Supplemental Material*

Nonlinear interactions between vibration modes with vastly different eigenfrequencies

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S.1 A generalized model for VDE modal interactions

To generalize the model of the main text, we consider the Hamiltonian $H = H_{\rm lf} + H_{\rm hf} + H_{\rm int}$, where $H_{\rm lf} = \sum_{i=1}^{n} p_{L_i}^2/2+\omega_{L_i}^2 q_{L_i}^2/2+\beta_i q_{L_i}^4/4-q_{L_i}F_{L_i}\cos(\omega_{F_{L_i}}t)$ is the Hamiltonian of n low-frequency (LF) modes and each of these modes can have a Duffing nonlinearity (when $\beta_i \neq 0$), $H_{\rm hf} = \sum_{i=1}^{m} p_{H_i}^2/2+\omega_{H_i}^2 q_{H_i}^2/2-q_{H_i}F_{H_i}\cos(\omega_{F_{H_i}}t)$ is the Hamiltonian of m high-frequency (HF) modes, and $H_{\rm int} = \sum_{i,j,j'} \alpha_{ijj'} q_{L_i} q_{H_j} q_{H_{j'}}$ is the interaction Hamiltonian and its coefficients $\alpha_{ijj'}$ are symmetric with respect to j and j', i.e., $\alpha_{ijj'} = \alpha_{ij'j}$. With the inclusion of linear dissipation and thermal noise terms (which are connected via the fluctuation-dissipation theorem [1]), we obtain the following dynamical system

$$\ddot{q}_{L_i} + 2\Gamma_{L_i}\dot{q}_{L_i} + \omega_{L_i}^2 q_{L_i} + \beta_i q_{L_i}^3 + \sum_{j,j'} \alpha_{ijj'} q_{H_j} q_{H_{j'}} = F_{L_i}\cos(\omega_{F_{L_i}}t) + \xi_{L_i}(t), \tag{S1}$$

$$\ddot{q}_{H_i} + 2\Gamma_{H_i}\dot{q}_{H_i} + \omega_{H_i}^2 q_{H_i} + 2q_{H_i}\sum_{j} \alpha_{jii} q_{L_j} + 2\sum_{j,j'\neq i} \alpha_{jij'} q_{L_j} q_{H_{j'}} = F_{H_i}\cos(\omega_{F_{H_i}}t) + \xi_{H_i}(t), \tag{S2}$$

where the Γ_{L_i} and Γ_{H_i} are the modal decay rates, and ξ_{L_i} and ξ_{H_i} are zero-mean delta-correlated independent noise terms, so that $\langle \xi_{L_i}(t)\rangle = \langle \xi_{H_i}(t)\rangle = 0$, $\left\langle \xi_{L_i}(t)\xi_{L_j}(t+\tau)\right\rangle = 2\delta_{ij}\delta(\tau)\mathcal{D}_{\xi_{L_i}}$, and $\left\langle \xi_{H_i}(t)\xi_{H_j}(t+\tau)\right\rangle = 2\delta_{ij}\delta(\tau)\mathcal{D}_{\xi_{H_i}}$. The above idealization of the noises also applies to general non-Gaussian noises, as long as their correlation times are considerably shorter than the relaxation time of the modes [2].

In the spirit of the analysis from the main text, we make the ansatz

$$\begin{aligned} q_{L_i}(t) &= \bar{q}_{L_i} + A_{L_i}(t)\mathrm{e}^{\mathrm{i}\omega_{F_{L_i}}t} + \mathrm{cc},\ \dot{q}_{L_i}(t) = \mathrm{i}\omega_{F_{L_i}} A_{L_i}(t)\mathrm{e}^{\mathrm{i}\omega_{F_{L_i}}t} + \mathrm{cc}, \\ q_{H_i}(t) &= A_{H_i}(t)\mathrm{e}^{\mathrm{i}\omega_{F_{H_i}}t} + \mathrm{cc},\ \dot{q}_{H_i}(t) = \mathrm{i}\omega_{F_{H_i}} A_{H_i}(t)\mathrm{e}^{\mathrm{i}\omega_{F_{H_i}}t} + \mathrm{cc}, \end{aligned} \tag{S3}$$

where $\bar{q}_{L_i} = -\langle \sum_{j,j'} \alpha_{ijj'} q_{H_j} q_{H_{j'}}\rangle/\omega_{L_i}^2$ are the DC deflections of the LF modes that arise from the time-independent component of $q_{H_j} q_{H_{j'}}$, and A_{L_i} (A_{H_i}) are the complex-amplitudes of the LF (HF) modes. Treating the q_{L_i} as quasi-static variables in Eq. (S2) and applying the rotating wave approximation (RWA), we obtain the following complex-amplitude equations

$$\dot{A}_{L_i} = -\left[\Gamma_{L_i} + \mathrm{i}\left(\Delta\omega_{L_i} - \frac{3\beta_i}{2\omega_{F_{L_i}}}|A_{L_i}|^2\right)\right] A_{L_i} + \frac{\mathrm{i}}{2\pi}\sum_{j,j'}\alpha_{ijj'}\int_0^{\frac{2\pi}{\omega_{F_{L_i}}}} A_{H_j}A^*_{H_{j'}}\mathrm{e}^{\mathrm{i}(\omega_{F_{H_j}}-\omega_{F_{H_{j'}}})t}\mathrm{e}^{-\mathrm{i}\omega_{F_{L_i}}t}dt$$
$$-\frac{\mathrm{i}F_{L_i}}{4\omega_{F_{L_i}}} - \frac{\mathrm{i}}{2\omega_{F_{L_i}}}\langle\xi_{L_i}\mathrm{e}^{-\mathrm{i}\omega_{F_{L_i}}t}\rangle, \tag{S4}$$

$$\dot{A}_{H_i} = -(\Gamma_{H_i} + \mathrm{i}\Delta\omega_{H_i})A_{H_i} + \frac{\mathrm{i}}{2\omega_{F_{H_i}}}\sum_j \alpha_{jii}(A_{L_j}\mathrm{e}^{\mathrm{i}\omega_{F_{L_j}}t} + A^*_{L_j}\mathrm{e}^{-\mathrm{i}\omega_{F_{L_j}}t})A_{H_i}$$
$$+\frac{\mathrm{i}}{2\pi}\sum_{j,j'\neq i}\alpha_{jij'}A^*_{L_j}A_{H_{j'}}\int_0^{\frac{2\pi}{\omega_{F_{H_i}}}}\mathrm{e}^{\mathrm{i}(\omega_{F_{H'_j}}-\omega_{F_{L_j}})t}\mathrm{e}^{-\mathrm{i}\omega_{F_{H_i}}t}dt - \frac{\mathrm{i}F_{H_i}}{4\omega_{F_{H_i}}} - \frac{\mathrm{i}}{2\omega_{F_{H_i}}}\langle\xi_{H_i}\mathrm{e}^{-\mathrm{i}\omega_{F_{H_i}}t}\rangle, \tag{S5}$$

where $\Delta\omega_{L_i} = \omega_{F_{L_i}} - \omega_{L_i}$ and $\Delta\omega_{H_i} = \omega_{F_{H_i}} - \omega_{H_i} - \sum_j \alpha_{jii}\bar{q}_{L_j}/(2\omega_{F_{H_i}})$ are the frequency detunings of the LF and HF modes.

Similar to Eq. (5) of the main text, Eq (S5) contains certain slowly varying excitations after the RWA, with frequencies $\omega_{F_{L_i}}$. In particular, for constant A_{L_i}'s, the A_{H_i} oscillate with frequencies $\omega_{F_{L_i}}$. Moreover, from Eq. (S5), we see that a pair of HF modes are coupled when they are separated in frequency by one of the LF mode's frequencies, i.e., $|\omega_{H_j} - \omega_{H'_j}| \approx \omega_{L_i}$. Refs. [3, 4] show experimental observations of this type of resonant coupling in cavity optomechanical systems. Consequently, Eq. (S5) represents a set of linearly coupled equations in A_{H_i}. We can formally solve Eq. (S5) in terms of the yet unknown complex amplitudes of the LF modes (A_{L_i}). Then, in the adiabatic tracking regime of A_{H_i} (under the assumption that $\Gamma_{H_i} \gg \Gamma_{L_i}$), we can obtain a set coupled Stuart–Landau oscillators for the complex amplitude of the LF modes. While we leave the details of such an analysis for future study, it is worth noting that even in the case of a single HF mode (and multiple LF modes), its backaction leads to linear and nonlinear coupling between the LF modes. This type of LF modal coupling, which is mediated by the HF modes, has been experimentally observed in Refs. [5, 6] in cavity optomechanics. Therefore, Eqs. (S4)-(S5), which account for the important effects of nonlinearity of the LF modes and noise [7, 8], can be viewed as a direct extension of the simplified model of the main text, which is clearly relevant to a wider class of systems.

S.2 Modal equations of the pendulums system

The equation of motion of the left and right pendulums that are shown in Fig. S1 are given by

$$m\ell^2\ddot{\theta}_L + \kappa(\theta_L)\theta_L - k_c\theta_R = k(\phi_{L_1} + \phi_{L_2}), \tag{S6}$$
$$m\ell^2\ddot{\theta}_R + \kappa(\theta_R)\theta_R - k_c\theta_L = k(\phi_{R_1} + \phi_{R_2}), \tag{S7}$$

where $\kappa(\theta_{L,R}) = k_c + k + mg\ell\sin\theta_{L,R}/\theta_{L,R}$, which in the leading order nonlinear approximation reduces to $\kappa(\theta_{L,R}) \approx k_c + k + mg\ell(1 - \theta^2_{L,R}/6)$. By setting $\phi_{L_1} = \phi_{R_1} = u_{DC} + u_0\cos(\omega_{F_0}t)$, $\phi_{L_2} = -\phi_{R_2} = u_1\cos(\omega_{F_1}t)$, and using the modal transformation $\theta_L = (q_0 + \delta_{DC} + q_1)/\sqrt{2}$, and $\theta_R = (q_0 + \delta_{DC} - q_1)/\sqrt{2}$, we find from Eqs. (S6)-(S7) that for $\delta_{DC} \ll q_0$, the lowest order (quadratic) nonlinear modal equations have the following form

$$\ddot{q}_0 + \omega_0^2 q_0 + \alpha\left(q_0^2 + q_1^2\right) = F_0\cos(\omega_{F_0}t), \tag{S8}$$
$$\ddot{q}_1 + \omega_1^2 q_1 + 2\alpha q_0 q_1 = F_1\cos(\omega_{F_1}t), \tag{S9}$$

where

$$\omega_0^2 = \frac{g}{\ell} + \frac{k}{m\ell^2},\ \omega_1^2 = \frac{g}{\ell} + \frac{k_1 + 2k_c}{m\ell^2},\ \alpha = -\frac{g\delta_{DC}}{4\ell},\ F_0 = \frac{\sqrt{2}ku_0}{m\ell^2},\ F_1 = \frac{\sqrt{2}ku_1}{m\ell^2},$$

and $\delta_{DC} \approx \frac{\sqrt{2}k}{k+mg\ell}u_{DC}$ for $\delta_{DC} \ll 1$.

S.3 The parameters of the driven van der Pol–Duffing oscillator

$$\epsilon = \frac{4\omega_{F_0}\left[\alpha^2F_1^2\Gamma_1\Delta\omega_1 - 8\Gamma_0\omega_{F_1}^3\prod_{n=-1}^1\Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2\right]}{\alpha^2F_1^2\Delta\omega_1(\Gamma_1^2 + \Delta\omega_1^2 - \omega_{F_0}^2) + 16\omega_0\omega_{F_0}\omega_{F_1}^3\prod_{n=-1}^1\Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2},$$

$$\gamma = \frac{64\omega_{F_0}^2\omega_{F_1}^3(\Gamma_1^4 + \omega_{F_0}^2(\Gamma_1^2 + 5\Delta\omega_1^2) - \Delta\omega_1^4 - 4\omega_{F_0}^4)\left[\alpha^2F_1^2\Gamma_1\Delta\omega_1 - 8\Gamma_0\omega_{F_1}^3\prod_{n=-1}^1\Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2\right]}{3\Gamma_1(3\Gamma_1^2 - 5\Delta\omega_1^2 + 8\omega_{F_0}^2)\left[\alpha^2F_1^2\Delta\omega_1(\Gamma_1^2 + \Delta\omega_1^2 - \omega_{F_0}^2) + 16\omega_0\omega_{F_0}\omega_{F_1}^3\prod_{n=-1}^1\Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2\right]^2}$$

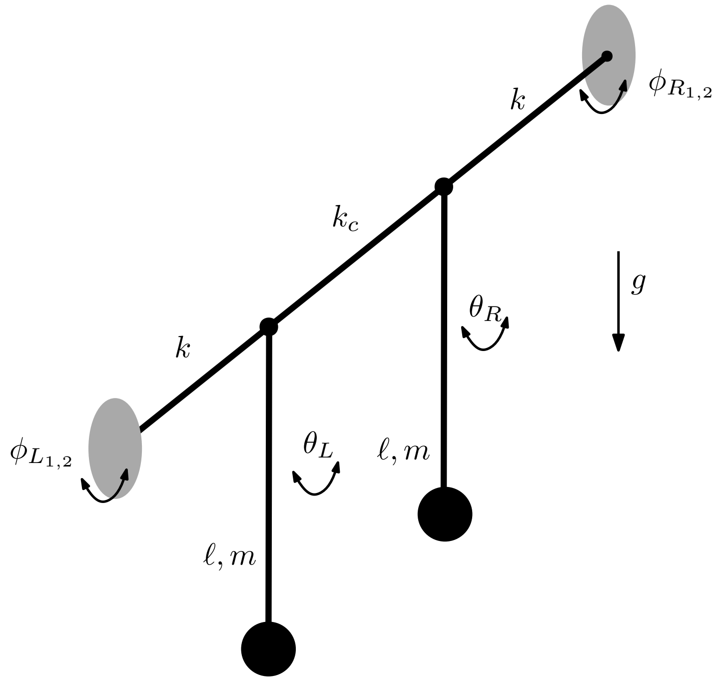

Fig. S1: Definition sketch of the coupled pendulums system. The two pendulums can oscillate transversely with respect to the shafts. The center shaft, which couples the motion of the pendulums, has a torsional stiffness of k_c, whereas the two side shafts are identical and have a torsional stiffness of $k \ll k_c$. The ends of the two side shafts are connected to drivers that impose angular motions $\phi_{L,R_{1,2}}$. The angular motions from the drivers satisfy the relations $\phi_{L_1} = \phi_{R_1}$ and $\phi_{L_2} = -\phi_{R_2}$, which generate the torques $k\phi_{L_1}/\sqrt{2}$ and $k\phi_{L_2}/\sqrt{2}$ in the symmetric and antisymmetric modes.

$$\times \prod_{n=-1}^{1} \Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2,$$

$$F = \left(\frac{3\Gamma_1 \Delta\omega_1 (3\Gamma_1^2 - 5\Delta\omega_1^2 + 8\omega_{F_0}^2)}{\left[\alpha^2 F_1^2 \Gamma_1 \Delta\omega_1 - 8\Gamma_0 \omega_{F_1}^3 \prod_{n=-1}^{1} \Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2\right] (\Gamma_1^2 + (\Delta\omega_1 - 2\omega_{F_0})^2)(\Gamma_1^2 + (\Delta\omega_1 + 2\omega_{F_0})^2)} \right)^{1/2}$$

$$\times \frac{\alpha F_0 F_1}{2\omega_{F_1}} \left(\omega_0 + \frac{\alpha^2 F_1^2 \Delta\omega_1 (\Gamma_1^2 + \Delta\omega_1^2 - \omega_{F_0}^2)}{16\omega_{F_0} \omega_{F_1}^3 \prod_{n=-1}^{1} \Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2} \right)^{-2},$$

$$\Omega_F = \omega_{F_0} \left(\omega_0 + \frac{\alpha^2 F_1^2 \Delta\omega_1 (\Gamma_1^2 + \Delta\omega_1^2 - \omega_{F_0}^2)}{16\omega_{F_0} \omega_{F_1}^3 \prod_{n=-1}^{1} \Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2} \right)^{-1},$$

$$L = \frac{2\omega_{F_1}}{\alpha^2 F_1} \left(\frac{\left[\alpha^2 F_1^2 \Gamma_1 \Delta\omega_1 - 8\Gamma_0 \omega_{F_1}^3 \prod_{n=-1}^{1} \Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2\right] (\Gamma_1^2 + (\Delta\omega_1 - 2\omega_{F_0})^2)(\Gamma_1^2 + (\Delta\omega_1 + 2\omega_{F_0})^2)}{3\Gamma_1 \Delta\omega_1 (3\Gamma_1^2 - 5\Delta\omega_1^2 + 8\omega_{F_0}^2)} \right)^{1/2},$$

$$T = \left(\omega_0 + \frac{\alpha^2 F_1^2 \Delta\omega_1 (\Gamma_1^2 + \Delta\omega_1^2 - \omega_{F_0}^2)}{16\omega_{F_0} \omega_{F_1}^3 \prod_{n=-1}^{1} \Gamma_1^2 + (\Delta\omega_1 + n\omega_{F_0})^2} \right)^{-1}.$$

S.4 The Adler equation

From Eq. (7) of the main text, we find that the polar notation $A_0 = -\mathrm{i}a_0\mathrm{e}^{\mathrm{i}[\varphi_0+\arg(\ell)]}/2$ yields the following pair of equations

$$\dot{a}_0 = \left(\Re\{\sigma\} - \Re\{\ell\}\frac{a_0^2}{4}\right)a_0 - \frac{\varepsilon f_0}{2\omega_{F_0}}\cos\theta, \tag{S10}$$

$$\dot{\varphi}_0 = \Im\{\sigma\} - \Im\{\ell\}\frac{a_0^2}{4} - \frac{\varepsilon f_0}{2\omega_{F_0}a_0}\sin\theta, \tag{S11}$$

where $\theta = \varphi_0 + \arg(\ell)$, and $\varepsilon f_0 = F_0$, which is used to explicitly denote the smallness of F_0 ($\varepsilon \ll 1$). For $\varepsilon = 0$, the amplitude of the LF mode reaches the steady-state value $a_{0\mathrm{ss}} = 2\sqrt{\Re\{\sigma\}/\Re\{\ell\}}$. Thus, in the presence of weak injection, we make the ansatz $a_0(t) = a_{0\mathrm{ss}} + \varepsilon\eta(t)$, and obtain the following evolution equation to the perturbation $\dot{\eta} = -2\Re\{\sigma\}\eta - [f_0/(2\omega_{F_0})]\cos\theta$. We see that the perturbation η is strongly damped; hence, for $t \gg 1/\Re\{\sigma\}$, we can set $\dot{\eta} = 0$ to obtain

$$a_0 = 2\sqrt{\frac{\Re\{\sigma\}}{\Re\{\ell\}}} - \frac{\varepsilon f_0}{4\omega_{F_0}\Re\{\sigma\}}\cos\theta. \tag{S12}$$

Substitution of Eq. (S12) into Eq. (S11) yields

$$\dot{\varphi}_0 = \Im\{\sigma\} - \Re\{\sigma\}\frac{\Im\{\ell\}}{\Re\{\ell\}} - \frac{\varepsilon f_0|\ell|}{4\omega_{F_0}\sqrt{\Re\{\sigma\}\Re\{\ell\}}}\sin\varphi_0 + O(\varepsilon^2), \tag{S13}$$

which is the well-known Adler equation.

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