Effects of Quantum Flux on Generalized KG-Oscillator in Ellis-Bronnikov-Type Wormhole with Point-Like Defect

F. AHMED
University of Science & Technology Meghalaya

H. AOUNALLAH (✉ houcine.aounallah@univ-tebessa.dz)
Echahid Cheikh Larbi Tebessi University - Tebessa, Algeria

P. RUDRA
Department of Mathematics, Asutosh College, Kolkata-700026,

Article

**Keywords:** Relativistic wave equation, solutions of wave-equation: bound-state, Magnetic Monopoles, Modified Theories of Gravity, special functions

**Posted Date:** March 27th, 2023

**DOI:** [https://doi.org/10.21203/rs.3.rs-2638631/v1](https://doi.org/10.21203/rs.3.rs-2638631/v1)

**License:** ☑️ ❬ This work is licensed under a Creative Commons Attribution 4.0 International License. [Read Full License](https://creativecommons.org/licenses/by/4.0/)

**Additional Declarations:** No competing interests reported.
Effects of Quantum Flux on Generalized KG-Oscillator in Ellis-Bronnikov-Type Wormhole with Point-Like Defect

F. AHMED¹, H. AOUNALLAH², P. RUDRA³

¹Department of Physics, University of Science & Technology Meghalaya, Ri-Bhoi, Meghalaya-793101, India
²Department of Science and Technology, Echahdh Cheikh Larbi Tebessi University - Tebessa, Algeria
³Department of Mathematics, Asutosh College, Kolkata-700026, India

(Corresponding author e-mail(s): houcine.aounallah@univ-tebessa.dz; houcine12400@gmail.com)

Received: date / Accepted: date

Abstract The relativistic quantum dynamics of oscillator field via the generalized Klein-Gordon oscillator in the background of Ellis-Bronnikov-type wormhole space-time with topological defect under the influence of quantum flux are investigated. In fact, we derived the radial wave equation for a Coulomb- and Cornell-type potential form functions and solved the equation analytically. The generalized KG-oscillator is introduced by replacing the momentum operator $\partial_\mu \rightarrow (\partial_\mu + M \omega X_\mu)$ in the wave equation, where $X_\mu = (0, f(x), 0, 0)$ with $f(x)$ is an arbitrary function other than $x$. We show that the topological defect and the radius of wormhole throat influences the eigenvalue solution and modified the result. Furthermore, the quantum flux shifted the energy levels of the oscillator and shows the gravitational analogue to the Aharonov-Bohm effect.

Keywords Relativistic wave equation · solutions of wave-equation: bound-state · Magnetic Monopoles · Modified Theories of Gravity · special functions

PACS 03.65.Ge · 03.65.Pm · 14.80.Hv · 04.50.Kd · 02.30.Gp

1 Introduction

In the early universe, there was a phase transition period that was accompanied by the decoupling of fundamental interactions. This resulted in the breaking of symmetry in the universe and gave rise to strange and abnormal geometrical objects. These are known as topological defects [1] and these defects appear in the universe in the form of exotic cosmological objects. There are numerous candidates for such objects such as the domain wall [2], cosmic string [3–5], global monopole [6], etc. These strange objects have not been observationally verified yet, and so they are considered as hypothetical to date. But some of them are theoretically so sound and obvious that their existence is almost guaranteed. One such defect is the global monopole (GM) which is expected to be observationally verified in near future. Due to this a lot of interest has been generated in the research of GM during the past few years [7–13]. Other works on topological defects include [14–16].

In quantum mechanics, oscillators are considered to be standard systems that can be used to study various quantum phenomena. The Klein-Gordon oscillator (KGO) is one of the most popular forms of a quantum oscillator used in the relativistic quantum mechanics. This is because KG-oscillator can be used to describe the corresponding non-relativistic quantum harmonic oscillator given by Schrödinger’s equation [17]. Initially, inspired by the Dirac oscillator [18] KG-oscillator has recently been probed under different theoretical set-ups. Some of the set-ups under which KG-oscillator has been explored are in the Kaluza-Klein theory [19], anti-de-Sitter space-time [20], space-time with torsion [21], cosmic string space-time [22], under the effect of central potentials [23, 24], and under Lorentz symmetry violation effects [25–29].
We have already seen that exotic cosmological objects can be formed as a result of the symmetry breaking in the early universe. One such object is a wormhole, and GM being a topological defect of spacetime can trigger the formation of such a structure. Wormholes are tunnels or shortcuts between two different regions of spacetime in the same universe or even two parallel universes [30, 31]. They were first detected theoretically in Einstein’s equations of general relativity (GR). Of all the known wormhole solutions of GR, the simplest one is the Ellis-Bronnikov (EB) wormhole [32, 33]. The Raychaudhuri equation for the congruence of a radial null geodesic in the EB wormhole spacetime is given by,

\[
\frac{d\Theta}{d\lambda} = -R_{\mu\nu}k^\mu k^\nu - \frac{\Theta^2}{2}
\]  

(1)

where \(\Theta\) is the congruence expansion, \(\lambda\) is the affine parameter connected with the geodesics, \(R_{\mu\nu}\) is the Ricci tensor and \(k^\mu\) is a null tangent vector to the geodesic. Near the throat of the wormhole, the congruence expansion is reduced to zero, i.e., \(\Theta = 0\), but \(\frac{d\Theta}{d\lambda} \geq 0\), which implies from eqn.(1) \(R_{\mu\nu}k^\mu k^\nu \leq 0\) which is a clear violation of the null convergence condition (NCC) [34]. When we take GR into consideration a violation of the NCC in equivalent to a violation of the null energy condition (NEC). In GR it is known that the energy conditions basically specify the attractive nature of the gravitational fields which are generated from the conventional sources of matter. For a wormhole, there is a direct violation of these conditions near its throat. This indicates that there is a requirement for some form of anti-gravitational effects to keep the throat of the wormhole open thus sustaining it. It is theorized that these anti-gravitational effects can be generated by some form of exotic matter known as dark energy (DE). EB wormhole is actually a solution obtained in the background of exotic matter known as dark energy (DE). EB wormhole is actually a solution obtained in the background of exotic matter known as dark energy (DE). EB wormhole is actually a solution obtained in the background of Eddington-inspired Born-Infeld (EiBI) gravity [35], which describes a static and spherically symmetric spacetime with the topological charge of GM [36, 37]. This solution was obtained by coupling the energy-momentum tensor of the exterior region of the GM core with the geometry of spacetime. EB wormhole has been explored under various set-ups in literature [38–40]. Therefore, a static and spherically symmetric space-time describing the Ellis-Bronnikov-type wormhole with a point-like defect is represented by the following line-element

\[
ds^2 = -dt^2 + \frac{dx^2}{a^2} + (x^2 + a^2)(d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(2)

Our motivation is the result presented in Ref. [39], where quantum effects of KG-oscillator in a topologically charged EB-type wormhole space-time were studied. As a natural extension, we study the quantum motions of the oscillator field via the generalized Klein-Gordon oscillator in this wormhole space-time background under the influences of the quantum flux field. We consider two different function for these studies, such as a Coulomb- and Cornell-type potential form functions and analyze the effects on the eigenvalue solutions. In fact, it is shown that the energy levels and wave functions presented here are different from the result obtained earlier in [39] for the KG-oscillator and gets modified compared to the known eigenvalue solutions. In addition, the quantum flux field also modified the eigenvalue solutions more and shows an analogue of the Aharonov-Bohm effect.

The structure of this paper is organized as follows: In Sect. 2, we write down the wave equation for the generalized KG-oscillator in a topologically charged EB-type wormhole for the function \(f(x)\). Then we choose a Coulomb-type (sub-section 2.1) and a Cornell-type (sub-section 2.2) potential form functions and present the eigenvalue solutions; in Sect. 3, we present our conclusions. Throughout the analysis, we choose the system of units as \(c = 1 = \hbar = G\).

2 Eigenvalue Solution of Generalized KG-oscillator in Wormhole Space-time with Point-like Defect

In this section, we study the relativistic quantum motions of generalized KG-oscillator in wormhole space-time background with a point-like defect. The relativistic quantum dynamics of spin-0 scalar particles in curved space-time is described by

\[
\left[-\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) + M^2\right] \Psi = 0,
\]  

(3)
where $M$ is the rest mass of the particles.

The Klein-Gordon oscillator field is studied by replacing the operator $\partial_\mu \rightarrow (\partial_\mu + M \omega X_\mu)$, where $\omega$ is the oscillator frequency, and $X_\mu = (0, x, 0, 0)$. Its generalized version can be studied by replacing $X_\mu = (0, x, 0, 0) \rightarrow (0, f(x), 0, 0)$, where $f(x)$ is an arbitrary function other than $x$. Furthermore, if one introduce an electromagnetic potential $A_\mu$ through a minimal substitution in the KG-oscillator equation, then the replacement will be $(\partial_\mu + M \omega X_\mu) \rightarrow (\partial_\mu + M \omega X_\mu - i q A_\mu)$, where $q$ is the electric charges.

Therefore, the relativistic generalized KG-oscillator with an electromagnetic potential $A_\mu$ is described by the following wave equation (system of units are chosen where $c = 1 = \hbar = G$)

$$\left[ \frac{1}{\sqrt{-g}} (D_\mu + M \omega X_\mu)(\sqrt{-g} g^{\mu \nu}) (D_\nu - M \omega X_\nu - M^2) \right] \Psi = 0,$$

(4)

where $D_\mu \equiv (\partial_\mu - i q A_\mu)$. In this analysis, we choose the electromagnetic three-vector potential $\vec{A}$ given by [42–47]

$$A_x = 0 = A_\theta, \quad A_\phi = \frac{\Phi_{AB}}{2 \pi \sin \theta},$$

(5)

where $\Phi_{AB} = \text{const} = \Phi_0 \Phi$ is the quantum flux field, $\Phi_0 = 2 \pi q^{-1}$ is the quantum of magnetic flux, and $\Phi$ is the amount of magnetic flux which is a positive integer. It is worth mentioning that the Aharonov–Bohm effect which is a quantum mechanical phenomena has been investigated in several branches of physics Refs. [42–49].

Thereby, expressing the wave equation (4) in the space-time background (2) and using the electromagnetic three-vector potential (5), we obtain the following equation

$$\left[ -\frac{\partial^2}{\partial t^2} + \frac{\alpha^2}{(x^2 + a^2)} \left( \frac{\partial}{\partial x} + M \omega f(x) \right) \left\{ (x^2 + a^2) \left( \frac{\partial}{\partial x} - M \omega f(x) \right) \right\} + \frac{1}{(x^2 + a^2)} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right\} + \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} - i \Phi \right)^2 \right] \Psi = M^2 \Psi.$$

(6)

In quantum mechanical systems, the wave function $\Psi$ is always expressible in terms of different variables by the method of separation of the variables. Here, we choose one such possible function $\Psi$ in terms of the function $\psi(x)$ given by

$$\Psi(x, \theta, \phi) = e^{-iEY(\theta, \phi)} \psi(x),$$

(7)

where $l, m$ are the orbital and magnetic quantum numbers, respectively, and $Y(\theta, \phi)$ is the spherical harmonics.

Thereby, substituting the wave function (7) in the Eq. (6), we obtain the following differential equation in terms of the wave function $\psi(x)$ as:

$$\psi''(x) + \frac{2x}{(x^2 + a^2)} \psi'(x) + \left[ b^2 - \frac{2M \omega x}{(x^2 + a^2)} f(x) - M \omega f'(x) - M^2 \omega^2 f^2(x) - \frac{l^2}{(x^2 + a^2)} \right] \psi = 0,$$

(8)

where different parameters are defined by

$$b^2 = \frac{E^2 - M^2}{\alpha^2}, \quad l^2 = \frac{l'(l' + 1)}{\alpha^2}, \quad l' = (l - \Phi).$$

(9)

Here, one can see that the orbital quantum number $l$ is shifted, $l \rightarrow l' = (l - \Phi)$, an effective magnetic quantum number due to the presence of the quantum flux in the quantum system.

Since, we are interested in the generalized version of the KG-oscillator field. Below, we will consider two different types of function $f(x)$ in the Eq. (8) and solve the equation using the special functions.
2.1 A Coulomb-Type Function

We consider the function $f(x)$ here to be a Coulomb-type potential form function. The Coulomb-type potential ($\propto \frac{1}{x}$) is responsible for short-range interactions, and has studied in H-atom, quark-antiquark interaction etc.. Therefore, the Coulomb-type potential function is given by

$$f(x) = \frac{\eta}{x}, \quad \eta > 0. \quad (10)$$

Substituting this function (10) in the Eq. (8), we obtain the following equation

$$\psi''(x) + \frac{2x}{(x^2 + a^2)} \psi'(x) + \left[ b^2 - \frac{(j^2 - \frac{1}{4})}{x^2} - \frac{\nu^2}{(x^2 + a^2)} \right] \psi(x) = 0, \quad (11)$$

where we set the parameters

$$j = \left( M \omega \eta - \frac{1}{2} \right), \quad \nu = \sqrt{2M \omega \eta + i^2}. \quad (12)$$

Our aim is to obtain the bound-state solution of the quantum system under investigation. Let us consider the function $\psi(x)$ as follows:

$$\psi(x) = x^{\frac{|j|}{2} + \frac{1}{2}} G(x), \quad (13)$$

where $G(x)$ is an unknown function.

Thereby, substituting this function (13) in the Eq. (12), we obtain

$$G''(x) + \left[ \frac{2x}{x^2 + a^2} + \frac{(1 + 2|j|)}{x} \right] G'(x) + \left[ b^2 + \frac{1 + 2|j| - \nu^2}{x^2 + a^2} \right] G(x) = 0. \quad (14)$$

Finally, introducing a new variable via $u = -\frac{x^2}{a^2}$ in the Eq. (14), we obtain the following differential equation for $G$ as:

$$G''(u) + \left[ \frac{1 + |j|}{u} + \frac{1}{u - 1} \right] G'(u) + \left[ -\frac{1}{4} \left( a^2 b^2 + 1 + 2|j| - \nu^2 \right) + \frac{1}{4} \left( 1 + 2|j| - \nu^2 \right) \right] G(u) = 0. \quad (15)$$

which is the confluent Heun equation form \cite{55, 56} and $G(u)$ is the confluent function given by

$$G(u) = H_c \left( 0, |j|, 0, -\frac{a^2 b^2}{4}, \frac{a^2 b^2 + 1 - \nu^2}{4}; u \right). \quad (16)$$

To solve the above differential equation (15), we use the Fröbenius power series solution given by \cite{57}

$$G(u) = \sum_{i=0}^{\infty} c_i u^i. \quad (17)$$

Substituting this power series in the Eq. (15), we obtain the following recurrence relation

$$c_{k+2} = \frac{1}{4(k+2)(k+2+|j|)} \left\{ 4(k+1)(k+2+|j|) + a^2 b^2 + 1 + 2|j| - \nu^2 \right\} c_{k+1} - a^2 b^2 c_k \quad (18)$$

with the coefficient

$$c_1 = \frac{(a^2 b^2 + 1 + 2|j| - \nu^2)}{4(1+|j|)} c_0. \quad (19)$$
To obtain the bound-state solution of the quantum system, we must truncate the power series (17) to a finite degree polynomial such that the wave function $\psi$ is regular everywhere. Let us consider $k = (n - 1)$ where $c_{n+1} = 0$ such that the wave function $\psi$ is regular everywhere. Thus, we obtain from (18)

$$c_n = \frac{a^2 b^2}{4n(n + 1 + |j|)} + a^2 b^2 + 1 + 2|j| - \eta^2} c_{n-1}. \quad (20)$$

Now, we obtain the individual energy levels and the wave functions for the mode $n = 1, 2, 3, ...$ In this work, we consider one such mode defined $n = 1$ called the ground state of the system. Thus, from (20) we obtain

$$c_1 = \frac{a^2 b^2}{4(2 + |j|)} + a^2 b^2 + 1 + 2|j| - \nu^2} c_0 \quad (21)$$

Now, comparing the Eqs. (19) and (21), we obtain the energy level as follows:

$$E_{1,l} = \left[ M^2 + \frac{2a^2}{a^2} \left( \frac{l^2}{2} + M \omega \eta - |j| - \frac{3}{2} \right) \pm \sqrt{(l + |j|) (l^2 + 2M \omega \eta) - (3 + 2|j|) |j|} \right]^{1/2}, \quad (22)$$

where $i^2 = \frac{1}{a^2} (l - \Phi) (l - \Phi + 1)$.

The ground state wave function will be

$$\psi_{1,l} = u \left( \frac{l + \frac{1}{2}}{2 (1 + |j|)} \right) \left[ 1 + \frac{u}{2} \right] \left( -1 \pm \sqrt{(l + |j|)(l^2 + 2M \omega \eta) - (3 + 2|j|) |j|} \right)] c_0.$$

Equation (22) is the ground state energy level and (23) is the corresponding wave function of the oscillator field by choosing a Coulomb-type potential function in the background of a topologically charged EB-type wormhole under the influence of the quantum flux field. We see that this energy expression depends on the topological defect parameter $\alpha$, the radius $a = \text{const}$ of the wormhole throat, and the quantum flux field $\Phi_{AB}$ and gets modified.

### 2.2 A Cornell-Type Function

Here, we consider the function $f(x)$ a Cornell-type potential form function which is a special case of quark-antiquark interactions which has one more term [58]. This Cornell-type potential function is given by

$$f(x) = \left( \frac{\eta}{x} + \delta x \right), \quad \eta > 0, \delta > 0. \quad (23)$$

Noted that for $\beta \to 0$, the function becomes a linear in $x$ and the quantum system is called the KG-oscillator. For $\delta \to 0$, the function becomes Coulomb-type which we discussed earlier. This type of potential function has widely been investigated in the context of quantum systems both in the relativistic and non-relativistic limit [49–54].

Thereby, substituting this function (23) in the Eq. (8), we obtain

$$\psi''(x) + \frac{2x}{(x^2 + a^2)} \psi'(x) + \left[ \frac{\epsilon^2}{x^2} - \frac{\sigma^2}{(x^2 + a^2)} - M^2 \omega^2 \delta^2 x^2 \right] \psi(x) = 0, \quad (24)$$

where different parameters are

$$\epsilon^2 = b^2 - M \omega \delta (3 + 2M \omega \eta), \quad \sigma = \sqrt{\nu^2 - 2M \omega \delta a^2}. \quad (25)$$
As stated earlier, let us consider a suitable wave function $\psi(x)$ as follows:

$$\psi(x) = x^{\left|\frac{j}{2}\right|} e^{-\frac{1}{2} M \omega \delta x^2} G(x). \quad (26)$$

Substituting this wave function in the Eq. (24), we obtain the following differential equation

$$G''(x) + \left(\frac{1 + 2 |j|}{x} - 2 M \omega \delta x + \frac{2x}{(x^2 + a^2)}\right) G'(x) + \left[\Pi + \sum \right] G(x) = 0, \quad (27)$$

where

$$\Pi = \delta^2 - M \omega \delta \left(7 + 2 M \omega \eta + 2 |j|\right), \quad \sum = 1 + 2 |j| - \Theta^2 + 2 M \omega \delta a^2. \quad (28)$$

By changing to a new variable via $u = -\frac{x^2}{a^2}$ in the Eq. (27), we obtain

$$G''(u) + \left[M \omega \delta a^2 + \frac{1 + |j|}{u} + \frac{1}{u-1}\right] G'(u) + \left[-\frac{1}{4} \left(\Pi a^2 + \sum\right) + \frac{(\sum/4)}{u-1}\right] G(u) = 0 \quad (29)$$

which is the confluent Heun equation form [55, 56] and the Heun function is given by

$$G(u) = H_c \left(M \omega \delta a^2, |j|, 0, -\frac{a^2 e^2}{4}, \frac{a^2 e^2 + 1 - \Theta^2}{4}; u\right). \quad (30)$$

To solve the differential Eq. (29), we consider the same power series solution (17) in the Eq. (29), we obtain the recurrence relation

$$c_{k+2} = \frac{1}{4(k+2)(k+2 + |j|)} \left[\left\{4(k+1)(k+2 + |j| - M \omega \delta a^2) + \Pi a^2 + \sum\right\} c_{k+1} + (4 M \omega \delta a^2 k - \Pi a^2) c_k\right] \quad (31)$$

with the coefficient

$$c_1 = \frac{\Pi a^2 + \sum}{4(1 + |j|)} c_0. \quad (32)$$

Similar to the previous analysis, let us consider $k = (n - 1)$ where $c_{n+1} = 0$, we obtain

$$c_n = \frac{4 M \omega \delta a^2 (n - 1) - \Pi a^2}{4 n (n + 1 + |j| - M \omega \delta a^2) + \Pi a^2 + \sum} c_{n-1}. \quad (33)$$

For the ground state system defined by $n = 1$, we obtain

$$c_1 = \frac{\Pi a^2}{4 (2 + |j| - M \omega \delta a^2) + \Pi a^2 + \sum} c_0. \quad (34)$$

Comparing Eqs. (32) with (34), we obtain the following energy level given by

$$E_{1,1} = \pm \sqrt{M^2 + \frac{\alpha^2}{a^2} \left\{\Theta + M \omega \delta a^2 (7 + 2 M \omega \eta)\right\}}, \quad (35)$$
where we have set the parameter
\[
\Theta = 2M \omega \delta a^2 |j| - 2|j| + \Theta^2 - 3
\]
\[
\pm 2[-3|j| - 2j^2 + \Theta^2(1 + |j|) - 2M \omega \delta a^2(|j| + 2) + M^2 \omega^2 \delta^2 a^4]^{1/2},
\]
(36)

with \( \Theta^2 = t^2 + 2M \omega (\eta - \delta a^2) \) and \( t \) is given in the Eq. (9).

The ground state wave function will be
\[
\psi_{1,1} = u (|j| + \frac{1}{2}) \exp \left( -\frac{1}{2} M \omega \delta a^2 \right) \left[ 1 + u \frac{(M \omega \delta a^2 - 1)}{2(1 + |j|)} \pm u \frac{1}{2(1 + |j|)} \times \right.
\]
\[
\left. \left\{-3|j| - 2j^2 + \Theta^2(1 + |j|) - 2M \omega \delta a^2(|j| + 2) + M^2 \omega^2 \delta^2 a^4 \right\}^{1/2} \right].
\]
(37)

Equation (35)–(36) is the ground state energy level and (37) is the corresponding wave function of the oscillator field by choosing a Cornell-type potential form function in the background of a topologically charged EB-type wormhole under the influences of the quantum flux field. We see that this eigenvalue solution depends on the topological parameter \( \alpha \), the radius \( a = \text{const} \) of the wormhole throat, and the quantum flux field \( \Phi_{AB} \) and gets modified.

Now, we comparing our result with the one obtained in Ref. [39]. Let \( \eta \to 0 \), and hence, the function \( f(x) \propto x \), that is, a linear function. In that case, the quantum system is called the KG-oscillator. Therefore, the energy eigenvalue will be
\[
E_{l,1} = \pm \sqrt{M^2 + \frac{\alpha^2}{a^2}} (\Theta + 7M \omega \delta),
\]
(38)

where
\[
\Theta = t^2 - M \omega \delta a^2 - 4 \pm 2 \left( -2 + \frac{3}{2} t^2 - 8M \omega \delta a^2 + M^2 \omega^2 \delta^2 a^4 \right)^{1/2},
\]
(39)

with \( t \) given in the Eq. (9).

The corresponding ground state wave function will be
\[
\psi_{1,1} = u \exp \left( -\frac{1}{2} M \omega \delta a^2 \right) \left[ 1 + u \frac{(M \omega \delta a^2 - 1)}{3} \pm u \frac{1}{3} \left\{-2 + \frac{3}{2} t^2 - 8M \omega \delta a^2 + M^2 \omega^2 \delta^2 a^4 \right\}^{1/2} \right].
\]
(40)

Equation (38)–(39) is the ground state energy level and (40) is the corresponding wave function of the KG-oscillator field in the background of a topologically charged EB-type wormhole under the influences of the quantum flux field.

3 Conclusions

The investigation of quantum mechanical problems in the background of topological defect is an interesting topic and have significant in the literature. Because the topological defect changes the geometrical properties and hence, changes the physical properties of a quantum system under investigation. The energy eigenvalue and wave function of quantum particles is influenced by the topological defect of a geometry and get them modified compared to the flat space result and breaks the degeneracy of the energy levels. In addition, dynamics of quantum particles under the influences of the quantum flux field is also significant because even though there is no direct interactions of the particles with external magnetic field but the eigenvalue solution gets modified by it.
In this analysis, the wave equation of the generalized Klein-Gordon oscillator in the background of Ellis-Bronnikov-type wormhole space-time with a point-like defect under the influences of the quantum flux field was derived. Then, we have chosen a Coulomb-type potential form function \( f(x) \propto \frac{1}{x} \) which is responsible for short-range interactions and arrived at the confluent Heun differential equation form after a few mathematical steps which can be solved using a power series solutions method. As particular cases, we have presented the ground state energy level \( E_{1,m} \) by the expression (22) and wave function by the Eq. (23) of the oscillator field and others are in the same way. Afterwards, we have considered a Cornell-type potential form function (a linear plus Coulomb function) and arrived at the confluent Heun equation form and finally solved it using the same procedure. Here also, we presented the ground state energy level \( E_{1,m} \) by the expression (35)–(36) and wave function by the Eq. (37) of the oscillator field. In fact, it has shown there that the presence of a Cornell-type potential function considered in this quantum system and the quantum flux field modified the energy levels and wave function of the oscillator field compared to those result obtained in Ref. [39].

Throughout the analysis, we have shown that the orbital quantum number \( l \) is shifted, that is, \( l \to l' = \left(l - \frac{\Phi_{AB}}{2\pi}\right) \) due to the quantum flux field. Hence, the energy eigenvalue depends on the geometric quantum phase and this dependence of the eigenvalue shows the gravitational analogue of the Aharonov-Bohm effect [59, 60]. Since the geometry under consideration is a wormhole space-time, therefore, the presented eigenvalue solutions is also influenced by the radius of the wormhole throat.

Acknowledgement

P.R. acknowledges the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for granting visiting associateship.

Data Availability Statement

All data generated or analysed during this study are included in this published article.

Conflict of Interest

There are no conflicts of interest regarding publication of this paper.

References

11. E. R. Bezerra de Mello and A. A. Saharian, Class. Quantum Grav. 29, 135007 (2012).