Supplementary Information:

Note 1: Derivation of nondiffracting supertoroidal pulses.

We start from the scalar generating function derived from modified power spectrum method [1,2]:

$$f = f_0 \frac{e^{-s/q_3}}{(q_1 + i\tau)(s + q_2)^{\alpha}} \tag{S1}$$

Where $s \equiv r^2/(q_1 + i\tau) - i\sigma$, $\tau = z - ct$, $\sigma = z + ct$, $r = \sqrt{x^2 + y^2}$, $c = 1/\sqrt{\mu_0 \varepsilon_0}$ is the speed of light, the parameters q_1, q_2, q_3 are real positive with units of length, and the real dimensionless parameter α must satisfy $\alpha \geq 1$ in order for the electromagnetic pulse to fulfill finite energy. In conventional method, it always assumes that $q_3 \to \infty$ and $\alpha = 1$. Here we break the limits that the parameter α can be any real number no less than one and q_3 can be any value in length dimension. The azimuthally polarized pulse is derived under a curled vector Hertz potential $\mathbf{\Pi} = \nabla \times \hat{\mathbf{z}} f(\mathbf{r}, t)$ in cylindrical coordinate (r, θ, z), the TE-mode electromagnetic field can be generated from Hertz potential by:

$$\begin{cases} \mathbf{E}(\mathbf{r}, t) = -\mu_0 \dfrac{\partial}{\partial t} \nabla \times \mathbf{\Pi} = \hat{\boldsymbol{\theta}} \mu_0 \partial_r \partial_t f \\ \mathbf{H}(\mathbf{r}, t) = \nabla \times (\nabla \times \mathbf{\Pi}) = \hat{\mathbf{r}} \partial_r \partial_z f + \hat{\mathbf{z}} \left(\partial_z^2 - \dfrac{1}{c^2} \partial_t^2 \right) f \end{cases} \tag{S2}$$

Electric field: Based on equation set (S2), the amplitude of azimuthally polarized electric field can be derived as:

$$\begin{aligned} E_\theta &= \mu_0 \partial_z \partial_t f = \mu_0 f_0 \partial_\rho \partial_t \frac{e^{-s/q_3}}{(q_1 + i\tau)(s + q_2)^{\alpha}} \\ &= \mu_0 f_0 2ice^{-s/q_3} \left[\frac{(\alpha+1)\alpha r u}{(q_1 + i\tau)^2 (q_2 + s)^{\alpha+2}} - \frac{2\alpha r}{(q_1 + i\tau)^3 (q_2 + s)^{\alpha+1}} + \frac{2\alpha r u}{q_3 (q_1 + i\tau)^2 (q_2 + s)^{\alpha+1}} \right. \\ &\quad \left. - \frac{2r}{q_3 (q_1 + i\tau)^3 (q_2 + s)^{\alpha}} - \frac{ru}{q_3^2 (q_1 + i\tau)^2 (q_2 + s)^{\alpha}} \right] \\ &= \frac{2i\mu_0 f_0 cre^{-s/q_3}}{(q_1 + i\tau)^2 (q_2 + s)^{\alpha}} \left[\frac{(\alpha+1)\alpha u (q_1 + i\tau) - 2\alpha (q_2 + s)}{(q_1 + i\tau)(q_2 + s)^2} + \frac{2\alpha u (q_1 + i\tau) - 2(q_2 + s)}{q_3 (q_1 + i\tau)(q_2 + s)} - \frac{u}{q_3^2} \right] \end{aligned} \tag{S3}$$

where $u = r^2/(q_1 + i\tau)^2 - 1$. When $q_3 \to \infty$ and $\alpha = 1$, the equation (S3) will be reduced into:

$$E_\theta\big|_{\alpha=1, q_3 \to \infty} = 4i\mu_0 f_0 cr \frac{u(q_1 + i\tau) - (q_2 + s)}{(q_1 + i\tau)^3 (q_2 + s)^3} \tag{S4}$$

Then we prove that the equation (S4) is exactly the expression of the azimuthal electric field of fundamental flying doughnut. Substitute $u = r^2/(q_1 + i\tau)^2 - 1$, $s \equiv r^2/(q_1 + i\tau) - i\sigma$, $\tau = z - ct$, $\sigma = z + ct$ into equation (S4), we get:

$$
\begin{aligned}
E_\theta &= 4i\mu_0 f_0 cr \frac{\left[\frac{r^2}{(q_1+i\tau)^2}-1\right](q_1+i\tau)-\left(q_2+\frac{r^2}{q_1+i\tau}-i\sigma\right)}{(q_1+i\tau)^3\left(q_2+\frac{r^2}{q_1+i\tau}-i\sigma\right)^3} \\
&= 4i\mu_0 f_0 cr \frac{\left[r^2-(q_1+i\tau)^2\right]-\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]}{(q_1+i\tau)\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} \\
&= 4i\mu_0 f_0 cr \frac{-(q_1+i\tau)-(q_2-i\sigma)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} = -4if_0\mu_0 c \frac{r(q_1+q_2-2ict)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3}
\end{aligned}
\tag{S5}
$$

This equation (S5) is exactly the expression of the electric field of the fundamental flying doughnut pulse [2].

Magnetic field: The magnetic field of the general transversely divergent pulse includes the radial and longitudinal components, based on equation set (S2), the radial component can be derived as

$$
\begin{aligned}
H_r &= \partial_r\partial_z f = f_0\partial_r\partial_z \frac{e^{-s/q_3}}{(q_1+i\tau)(s+q_2)^\alpha} \\
&= \frac{2if_0 re^{-s/q_3}}{(q_1+i\tau)^2(q_2+s)^\alpha}\left[\frac{-(\alpha+1)\alpha v}{(q_2+s)^2}+\frac{2\alpha}{(q_1+i\tau)(q_2+s)}-\frac{2\alpha v}{q_3(q_2+s)}+\frac{2}{q_3(q_1+i\tau)}-\frac{v}{q_3^2}\right] \\
&= \frac{2if_0 re^{-s/q_3}}{(q_1+i\tau)^2(q_2+s)^\alpha}\left[\frac{2\alpha(q_2+s)-(\alpha+1)\alpha v(q_1+i\tau)}{(q_1+i\tau)(q_2+s)^2}+\frac{2(q_2+s)-2\alpha v(q_1+i\tau)}{q_3(q_1+i\tau)(q_2+s)}-\frac{v}{q_3^2}\right]
\end{aligned}
\tag{S6}
$$

where $v = r^2/(q_1+i\tau)^2+1$. When $q_3\to\infty$ and $\alpha=1$, the equation (S6) will be reduced into:

$$
H_r\big|_{\alpha=1,q_3\to\infty} = 4if_0 r\frac{(q_2+s)-v(q_1+i\tau)}{(q_1+i\tau)^3(q_2+s)^3} \tag{S7}
$$

Then we prove that the equation (S7) is exactly the radial magnetic field of fundamental flying doughnut. Substitute $v = r^2/(q_1+i\tau)^2+1$, $s\equiv r^2/(q_1+i\tau)-i\sigma$, $\tau = z-ct$, $\sigma = z+ct$ into equation (S7), we get:

$$
\begin{aligned}
H_r &= 4if_0 r\frac{\left(q_2+\frac{r^2}{q_1+i\tau}-i\sigma\right)-\left[\frac{r^2}{(q_1+i\tau)^2}+1\right](q_1+i\tau)}{(q_1+i\tau)^3\left(q_2+\frac{r^2}{q_1+i\tau}-i\sigma\right)^3} \\
&= 4if_0 r\frac{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]-\left[r^2+(q_1+i\tau)^2\right]}{(q_1+i\tau)\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} \\
&= 4if_0 r\frac{(q_2-i\sigma)-(q_1+i\tau)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} = 4if_0 r\frac{r(q_2-q_1-2iz)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3}
\end{aligned}
\tag{S8}
$$

This equation (S8) is exactly the closed-form expression of the radial magnetic field of the fundamental toroidal pulse [2].

The longitudinal component of magnetic field can be divided into two terms as follow:

$$\partial_z^2 f = f_0 \partial_z^2 \frac{e^{-s/q_3}}{(q_1+i\tau)(s+q_2)^{\alpha}}$$
$$= \frac{f_0 e^{-s/q_3}}{(q_1+i\tau)(q_2+s)^{\alpha+2}}\left[-(1+\alpha)\alpha v^2 + \frac{2\alpha v(q_2+s)}{q_1+i\tau} + \frac{2\alpha r^2(q_2+s)}{(q_1+i\tau)^3} - \frac{2(q_2+s)^2}{(q_1+i\tau)^2}\right. \quad \text{(S9)}$$
$$\left. - \frac{2\alpha v^2(q_2+s)}{q_3} + \frac{2v(q_2+s)^2}{q_3(q_1+i\tau)} + \frac{2r^2(q_2+s)^2}{q_3(q_1+i\tau)^3} - \frac{v^2(q_2+s)^2}{q_3^2}\right]$$

$$-\frac{1}{c^2}\partial_t^2 f = -\frac{f_0}{c^2}\partial_t^2 \frac{e^{-s/q_3}}{(q_1+i\tau)(s+q_2)^{\alpha}}$$
$$== -\frac{f_0 e^{-s/q_3}}{(q_1+i\tau)(q_2+s)^{\alpha+2}}\left[-(1+\alpha)\alpha u^2 + \frac{2\alpha u(q_2+s)}{q_1+i\tau} + \frac{2\alpha r^2(q_2+s)}{(q_1+i\tau)^3} - \frac{2(q_2+s)^2}{(q_1+i\tau)^2}\right. \quad \text{(S10)}$$
$$\left. - \frac{2\alpha u^2(q_2+s)}{q_3} + \frac{2u(q_2+s)^2}{q_3(q_1+i\tau)} + \frac{2r^2(q_2+s)^2}{q_3(q_1+i\tau)^3} - \frac{u^2(q_2+s)^2}{q_3^2}\right]$$

Then the longitudinal component of magnetic field can be derived as the sum of equations (S9) & (S10):

$$H_z = \left(\partial_z^2 - \frac{1}{c^2}\partial_t^2\right) f$$
$$= \frac{f_0 e^{-s/q_3}\left[(1+\alpha)\alpha\left(u^2-v^2\right) + \frac{2\alpha(v-u)(q_2+s)}{q_1+i\tau} - \frac{2\alpha\left(v^2-u^2\right)(q_2+s)}{q_3} + \frac{2(v-u)(q_2+s)^2}{q_3(q_1+i\tau)} - \frac{\left(v^2-u^2\right)(q_2+s)^2}{q_3^2}\right]}{(q_1+i\tau)(q_2+s)^{\alpha+2}} \quad \text{(S11)}$$

Where $u = r^2/(q_1+i\tau)^2 - 1$ and $v = r^2/(q_1+i\tau)^2 + 1$, thus $v-u=2$ and $v^2-u^2 = (v-u)(v+u) = 4r^2/(q_1+i\tau)^2$, then the equation (S11) can be further simplified as:

$$H_z = \frac{f_0 e^{-s/q_3}\left[-\frac{(1+\alpha)\alpha 4r^2}{(q_1+i\tau)^2} + \frac{4\alpha(q_2+s)}{q_1+i\tau} - \frac{8\alpha r^2(q_2+s)}{q_3(q_1+i\tau)^2} + \frac{4(q_2+s)^2}{q_3(q_1+i\tau)} - \frac{4r^2(q_2+s)^2}{q_3^2(q_1+i\tau)^2}\right]}{(q_1+i\tau)(q_2+s)^{\alpha+2}} \quad \text{(S12)}$$
$$= -\frac{4 f_0 e^{-s/q_3}}{(q_1+i\tau)^3(q_2+s)^{\alpha+2}}\left[(1+\alpha)\alpha r^2 - \alpha(q_1+i\tau)(q_2+s) + \frac{2\alpha r^2(q_2+s) - (q_1+i\tau)(q_2+s)^2}{q_3} + \frac{r^2(q_2+s)^2}{q_3^2}\right]$$

When $q_3 \to \infty$ and $\alpha = 1$, the equation (S12) will be reduced into:

$$H_z\big|_{\alpha=1, q_3\to\infty} = -4 f_0 \frac{2r^2 - (q_1+i\tau)(q_2+s)}{(q_1+i\tau)^3(q_2+s)^3} \quad \text{(S13)}$$

Substitute $s \equiv r^2/(q_1+i\tau) - i\sigma$, $\tau = z-ct$, $\sigma = z+ct$ into equation (S13), we get:

$$
\begin{aligned}
H_z\big|_{\alpha=1,q_3\to\infty} &= -4f_0\frac{2r^2-(q_1+i\tau)\left(q_2+\dfrac{r^2}{q_1+i\tau}-i\sigma\right)}{(q_1+i\tau)^3\left(q_2+\dfrac{r^2}{q_1+i\tau}-i\sigma\right)^3} \\
&= -4f_0\frac{2r^2-\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} = -4f_0\frac{r^2-(q_1+i\tau)(q_2-i\sigma)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3}
\end{aligned}
\tag{S14}
$$

This equation (S14) is exactly the closed-form expression of the longitudinal magnetic field of the fundamental flying doughnut pulse [2].

Electromagnetic field: In summary, combining equations (S3), (S6), and (S12), the electromagnetic field of the general transversely divergent pulse related to q_1, q_2, q_3, and α can be given by.

$$
\begin{cases}
E_\theta = \dfrac{2i\mu_0 f_0 cre^{-s/q_3}}{(q_1+i\tau)^2(q_2+s)^\alpha}\left[\dfrac{(\alpha+1)\alpha u(q_1+i\tau)-2\alpha(q_2+s)}{(q_1+i\tau)(q_2+s)^2}+\dfrac{2\alpha u(q_1+i\tau)-2(q_2+s)}{q_3(q_1+i\tau)(q_2+s)}-\dfrac{u}{q_3^2}\right] \\
H_r = \dfrac{2if_0 re^{-s/q_3}}{(q_1+i\tau)^2(q_2+s)^\alpha}\left[\dfrac{2\alpha(q_2+s)-(\alpha+1)\alpha v(q_1+i\tau)}{(q_1+i\tau)(q_2+s)^2}+\dfrac{2(q_2+s)-2\alpha v(q_1+i\tau)}{q_3(q_1+i\tau)(q_2+s)}-\dfrac{v}{q_3^2}\right] \\
H_z = -\dfrac{4f_0 e^{-s/q_3}}{(q_1+i\tau)^3(q_2+s)^{\alpha+2}}\left[(1+\alpha)\alpha r^2-\alpha(q_1+i\tau)(q_2+s)+\dfrac{2\alpha r^2(q_2+s)-(q_1+i\tau)(q_2+s)^2}{q_3}+\dfrac{r^2(q_2+s)^2}{q_3^2}\right]
\end{cases}
\tag{S15}
$$

When $q_3\to\infty$ and $\alpha=1$, the electromagnetic field of general transversely divergent pulse will be reduced into that of the fundamental toroidal pulse [2]:

$$
\begin{cases}
E_\theta = -4if_0\mu_0 c\dfrac{r(q_1+q_2-2ict)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} \\
H_r = 4if_0\dfrac{r(q_2-q_1-2iz)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3} \\
H_z = -4f_0\dfrac{r^2-(q_1+i\tau)(q_2-i\sigma)}{\left[r^2+(q_1+i\tau)(q_2-i\sigma)\right]^3}
\end{cases}
\tag{S16}
$$

When $q_3\to q_1$ and $\alpha=1$, the electromagnetic field of general transversely divergent pulse will be reduced into that of the nondiffracting supertoroidal pulse, see details in Note 2.

Note 2: Theoretical proving of nondiffraction.

Hereinafter, we prove that the supertoroidal pulse as Eq. (S15) when $q_3 = q_1$ and $\alpha = 1$ fulfills the classic definition of nondiffraction:

$$\left|E(r,\theta,z,t)\right| = \left|E(r,\theta,z+\Delta z,t+\Delta z/c)\right| \tag{S17}$$

Where Δz is a given propagation distance. To prove this, we only need to prove the generating function, Eq. (S1), fulfilling the nondiffraction relationship:

$$\left|f(r,\theta,z,t)\right| = \left|f(r,\theta,z+\Delta z,t+\Delta z/c)\right| \tag{S18}$$

when $q_3 = q_1$ and $\alpha = 1$, substituting the relations of $s \equiv r^2/(q_1+i\tau) - i\sigma$, $\tau = z - ct$, $\sigma = z + ct$, and $q_2 \gg q_1$, the generating function can be written as:

$$\begin{aligned} f &= f_0 \frac{e^{-s/q_1}}{(q_1+i\tau)(s+q_2)} = f_0 \frac{e^{-\frac{r^2}{q_1(q_1+i\tau)}+\frac{i\sigma}{q_1}}}{(q_1+i\tau)\left(\frac{r^2}{q_1+i\tau}-i\sigma+q_2\right)} = f_0 \frac{e^{-\frac{r^2(q_1-i\tau)}{q_1(q_1^2+\tau^2)}+\frac{i\sigma}{q_1}}}{(q_2-i\sigma)\left[\frac{r^2}{q_2-i\sigma}+(q_1+i\tau)\right]} \\ &= f_0 \frac{e^{-\frac{r^2}{q_1^2+\tau^2}+i\frac{\tau r^2+\sigma(q_1^2+\tau^2)}{q_1(q_1^2+\tau^2)}}}{(q_2-i\sigma)\left[q_1+\frac{r^2(q_1+i\sigma)}{q_2^2+\sigma^2}+i\tau\right]} = f_0 \frac{e^{-\frac{r^2}{q_1^2+\tau^2}} e^{i\frac{\tau r^2+\sigma(q_1^2+\tau^2)}{q_1(q_1^2+\tau^2)}}}{(q_2-i\sigma)\left[q_1+\frac{q_1 r^2}{q_2^2+\sigma^2}+i\left(\frac{\sigma r^2}{q_2^2+\sigma^2}+\tau\right)\right]} \end{aligned} \tag{S19}$$

Because the assumptions of $q_2 \gg q_1$ and the radial distribution of the wave function is effectively confined in a finite region, the term $\frac{q_1 r^2}{q_2^2+\sigma^2} = \frac{q_1}{q_2}\frac{r^2}{q_2+\sigma^2/q_2} \doteq 0$ can be neglected. Because $\tau = z - ct$, $\sigma = z + ct$, and the pulse has an finite duration, which is much smaller than q_2/c (pulse width is much smaller than q_2). Therefore, the intensity of the generating function fulfills:

$$\left|f(r,\theta,z,t)\right|^2 = f_0^2 \frac{e^{-\frac{2r^2}{q_1^2+\tau^2}}}{\left(q_2^2+\sigma^2\right)\left[q_1^2+\left(\frac{\sigma r^2}{q_2^2+\sigma^2}+\tau\right)^2\right]} \tag{S20}$$

When $z = ct$, i.e. $\tau = 0$, the exponential term $e^{-2r^2/(q_1^2+\tau^2)}$ rapidly tend to zero for $r > q_1$, thus the radially dependent term in $\left(\frac{\sigma r^2}{q_2^2+\sigma^2}+\tau\right)^2$ can be neglected. In addition, the function $\frac{1}{q_2^2+\sigma^2}$ is a slowly variant function when $|\sigma| < q_2$, and the value of q_2 is much larger than the pulse duration. Therefore, this term works as a constant, and Eq. (S20) can be derived as:

$$\left|f(r,\theta,z,t)\right| \approx f_0 \frac{e^{-\frac{r^2}{q_1^2+\tau^2}}}{\sqrt{q_1^2+\tau^2}} = f_0 \frac{e^{-\frac{r^2}{q_1^2+(z-ct)^2}}}{\sqrt{q_1^2+(z-ct)^2}} \tag{S21}$$

Therefore, the nondiffraction relation, Eq. (S18), can be verified by:

$$\left|f(r,\theta,z+\Delta z,t+\Delta z/c)\right| = f_0 \frac{e^{-\frac{r^2}{q_1^2+\left[z+\Delta z-c(t+\Delta z/c)\right]^2}}}{\sqrt{q_1^2+\left[z+\Delta z-c(t+\Delta z/c)\right]^2}} = f_0 \frac{e^{-\frac{r^2}{q_1^2+(z-ct)^2}}}{\sqrt{q_1^2+(z-ct)^2}} = \left|f(r,\theta,z,t)\right| \qquad \text{(S22)}$$

The generating function is a nondiffracting signal and the supertoroidal pulse is nondiffracting. Figure S1 shows the simulated distributions of the real oscillation and amplitude of the supertoroidal pulse of $q_3 = q_1$, $q_2 = 100q_1$, and $\alpha = 1$, and the amplitude distribution given by Eq. (S21) at three different times upon propagation, where they shows great agreement.

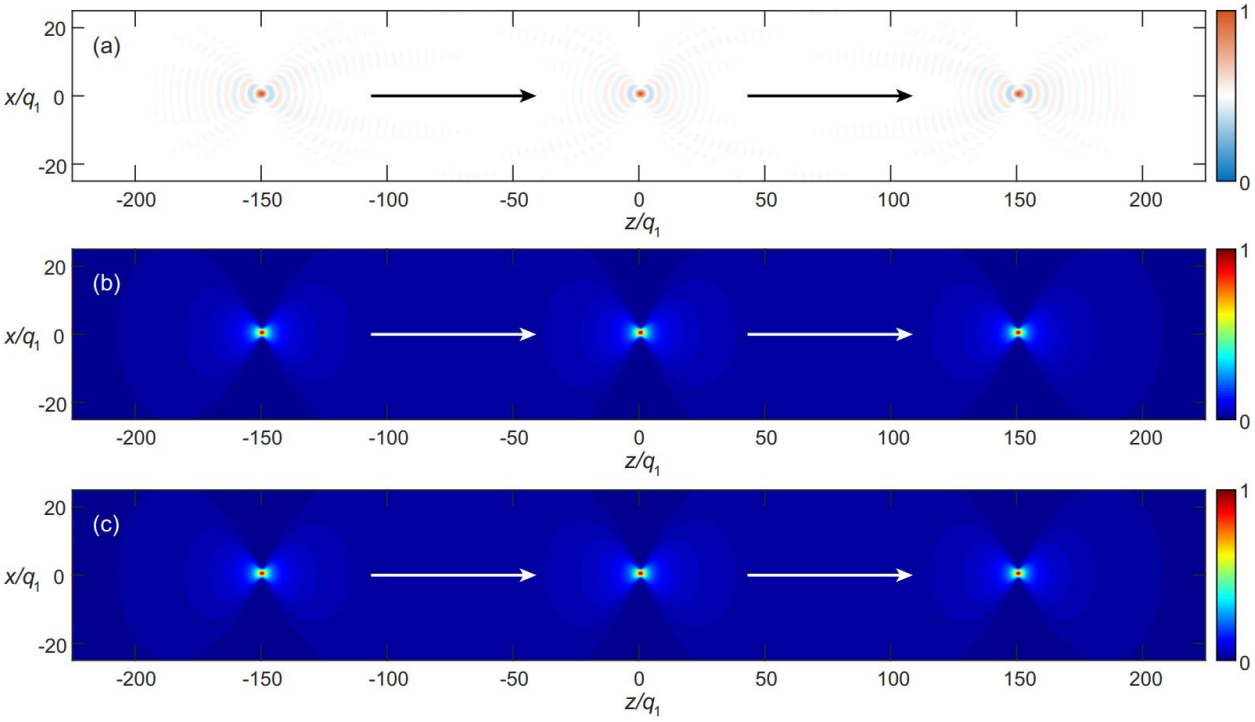


Figure S1 (a) The simulated real oscillation signal and (b) amplitude distribution of the supertoroidal pulse of $q_3 = q_1$, $q_2 = 100q_1$, and $\alpha = 1$, given by Eq. (S15), and (c) the amplitude distribution given by Eq. (S21) at three different times $t = -150\,q_1/c$, $t = 0$, and $t = 150\,q_1/c$ upon propagation.

Note 3: Topological numbers of skyrmions.

Topological properties of a skyrmionic configuration can be characterized by the skyrmion number [4]:

$$s = \frac{1}{4\pi}\iint_{\sigma} \mathbf{n}\cdot\left(\frac{\partial \mathbf{n}}{\partial x}\times\frac{\partial \mathbf{n}}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \frac{1}{4\pi}\int_0^{r_\sigma}\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{d}\beta(r)}{\mathrm{d}r}\frac{\mathrm{d}\alpha(\theta)}{\mathrm{d}\theta}\sin\beta(r)$$
$$= \frac{1}{4\pi}[\cos\beta(r)]_{r=0}^{r=r_\sigma}[\alpha(\theta)]_{\theta=0}^{\theta=2\pi} = p\cdot m \qquad \text{(S24)}$$

where $\mathbf{n}(x, y)$ represents the vector field to construct a skyrmion and σ the region to confine the skyrmion, which can be infinity (for an isolated skyrmion) also can be a cell of a periodic distribution (for skyrmion lattices). The skyrmion number is an integer counting how many times the vector $\mathbf{n}(x, y) = \mathbf{n}(r\cos\theta, r\sin\theta)$ wraps around the unit sphere, as the mapping shown in Fig. S3(a). For mapping to the unit sphere, the vector can be given by $\mathbf{n} = (\cos\alpha(\theta)\sin\beta(r), \sin\alpha(\theta)\sin\beta(r), \cos\beta(r))$. The skyrmion number can be separated into two integers: the polarity, $p = \frac{1}{2}[\cos\beta(r)]_{r=0}^{r=r_\sigma}$, means that the vector direction is down (up) at center $r = 0$ and up (down) at boundary $r \to r_\sigma$ for $p = 1$ ($p = -1$), and the vorticity, $m = \frac{1}{2\pi}[\alpha(\theta)]_{\theta=0}^{\theta=2\pi}$, controls distribution of the transverse field components. In the case of a helical distribution, an initial phase γ should be added, $\alpha(\theta) = m\theta + \gamma$. Figures S3(b-g) show the theoretical results of selective topological structures of skyrmions with various values of polarity, vorticity, and helicity, (p, m, γ).

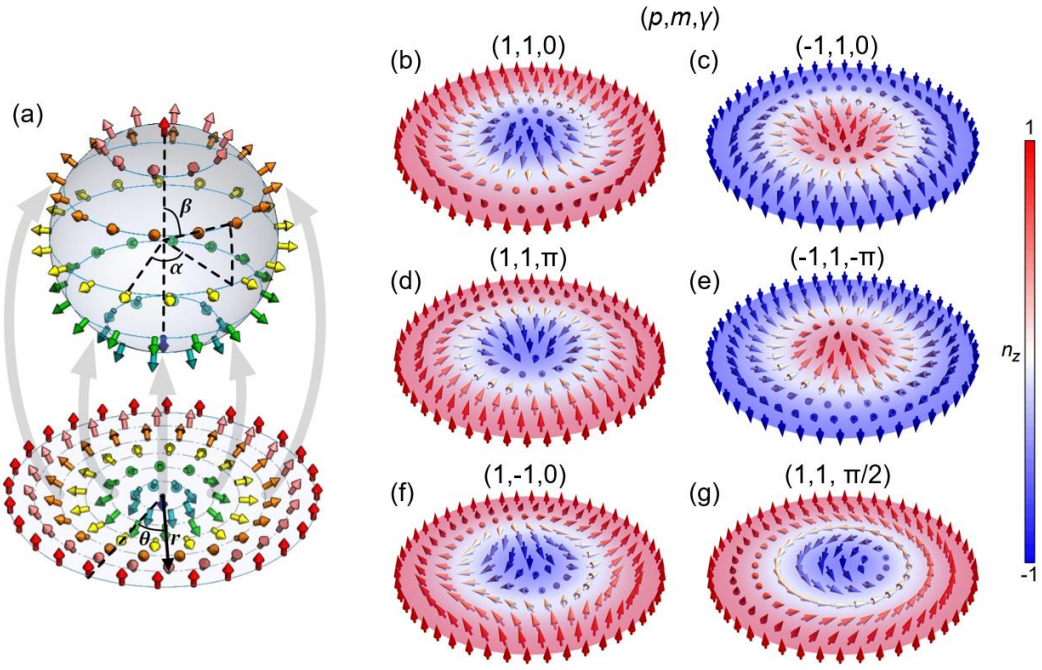


Fig. S3 (a) Representations of unit sphere wrapping for a skyrmion. (b-g) theoretical results of selective topological structures of skyrmions with various values of (p, m, γ) = (1,1,0), (-1,1,0), (1,1,π), (-1,1,-π), (1,-1,0), and (1,1, π/2), respectively.

Note 4: Plane wave expansion.

The spectrum of a supertoroidal light pulse, given by $E(r, z, t)$, can be directly obtained by Fourier transform from the space-time (r, z, t) to the wavevector-frequency domain (k_r, k_z, ω) [5]:

$$\tilde{E}(k_r, k_z, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(r, z, t)\exp[-i(k_r r + k_z z + \omega t)]\, dr dz dt \tag{S25}$$

Limited by the solution of Maxwell's equations, the entire spectrum of a space-time light pulse must be distributed on the surface of light cone, i.e the conic surface with unit slope of its generatrix in the coordinate $(k_r, k_z, \omega/c)$. The numerically evaluated results for selected toroidal, supertoroidal, and nondiffracting supertoroidal pulses with q_2=100q_1 are shown in Fig. S4. In all cases, we observe that the energy of the forward propagating components is over 99% of the total pulse energy. Especially, when the value of q_3 is decreased, the spectrum will approach a thin line along a conic section. If we further increase the parameter of q_2, the spectral line will become further thinner approaching the ideal infinitely thin line, corresponding to the case of exact nondiffraction.

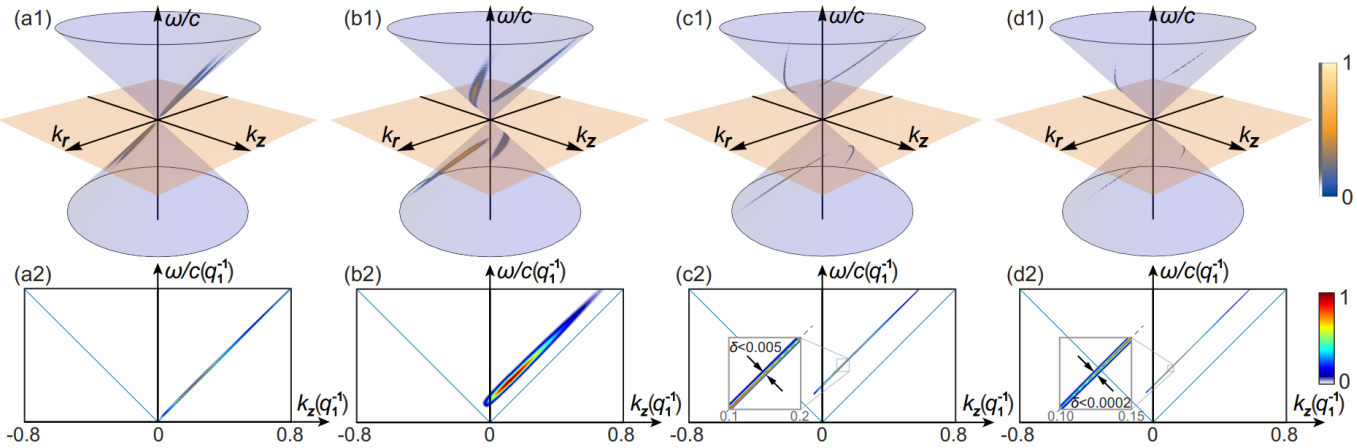


Fig. S4 The power spectrum distributions $\left|\tilde{E}(k_r, k_z, \omega)\right|^2$ on light cone via plane wave decomposition of toroidal (a1,a2), supertoroidal (b1,b2), and nondiffracting supertoroidal (c1,c2) pulses in the $(k_r, k_z, \omega/c)$ domain (a1-c1), and their projections onto the k_z–ω/c plane (a2-c2). The blue surface in the top row of graphs corresponds to the light cone. The following pulse parameters were used: (a1,a2) q_2=100q_1, q_3=∞, α=1; (b1,b2) q_2=100q_1, q_3=∞, α=50; (c1,c2) q_2=100q_1, q_3=q_1, α=1; (d1,d2) q_2=500q_1, q_3=q_1, α=1;

The plane wave expansion of NDSTPs allows us to study quantitatively the effects of finite apertures in their propagation properties. To this end, we simulated the propagation of a nondiffracting supertoroidal pulse (q_1=1μm, q_2=200μm, q_3=1μm, α=1) using plane wave expansion methods [6], the pulse propagation can be simulated by superimposing all the plane wave components and following truncation at focus by a finite circular aperture (see Fig. S5). We consider apertures with diameter of 0.4 mm, 0.3 mm, and 0.2 mm and calculate the electric field intensity integrated over the duration of the pulse as a function of propagation distance (see Figs. S5(a-c)). We quantify the effect of the aperture by calculating the propagation distance, d, at which the pulse FWHM increases by 1%. As the aperture size decreases from 0.4 mm to 0.3 mm and 0.2 mm, propagation distance d decreases from ~100 mm to ~70 mm and ~40 mm, respectively. For the results with aperture diameter of 0.4 mm, the FWHM changing is always in 1% difference in the 100 mm propagation. The records are at the same level to the previously endorsed experimental results of nondiffracting space-time wave packets of light, where FWHM size preserved over 100 mm propagation [7]. Moreover, in contrast to prior results of nondiffracting degree defined by "how thin is the line" (the rate of thickness of spectral line and the spectral bandwidth) [7], $\delta\lambda/\Delta\lambda \approx$ (0.3pm/0.8nm) ≈ 0.000375, a case of our result of NDSTP pulse in Fig.S4c (q_2=100q_1, q_3=q_1, α=1) give the rate of $\delta\lambda/\Delta\lambda \approx (0.005q_1)/(10q_1) \approx 0.0005$, which is at the same level to the prior result. And the case Fig.S4d (q_2=500q_1, q_3=q_1, α=1) improves it as $\delta\lambda/\Delta\lambda \approx (0.0002q_1)/(10q_1)$ ≈ 0.00002, better than the prior nondiffracting wave result. The results presented here illustrate practicality of experimental generation of ND-STPs.

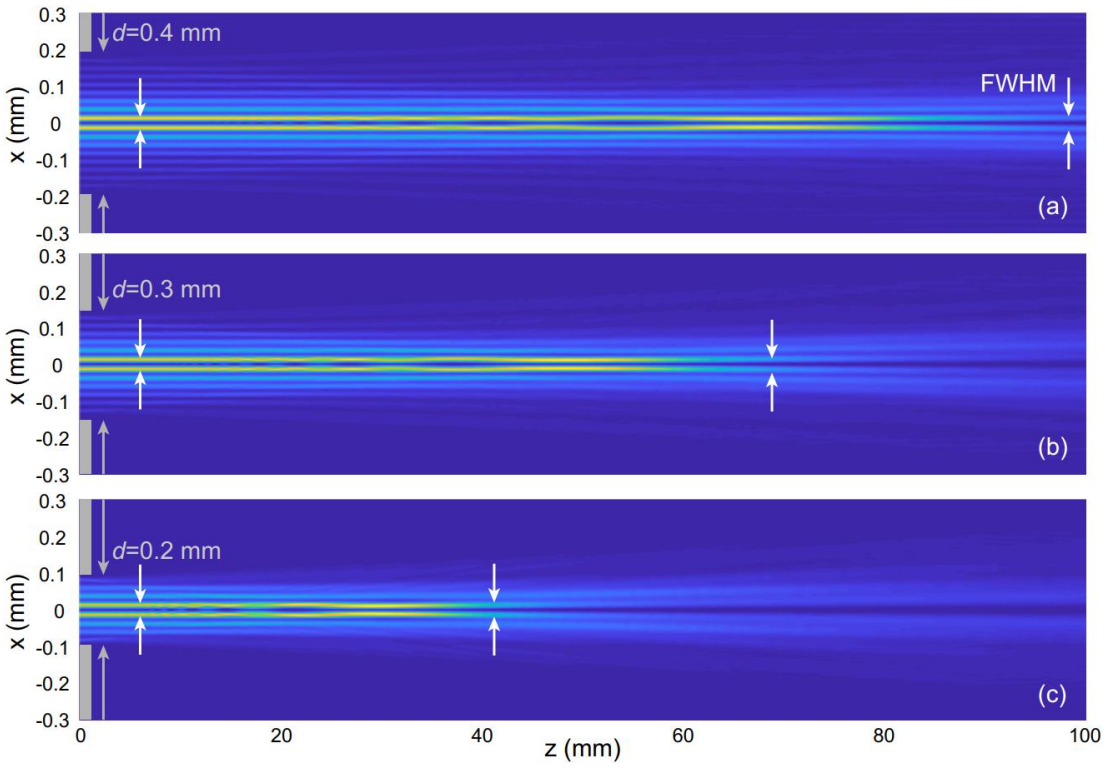


Fig. S5 The simulated results of the evolution of integrated intensity distribution upon propagation, $\int |E(r,z,t)|^2 dt$, of the nondiffracting supertoroidal pulse (q_1=1μm, q_2=200μm, q_3=1μm, α=1) over a distance of 100 mm (1000 times of theoretical Rayleigh range) under the truncation of practical aperture with diameter of (a) 0.4 mm, (b) 0.3 mm, (c) 0.2 mm, respectively.

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