_m \int_0^t \mathbf{B}_m(s) ds.$$

From (6) we can write

$$\int_0^t \phi_m(s) ds \approx \Phi_m \mathbf{P}_m \mathbf{B}_m(t),$$

and from (10) we have

$$\int_0^t \phi_m(s) ds \approx \Phi_m \mathbf{P}_m \left(\Phi_m \right)^{-1} \phi_m(t).$$

So we can conclude

$$\mathbf{P}_D = \Phi_m \mathbf{P}_m \left(\Phi_m \right)^{-1}.$$

4.2 Stochastic Integration Operational Matrix for the Scaling Function of the Daubechies Wavelet

Assume stochastic integration operational matrix for the scaling functions of the Daubechies wavelet has the following form

$$\int_0^t \phi_m(s) dW_s \approx \mathbf{M}_s \phi_m(t). \quad (12)$$

In the grid points, from (9) we can write

$$\int_0^t \phi_m(s) dW_s = \int_0^t \Phi_m \mathbf{B}_m(s) dW_s = \Phi_m \int_0^t \mathbf{B}_m(s) dW_s.$$

From (7) we can get

$$\int_0^t \phi_m(s) dW_s \approx \Phi_m \mathbf{R}_s \mathbf{B}_m(t),$$

and (10) results in

$$\int_0^t \phi_m(s) dW_s \approx \Phi_m \mathbf{R}_s \left(\Phi_m \right)^{-1} \phi_m(t).$$

So we have

$$\mathbf{M}_s = \Phi_m \mathbf{R}_s \left(\Phi_m \right)^{-1}.$$

5 Numerical Method

If we let $T = 1$, from Eq. (1) we have

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s, \quad 0 \leq t < 1. \quad (13)$$

We let the dyadic points in $[0, 1)$ as the grid points in (8).

5.1 The Linear case

Theorem 3 Assume $\mathbf{X}_m$ is a vector with m elements, then

$$\phi_m(t) \left(\phi_m(t) \right)^T \mathbf{X}_m = \tilde{\mathbf{Q}}_m \phi_m(t),$$

such that

$$\tilde{\mathbf{Q}}_m = \text{diag} \left((\Phi_m)^T \mathbf{X}_m \right).$$

Proof From (9) we have

$$\phi_m(t) \left(\phi_m(t) \right)^T \mathbf{X}_m = \Phi_m \mathbf{B}_m(t) \left(\mathbf{B}_m(t) \right)^T \mathbf{Q}_m, \quad (14)$$

where

$$\mathbf{Q}_m = (\Phi_m)^T \mathbf{X}_m.$$

Let $\mathbf{Q}_m = [q_1, q_2, \dots, q_m]^T$, therefore we have

$$\begin{aligned} \mathbf{B}_m(t) \left(\mathbf{B}_m(t) \right)^T \mathbf{Q}_m &= \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_m(t) \end{bmatrix} [b_1(t), b_2(t), \dots, b_m(t)] \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \\ &= \begin{bmatrix} (b_1(t))^2 & b_1(t)b_2(t) & \dots & b_1(t)b_m(t) \\ b_2(t)b_1(t) & (b_2(t))^2 & \dots & b_2(t)b_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_m(t)b_1(t) & b_m(t)b_2(t) & \dots & (b_m(t))^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \\ &= \begin{bmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_m(t) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \\ &= \begin{bmatrix} q_1 & 0 & \dots & 0 \\ 0 & q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_m \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_m(t) \end{bmatrix}. \end{aligned}$$

So, we can write

$$\mathbf{B}_m(t) \left(\mathbf{B}_m(t) \right)^T \mathbf{Q}_m = \tilde{\mathbf{Q}}_m \mathbf{B}_m(t),$$

which $\tilde{\mathbf{Q}}_m = \text{diag}(\mathbf{Q}_m)$. Now according to (14) we have

$$\phi_m(t) \left(\phi_m(t) \right)^T \mathbf{X}_m = \Phi_m \tilde{\mathbf{Q}}_m \mathbf{B}_m(t).$$

Due to definition of quasi Block-Pulse functions and since $\tilde{\mathbf{Q}}_m$ is a diagonal matrix, in grid points we can write

$$\phi_m(t) \left(\phi_m(t) \right)^T \mathbf{X}_m = \tilde{\mathbf{Q}}_m \Phi_m \mathbf{B}_m(t),$$

and (9) leads to

$$\phi_m(t) \left(\phi_m(t) \right)^T \mathbf{X}_m = \tilde{\mathbf{Q}}_m \phi_m(t).$$

It completes the proof. $\square$

In the linear case we consider $a(s, X_s)$ and $b(s, X_s)$ in the form of

$$a(s, X_s) = f_1(s)X_s, \quad b(s, X_s) = f_2(s)X_s, \quad (15)$$

such that the functions f_1 and f_2 are real-valued functions. Consider approximation of functions f_1 and f_2 by using the scaling function of the Daubechies wavelet in the following form

$$f_1(s) = \mathbf{D}_1^T \phi_m(s), \quad f_2(s) = \mathbf{D}_2^T \phi_m(s), \quad (16)$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are column vectors with m elements. In continue assume

$$X_s = \left(\phi_m(s) \right)^T \mathbf{X}_m = \mathbf{X}_m^T \phi_m(s), \quad (17)$$

which $\mathbf{X}_m$ is the unknowns column vector. By substituting (16) and (17) in Eq. (13) we get

$$\mathbf{X}_m^T \phi_m(t) = X_0 + \mathbf{D}_1^T \int_0^t \phi_m(s) \left(\phi_m(s) \right)^T \mathbf{X}_m ds + \mathbf{D}_2^T \int_0^t \phi_m(s) \left(\phi_m(s) \right)^T \mathbf{X}_m dW_s.$$

From Theorem 3 we have

$$\mathbf{X}_m^T \phi_m(t) = X_0 + \mathbf{D}_1^T \tilde{\mathbf{Q}}_m \int_0^t \phi_m(s) ds + \mathbf{D}_2^T \tilde{\mathbf{Q}}_m \int_0^t \phi_m(s) dW_s,$$

where

$$\tilde{\mathbf{Q}}_m = \text{diag} \left(\left(\Phi_m \right)^T \mathbf{X}_m \right).$$

Now from (9), (11) and (12), in grid points we have

$$\mathbf{X}_m^T \Phi_m \mathbf{B}_m(t) = [X_0, X_0, \dots, X_0] \mathbf{B}_m(t) + \mathbf{D}_1^T \tilde{\mathbf{Q}}_m \mathbf{P}_D \Phi_m \mathbf{B}_m(t) + \mathbf{D}_2^T \tilde{\mathbf{Q}}_m \mathbf{M}_s \Phi_m \mathbf{B}_m(t).$$

Finally, we can have a linear system of equations

$$\mathbf{X}_m^T \Phi_m = [X_0, X_0, \dots, X_0] + \mathbf{D}_1^T \tilde{\mathbf{Q}}_m \mathbf{P}_D \Phi_m + \mathbf{D}_2^T \tilde{\mathbf{Q}}_m \mathbf{M}_s \Phi_m. \quad (18)$$

As it is seen (18) is a system of linear algebraic equations included m equations and m unknowns. By solving this system $\mathbf{X}_m$ is determined and then by substituting in (17) the solution of Eq. (13) with coefficient functions in (15) is obtained.

5.2 The nonlinear case

Theorem 4 Assume that V is a row vector with m elements in the form of $V = [v_1, v_2, \dots, v_m]$, then

$$\mathbf{B}_m(t) V \mathbf{B}_m(t) = \text{diag}(V) \mathbf{B}_m(t).$$

Proof We have

$$\begin{aligned} \mathbf{B}_m(t) V \mathbf{B}_m(t) &= \mathbf{B}_m(t) [v_1, v_2, \dots, v_m] \mathbf{B}_m(t) = \begin{bmatrix} b_1(t)v_1 & b_1(t)v_2 & \dots & b_1(t)v_m \\ b_2(t)v_1 & b_2(t)v_2 & \dots & b_2(t)v_m \\ \vdots & \vdots & \ddots & \vdots \\ b_m(t)v_1 & b_m(t)v_2 & \dots & b_m(t)v_m \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_m(t) \end{bmatrix} \\ &= \begin{bmatrix} v_1 b_1(t) \\ v_2 b_2(t) \\ \vdots \\ v_m b_m(t) \end{bmatrix} \\ &= \begin{bmatrix} v_1 & & & \mathbf{0} \\ & v_2 & & \\ & & \ddots & \\ \mathbf{0} & & & v_m \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_m(t) \end{bmatrix}, \end{aligned}$$

and proof is completed. $\square$

In the nonlinear case we consider the functions $a(s, X_s)$ and $b(s, X_s)$ which have the following form

$$a(s, X_s) = f_1(s) X_s^\alpha, \quad b(s, X_s) = f_2(s) X_s^\beta, \quad (19)$$

such that $\alpha, \beta \geq 2$ and belong $\mathbb{N}$. The functions f_1 and f_2 are real-valued functions. Again we consider approximation of functions f_1 and f_2 by using the scaling function of Daubechies wavelet in the form of (16). Again let

$$X_s = \mathbf{X}_m^T \phi_m(s), \quad (20)$$

from (9) we can have the following result

$$X_s = \mathbf{X}_m^T \Phi_m \mathbf{B}_m(s). \quad (21)$$

It is clear that $\mathbf{X}_m^T \Phi_m$ is a row vector with m elements. Let

$$\mathbf{X}_m^T \Phi_m = [d_1, d_2, \dots, d_m],$$

therefore

$$X_s = [d_1, d_2, \dots, d_m] \mathbf{B}_m(s).$$

Now due to definition and properties of the quasi Block-Pulse functions, we can write

$$X_s^\alpha = [d_1^\alpha, d_2^\alpha, \dots, d_m^\alpha] \mathbf{B}_m(s), \quad (22)$$

and

$$X_s^\beta = [d_1^\beta, d_2^\beta, \dots, d_m^\beta] \mathbf{B}_m(s). \quad (23)$$

By substituting (9), (21), (22) and (23) in Eq. (13), in grid points we get

$$\begin{aligned} \mathbf{X}_m^T \Phi_m \mathbf{B}_m(t) = & [X_0, X_0, \dots, X_0] \mathbf{B}_m(t) \\ & + \mathbf{D}_1^T \Phi_m \int_0^t \mathbf{B}_m(s) [d_1^\alpha, d_2^\alpha, \dots, d_m^\alpha] \mathbf{B}_m(s) ds \\ & + \mathbf{D}_2^T \Phi_m \int_0^t \mathbf{B}_m(s) [d_1^\beta, d_2^\beta, \dots, d_m^\beta] \mathbf{B}_m(s) dW_s. \end{aligned}$$

Now, Theorem 4 leads to

$$\begin{aligned} \mathbf{X}_m^T \Phi_m \mathbf{B}_m(t) = & [X_0, X_0, \dots, X_0] \mathbf{B}_m(t) \\ & + \mathbf{D}_1^T \Phi_m \mathbf{V}_1 \int_0^t \mathbf{B}_m(s) ds \\ & + \mathbf{D}_2^T \Phi_m \mathbf{V}_2 \int_0^t \mathbf{B}_m(s) dW_s, \end{aligned}$$

where

$$\begin{aligned} \mathbf{V}_1 &= \text{diag}([d_1^\alpha, d_2^\alpha, \dots, d_m^\alpha]), \\ \mathbf{V}_2 &= \text{diag}([d_1^\beta, d_2^\beta, \dots, d_m^\beta]). \end{aligned}$$

Finally (6) and (7) results in

$$\begin{aligned} \mathbf{X}_m^T \Phi_m \mathbf{B}_m(t) = & [X_0, X_0, \dots, X_0] \mathbf{B}_m(t) \\ & + \mathbf{D}_1^T \Phi_m \mathbf{V}_1 \mathbf{P}_m \mathbf{B}_m(t) \\ & + \mathbf{D}_2^T \Phi_m \mathbf{V}_2 \mathbf{R}_s \mathbf{B}_m(t). \end{aligned}$$

Hence it is concluded that

$$\mathbf{X}_m^T \Phi_m = [X_0, X_0, \dots, X_0] + \mathbf{D}_1^T \Phi_m \mathbf{V}_1 \mathbf{P}_m + \mathbf{D}_2^T \Phi_m \mathbf{V}_2 \mathbf{R}_s. \quad (24)$$

It is easy to see that (24) is a system of nonlinear algebraic equations with m equations and m unknowns. By finding $\mathbf{X}_m$ and then substituting it in (20) the solution of Eq. (13) with coefficient functions defined in (19) is obtained.

6 Convergence of Method

Here, first we declare some theorems that we need them to prove convergence of the method introduced in the previous section, then we prove our theorem.

Theorem 5 *Assume $f(t)$ is L^∞ on a finite interval I , it means $f(t)$ is bounded on I if there is a number $M > 0$ such that $|f(t)| \leq M$ for all $t \in I$, then $f(t)$ is L^1 on I .*

Proof A complete proof can be found in [43]. $\square$

Theorem 6 *Any function that is L^∞ and L^1 on any interval I , is L^2 on I .*

Proof The proof is available in [43]. $\square$

Lemma 1 (Gronwall's lemma) *Let f and g be nonnegative, continuous functions defined on $0 \leq t \leq T$, and let $\delta, \epsilon \geq 0$ are constants. If*

$$f(t) \leq \delta + \epsilon \int_0^t f(s)g(s)ds, \quad 0 \leq t \leq T,$$

then

$$f(t) \leq \delta \exp \left(\epsilon \int_0^t g(s)ds \right), \quad 0 \leq t \leq T.$$

Theorem 7 *Let X_t and $\tilde{X}_t$ be the exact and numerical solution by the present method explained in Section 5 by using the scaling function of the Daubechies wavelet with N vanishing moments in space V_j for Eq. (13). Assume that the conditions of Theorem 1 are satisfied, then $\tilde{X}_t$ converges to X_t .*

Proof We consider functions $z_1(t) = a(t, X_t)$ and $z_2(t) = b(t, X_t)$ and their approximations by using scaling functions of Daubechies wavelet, $\tilde{z}_1(t) = \tilde{a}(t, \tilde{X}_t)$ and $\tilde{z}_2(t) = \tilde{b}(t, \tilde{X}_t)$, respectively. Also, we define $\hat{z}_1(t)$ and $\hat{z}_2(t)$ as follows

$$\hat{z}_1(t) = a(t, \tilde{X}_t),$$

$$\hat{z}_2(t) = b(t, \tilde{X}_t).$$

Now we let error function as

$$e(t) = \mathbb{E}|X_t - \tilde{X}_t|^2.$$

We can write

$$\begin{aligned} e(t) &= \mathbb{E} \left| \int_0^t (z_1(s) - \tilde{z}_1(s))ds + \int_0^t (z_2(s) - \tilde{z}_2(s))dW_s \right|^2 \\ &\leq 2\mathbb{E} \left(\left(\int_0^t (z_1(s) - \tilde{z}_1(s))ds \right)^2 + \left(\int_0^t (z_2(s) - \tilde{z}_2(s))dW_s \right)^2 \right). \end{aligned}$$

But we can rewrite

$$\begin{aligned}\mathbb{E}\left(\int_0^t (z_1(s) - \tilde{z}_1(s))ds\right)^2 &\leq \mathbb{E}\left(\int_0^t (z_1(s) - \tilde{z}_1(s))^2 ds\right) \\ &\leq \int_0^t \mathbb{E}(z_1(s) - \tilde{z}_1(s))^2 ds,\end{aligned}$$

and by using of (2) we can have

$$\mathbb{E}\left(\int_0^t (z_2(s) - \tilde{z}_2(s))dW_s\right)^2 = \left(\int_0^t \mathbb{E}(z_2(s) - \tilde{z}_2(s))^2 ds\right).$$

Therefore we can continue

$$\begin{aligned}e(t) &\leq 2\left(\int_0^t \mathbb{E}(z_1(s) - \tilde{z}_1(s))^2 ds + \int_0^t \mathbb{E}(z_2(s) - \tilde{z}_2(s))^2 ds\right) \\ &\leq 4\left(\int_0^t \mathbb{E}(z_1(s) - \hat{z}_1(s))^2 ds + \int_0^t \mathbb{E}(\hat{z}_1(s) - \tilde{z}_1(s))^2 ds\right) \\ &\quad + 4\left(\int_0^t \mathbb{E}(z_2(s) - \hat{z}_2(s))^2 ds + \int_0^t \mathbb{E}(\hat{z}_2(s) - \tilde{z}_2(s))^2 ds\right) \\ &= 4\left(\int_0^t \mathbb{E}(z_1(s) - \hat{z}_1(s))^2 ds + \mathbb{E}\left[\int_0^t (\hat{z}_1(s) - \tilde{z}_1(s))^2 ds\right]\right) \\ &\quad + 4\left(\int_0^t \mathbb{E}(z_2(s) - \hat{z}_2(s))^2 ds + \mathbb{E}\left[\int_0^t (\hat{z}_2(s) - \tilde{z}_2(s))^2 ds\right]\right).\end{aligned}$$

In addition, Theorem 2 about approximation of functions by using the scaling function of Daubechies wavelet leads to

$$\begin{aligned}e(t) &\leq 4\left(\int_0^t \mathbb{E}(z_1(s) - \hat{z}_1(s))^2 ds + \mathbb{E}(C_1 2^{-jN})^2\right) \\ &\quad + 4\left(\int_0^t \mathbb{E}(z_2(s) - \hat{z}_2(s))^2 ds + \mathbb{E}(C_2 2^{-jN})^2\right) \\ &\leq 4\left(\int_0^t \mathbb{E}(z_1(s) - \hat{z}_1(s))^2 ds + \int_0^t \mathbb{E}(z_2(s) - \hat{z}_2(s))^2 ds\right) + 8C^2 2^{-2jN},\end{aligned}$$

where $C = \max\{C_1, C_2\}$. We can conclude from Theorems 5 and 6 that X_t is $L^2([0, 1])$. Finally, the Lipschitz condition results in

$$e(t) \leq 8L_{Lip}^2 \left(\int_0^t \mathbb{E}|X_s - \tilde{X}_s|^2 ds\right) + 8C^2 2^{-2jN},$$

and hence

$$e(t) \leq 8C^2 2^{-2jN} + 8L_{Lip}^2 \left(\int_0^t e(s) ds\right).$$

In conclusion Lemma 1 results in

$$e(t) \leq 8C^2 2^{-2jN} \exp\left(8L_{Lip}^2\right).$$

The latter relation proves convergence as $j \rightarrow \infty$. □

7 Numerical Examples

In order to show application of the present method in practice, we provide five examples. In all examples we use the scaling function of the Daubechies wavelet of order 2. We present some errors such as: absolute error (e) and the Root-Mean-Square error (RMSE). Let X and x_j as the exact and approximate solution, respectively. Then

$$e_j(t_i) = |X(t_i) - x_j(t_i)|, \quad t_i \in [0, 1), \quad i = 1, 2, \dots, m,$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^m e_j^2(t_i)}{m}}.$$

All codes are written in MATLAB R2018b, on a pc system with an Intel® Pentium(R) CPU G4400 @ 3.30GHz $\times$ 2, 7.7GB(RAM), 64-bit under Ubuntu 20.04.1 LTS.

7.1 Linear case

Example 1 We consider the linear Itô stochastic differential equation

$$X_t = X_0 + c \int_0^t X_s ds + \sigma \int_0^t X_s dW_s, \quad 0 \leq s, t < 1, \quad (25)$$

where $c = 0.01$, $\sigma = 0.01$, and the initial condition is $X_0 = \frac{1}{2}$. In the stochastic integral Brownian motion and the process X are together. It is said Eq. (25) has *multiplicative noise*. The exact solution for Eq. (25) is [22]:

$$X_t = X_0 \exp\left((c - 0.5\sigma^2)t + \sigma W_t\right). \quad (26)$$

X_t in (26) is known as the geometric Brownian motion. The RMS error for several sample paths is shown in Table 1 for some different j . As we can see RMS error decreases when j increases which means convergence. Table 2 demonstrates numerical results obtained by the present method compared to the exact solution for the first path and $j = 7$. Also, graph of the absolute error for the first path and $j = 7$ is shown in Figure 1.

Table 1 The RMS error for $N = 2$, Example 1.

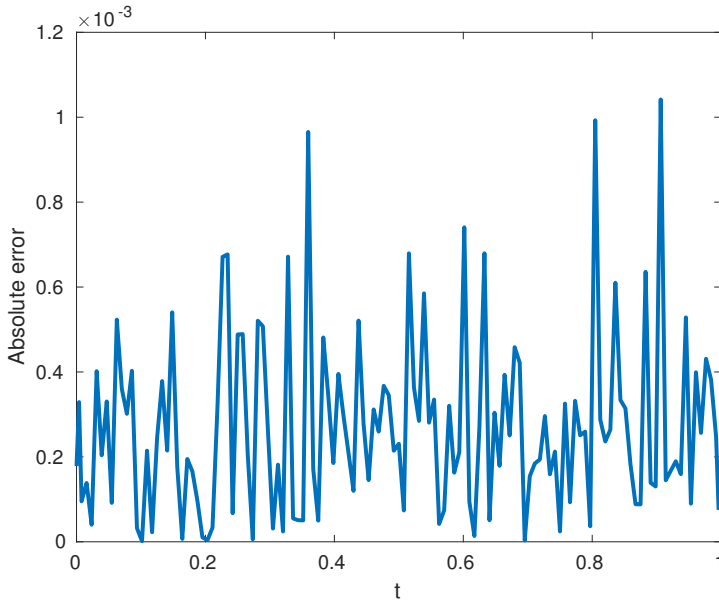
Path	j= 3	j= 5	j= 7
ω_1	1.189724e-03	5.598874e-04	3.420675e-04
ω_2	1.233056e-03	7.253622e-04	3.083418e-04
ω_3	1.529944e-03	6.141261e-04	3.330084e-04
ω_4	1.818255e-03	6.425052e-04	3.254339e-04
ω_5	1.176162e-03	6.467252e-04	3.210786e-04

Example 2 Consider the following linear Itô stochastic differential equation

$$X_t = X_0 + \int_0^t \ln(1+s) X_s ds + \int_0^t s X_s dW_s, \quad 0 \leq s, t < 1,$$

Table 2 Numerical solution based on the present method and the exact solution for ω_1 , $N = 2$ and $j = 7$, Example 1.

Time	Present Method	Exact	Absolute Error
0.00	0.499822	0.500000	1.783313e-04
0.25	0.504242	0.504309	6.728393e-05
0.50	0.503772	0.503986	2.144939e-04
0.75	0.503696	0.503909	2.125174e-04
1.00	0.503122	0.503047	7.499398e-05

**Fig. 1** Absolute error for ω_1 , $N = 2$ and $j = 7$, Example 1.

with $X_0 = \frac{1}{75}$. The exact solution is [34]:

$$X_t = \frac{1}{75} \exp \left((1+t) \ln(1+t) - t - \frac{t^3}{6} + \int_0^t s dW_s \right).$$

The RMS error for some sample paths is shown in Table 3 for some values of j . We can find the RMS error decreases as j increases. Table 4 indicates numerical results by the present method and the exact solution for the fifth path and $j = 8$. Figure 2 shows the numerical solution based on the present method compared to the exact solution for the fifth path and $j = 8$.

Example 3 Let us consider another linear Itô stochastic differential equation

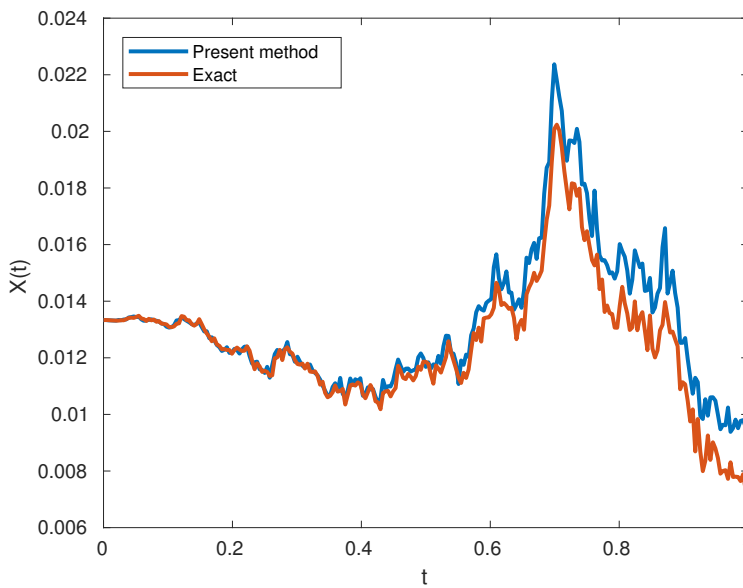
$$X_t = X_0 + \int_0^t \cos(s) X_s ds + \int_0^t \sin(s) X_s dW_s, \quad 0 \leq s, t < 1,$$

Table 3 The RMS error for $N = 2$, Example 2.

Path	j=4	j=6	j=8
ω_1	1.840026e-03	1.891139e-03	9.125506e-04
ω_2	4.147880e-03	3.154048e-03	1.276819e-03
ω_3	4.274300e-03	8.463968e-04	6.089297e-04
ω_4	7.260763e-03	2.499652e-03	7.411351e-04
ω_5	9.040620e-04	2.942558e-03	1.037556e-03

Table 4 Numerical solution based on the peresent method and the exact solution for ω_5 , $N = 2$ and $j = 8$, Example 2.

Time	Present Method	Exact	Absolute Error
0.00	0.013333	0.013333	1.166817e-10
0.25	0.011574	0.011501	7.302481e-05
0.50	0.011996	0.011866	1.298661e-04
0.75	0.018150	0.016157	1.993857e-03
1.00	0.010601	0.008044	2.556790e-03

**Fig. 2** Numerical solution based on the present method and the exact solution for ω_5 , $N = 2$ and $j = 8$, Example 2.

with the initial condition $X_0 = \frac{1}{120}$. The exact solution is [34]:

$$X_t = \frac{1}{120} \exp \left(-\frac{t}{4} + \sin(t) + \frac{\sin(2t)}{8} \int_0^t \sin(s) dW_s \right).$$

Table 5 indicates the RMS error for different j for several sample paths. Numerical results by using the present method and the exact solution for the forth path and

$j = 7$ is presented in Table 6. Figure 3 shows the numerical solution based on the present method compared to the exact solution for the same path and j .

Table 5 The RMS error for $N = 2$, Example 3.

Path	j=5	j=6	j=7
ω_1	3.340622e-03	1.666418e-03	8.682893e-04
ω_2	9.789401e-04	2.461135e-03	1.261553e-03
ω_3	1.832066e-03	1.809826e-03	9.267302e-04
ω_4	2.774917e-03	9.783834e-04	7.145506e-04
ω_5	3.059496e-03	7.803249e-04	9.578801e-04

Table 6 Numerical solution based on the present method and the exact solution for ω_4 , $N = 2$ and $j = 7$, Example 3.

Time	Present Method	Exact	Absolute Error
0.00	0.008350	0.008333	1.639841e-05
0.25	0.009634	0.009487	1.467543e-04
0.50	0.008743	0.008574	1.696685e-04
0.75	0.011009	0.010215	7.931460e-04
1.00	0.007040	0.005704	1.336274e-03

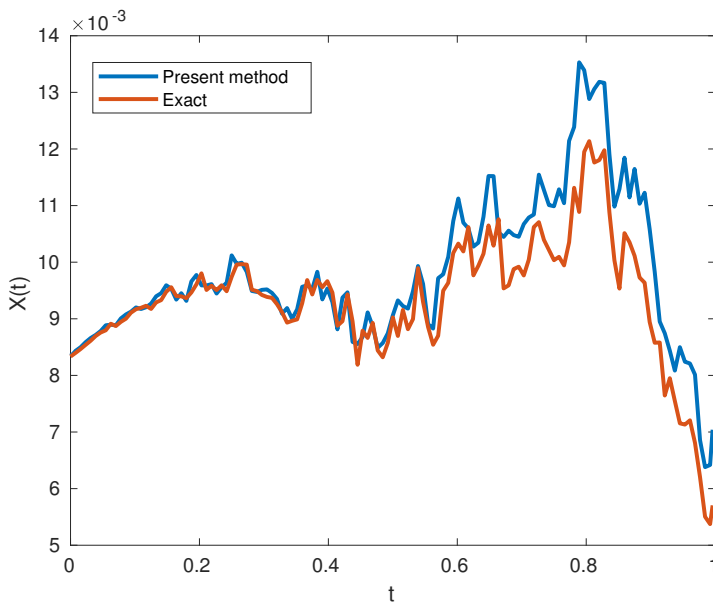


Fig. 3 Numerical solution based on the present method and the exact solution for ω_4 , $N = 2$ and $j = 7$, Example 3.

7.2 Nonlinear case

Example 4 We want to solve the nonlinear Itô stochastic differential equation

$$dX_t = \frac{1}{2}a^2n_1X_t^{2n_1-1}dt + aX_t^{n_1}dW_t, \quad 0 \leq t \leq 1, \quad n_1 \neq 1.$$

Here the initial condition is $X_0 = \frac{1}{2}$, $a = 0.1$, $n_1 = 2$ and the exact solution is [16]:

$$X_t = \left(X_0^{1-n_1} - a(n_1-1)W_t \right) \frac{1}{1-n_1}.$$

Table 7 shows the RMS error for some j for five sample paths. Numerical results based on the present method and the exact solution for the first path and $j = 7$ is demonstrated in Table 8. Figure 4 indicates the absolute error for the first path and $j = 7$.

Table 7 The RMSE error for $N = 2$, Example 4.

Path	j=3	j= 5	j=7
ω_1	8.317664e-03	2.710919e-03	1.465871e-03
ω_2	6.592069e-03	2.806349e-03	2.053098e-03
ω_3	6.103472e-03	3.229574e-03	1.889319e-03
ω_4	7.648277e-03	3.363171e-03	1.928895e-03
ω_5	8.969578e-03	3.206449e-03	1.514159e-03

Table 8 Numerical solution based on the present method and the exact solution for ω_1 , $N = 2$ and $j = 7$, Example 4.

Time	Present Method	Exact	Absolute Error
0.00	0.500704	0.500000	7.041799e-04
0.25	0.499755	0.496571	3.183721e-03
0.50	0.494611	0.495278	6.675488e-04
0.75	0.491089	0.491157	6.726415e-05
1.00	0.494719	0.493867	8.514789e-04

Example 5 We solve another nonlinear Itô stochastic differential equation with the initial condition

$$dX_t = \frac{1}{2}a(a-1)X_t^{1-\frac{2}{a}}dt + aX_t^{1-\frac{1}{a}}dW_t, \quad 0 \leq t < 1.$$

The initial condition is $X_0 = \frac{1}{2}$, $a = -\frac{1}{2}$ and the exact solution is [16]:

$$X_t = \left(X_0^{\frac{1}{a}} + W_t \right)^a.$$

The RMS error for some values of j for five sample paths is shown in Table 9. Numerical results obtained by the present method and the exact solution for the third path and $j = 7$ is presented in Table 10. Figure 5 shows the numerical solution based on the present method compared to the exact solution for the third path and $j = 7$. The absolute error for the same data is shown in Figure 6.

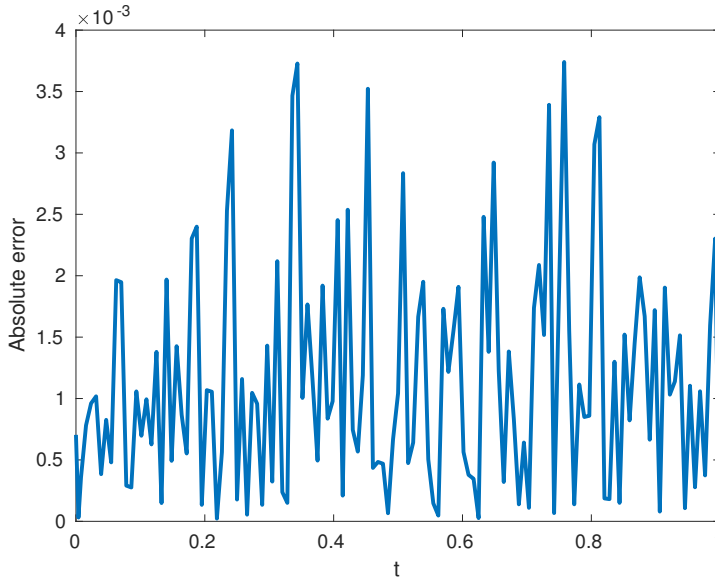


Fig. 4 Absolute error for ω_1 , $N = 2$ and $j = 7$, Example 4.

Table 9 The RMS error for $N = 2$, Example 5.

Path	j=5	j=6	j=7
ω_1	2.324759e-02	1.872409e-02	7.824655e-03
ω_2	3.610773e-02	1.020272e-02	8.480478e-03
ω_3	5.352925e-02	9.096077e-03	5.213692e-03
ω_4	3.375873e-02	1.324295e-02	5.581646e-03
ω_5	9.625489e-03	1.077801e-02	7.724419e-03

Table 10 Numerical solutions based on the present method and the exact solution for ω_3 , $N = 2$ and $j = 7$, Example 5.

Time	Present Method	Exact	Absolute Error
0.00	0.494968	0.500000	5.032328e-03
0.25	0.489550	0.486787	2.763175e-03
0.50	0.460865	0.458502	2.363386e-03
0.75	0.460560	0.459845	7.152582e-04
1.00	0.425205	0.418582	6.623136e-03

8 Conclusion

In this work, by using the quasi Block-Pulse functions, we introduce integration and stochastic integration operational matrices based on the scaling function of the Daubechies wavelet to solve some linear and nonlinear stochastic differential problems. According to this method, the problems solve easily and faster.

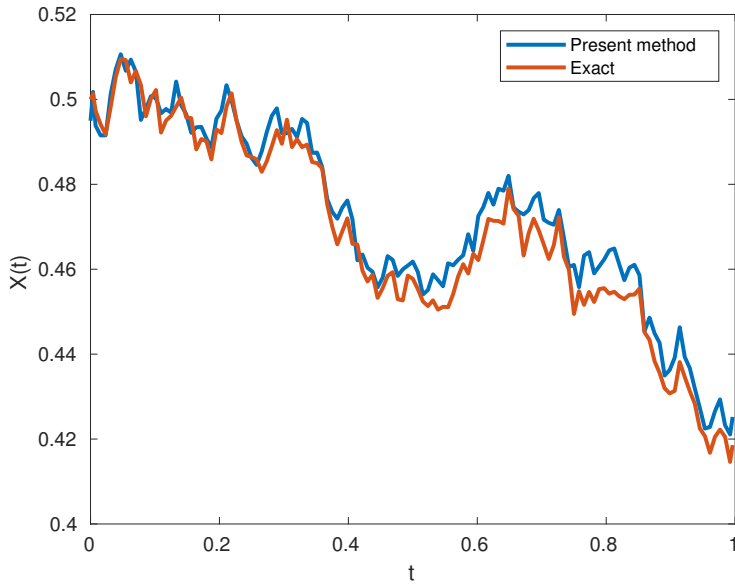


Fig. 5 Numerical solution based on the present method and the exact solution for ω_3 , $N = 2$ and $j = 7$, Example 5.

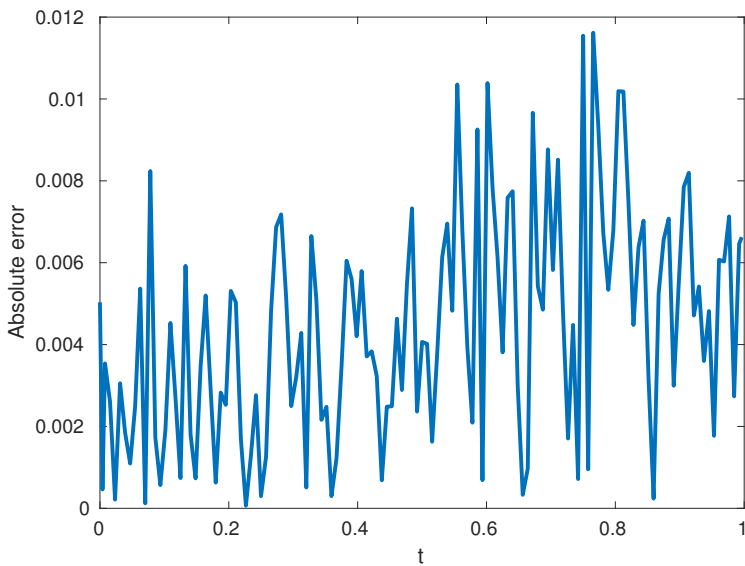


Fig. 6 Absolute error for ω_3 , $N = 2$ and $j = 7$, Example 5.

These operational matrices have good properties due to properties of the Daubechies wavelet such as orthogonality, compactly support and vanishing moments. Accuracy of the proposed method proved by some numerical examples.

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- Funding "Not Applicable"
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- Code availability "Not Applicable"
- Authors' contributions: All sections was written by the first author and all the authors reviewed paper.

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