除(29)和(30)外, 另一个具有普适性的经典极限定理是Donsker (弱)不变原理. 假设ξ_n, $n \geqslant 1$是一列 i.i.d. 随机变量, $\mathsf{E}\xi_1 = \mu$, $\mathsf{Var}\,(\xi_1) = \sigma^2 > 0$. 记$S_n = \sum\limits_{k=1}^{n} \xi_k$, 定义

$$X_n(t) = \begin{cases} \dfrac{1}{\sqrt{n}\,\sigma} S_k, & t = \dfrac{k}{n},\ 1 \leqslant k \leqslant n; \\ \text{线性插值}, & \dfrac{k-1}{n} < t < \dfrac{k}{n},\ 1 \leqslant k \leqslant n, \end{cases} \tag{33}$$

那么

$$X_n \Rightarrow \boldsymbol{B}, \qquad n \to \infty, \tag{34}$$

其中$\boldsymbol{B} = (B(t), 0 \leqslant t \leqslant 1)$为标准Brown运动.

Donsker不变原理被看作是20世纪60年代概率论学科所取得的代表性成果. 实际上, 在此之前, Chung、Erdös、Kac、Kolmogorov、Le Cam、Prohorov等人做了大量基础性的工作. (34)的证明通常由两部分组成: (i) X_n弱相对紧, 等价地, X_n一致胎紧; (ii) X_n的所有弱极限都相同, 并都等于$\boldsymbol{B}$, 等价地, X_n的任意有限维分布收敛到$\boldsymbol{B}$的有限维分布. (34)不仅证明了Brown运动$\boldsymbol{B}$的存在性, 而且有着广泛的应用. 假设$h: C[0,1] \mapsto \mathbb{R}^k$是连续映射(可以推广到不连续情形), 那么

$$h(X_n) \Rightarrow h(\boldsymbol{B}), \qquad n \to \infty. \tag{35}$$

一般来说, (35)的应用如下: (i) 通过选择某具体可解的随机变量ξ_n, 计算$h(X_n)$的极限分布, 从而得到$h(\boldsymbol{B})$的分布; (ii) 一旦知道$h(\boldsymbol{B})$的分布, 便可以得到由任意随机变量ξ_n所生成的$h(X_n)$的极限分布.

以上这些简单朴素的原理可以指导者许多概率问题的学习和研究, 特别适用于本节所介绍各个问题: 微观模型到宏观模型, 离散模型到连续模型.

附录: Tracy-Widom分布表

$$F_2(x) = \mathrm{e}^{-\int_x^{\infty}(u-x)q^2(u)\mathrm{d}u}$$

x	9	8	7	6	5	4	3	2	1	0
-3.8	0.005479	0.005691	0.005910	0.006137	0.006370	0.006612	0.006861	0.007119	0.007384	0.007659
-3.7	0.007941	0.008233	0.008534	0.008844	0.009164	0.009494	0.009833	0.010183	0.010544	0.010915
-3.6	0.011297	0.011691	0.012096	0.012513	0.012941	0.013383	0.013836	0.014303	0.014782	0.015275
-3.5	0.015782	0.016302	0.016837	0.017386	0.017950	0.018529	0.019123	0.019733	0.020358	0.021000
-3.4	0.021659	0.022334	0.023026	0.023736	0.024463	0.025209	0.025972	0.026755	0.027556	0.028376
-3.3	0.029216	0.030076	0.030956	0.031856	0.032777	0.033719	0.034683	0.035668	0.036675	0.037704
-3.2	0.038756	0.039831	0.040929	0.042050	0.043195	0.044364	0.045558	0.046776	0.048019	0.049287
-3.1	0.050581	0.051901	0.053247	0.054619	0.056017	0.057443	0.058895	0.060375	0.061883	0.063419
-3.0	0.064982	0.066574	0.068195	0.069845	0.071524	0.073232	0.074969	0.076736	0.078534	0.080361
-2.9	0.082219	0.084107	0.086026	0.087976	0.089957	0.091969	0.094013	0.096088	0.098194	0.100333
-2.8	0.102503	0.104706	0.106940	0.109207	0.111506	0.113838	0.116202	0.118599	0.121028	0.123490
-2.7	0.125984	0.128511	0.131071	0.133664	0.136290	0.138948	0.141639	0.144363	0.147120	0.149909
-2.6	0.152730	0.155585	0.158471	0.161390	0.164342	0.167325	0.170341	0.173389	0.176468	0.179579

x	9	8	7	6	5	4	3	2	1	0
-2.5	0.182721	0.185895	0.189100	0.192336	0.195603	0.198901	0.202229	0.205587	0.208975	0.212392
-2.4	0.215839	0.219316	0.222821	0.226355	0.229917	0.233507	0.237125	0.240771	0.244443	0.248142
-2.3	0.251868	0.255620	0.259397	0.263200	0.267028	0.270880	0.274756	0.278657	0.282580	0.286527
-2.2	0.290496	0.294487	0.298500	0.302534	0.306589	0.310664	0.314759	0.318873	0.323006	0.327157
-2.1	0.331327	0.335513	0.339717	0.343937	0.348173	0.352424	0.356691	0.360971	0.365265	0.369573
-2.0	0.373893	0.378225	0.382569	0.386924	0.391290	0.395665	0.400050	0.404444	0.408846	0.413256
-1.9	0.417673	0.422097	0.426527	0.430962	0.435402	0.439847	0.444295	0.448747	0.453201	0.457657
-1.8	0.462115	0.466574	0.471033	0.475492	0.479950	0.484407	0.488862	0.493315	0.497765	0.502211
-1.7	0.506654	0.511091	0.515524	0.519951	0.524372	0.528785	0.533192	0.537591	0.541982	0.546363
-1.6	0.550736	0.555099	0.559451	0.563792	0.568123	0.572441	0.576747	0.581040	0.585320	0.589587
-1.5	0.593839	0.598077	0.602300	0.606507	0.610698	0.614873	0.619031	0.623172	0.627296	0.631401
-1.4	0.635489	0.639557	0.643607	0.647637	0.651647	0.655637	0.659606	0.663554	0.667482	0.671388
-1.3	0.675272	0.679133	0.682973	0.686789	0.690583	0.694353	0.698100	0.701823	0.705522	0.709196
-1.2	0.712846	0.716471	0.720072	0.723646	0.727196	0.730720	0.734218	0.737690	0.741136	0.744555
-1.1	0.747948	0.751315	0.754654	0.757967	0.761252	0.764511	0.767742	0.770946	0.774122	0.777271
-1.0	0.780392	0.783485	0.786550	0.789588	0.792597	0.795579	0.798533	0.801458	0.804356	0.807225
-0.9	0.810066	0.812880	0.815665	0.818422	0.821150	0.823851	0.826524	0.829168	0.831785	0.834374
-0.8	0.836935	0.839467	0.841973	0.844450	0.846900	0.849322	0.851716	0.854084	0.856423	0.858736
-0.7	0.861021	0.863280	0.865511	0.867716	0.869894	0.872045	0.874170	0.876268	0.878340	0.880387
-0.6	0.882407	0.884401	0.886370	0.888313	0.890231	0.892124	0.893992	0.895834	0.897652	0.899446
-0.5	0.901215	0.902960	0.904681	0.906378	0.908052	0.909702	0.911328	0.912932	0.914512	0.916070
-0.4	0.917605	0.919118	0.920609	0.922078	0.923524	0.924950	0.926354	0.927737	0.929098	0.930439
-0.3	0.931760	0.933060	0.934340	0.935599	0.936840	0.938060	0.939261	0.940443	0.941606	0.942751
-0.2	0.943876	0.944984	0.946073	0.947145	0.948198	0.949235	0.950253	0.951255	0.952240	0.953208
-0.1	0.954160	0.955095	0.956015	0.956918	0.957806	0.958678	0.959535	0.960377	0.961204	0.962016
-0.0	0.962814	0.963598	0.964367	0.965123	0.965865	0.966593	0.967308	0.968010	0.968699	0.969375
0.0	0.969375	0.970038	0.970689	0.971328	0.971955	0.972570	0.973173	0.973764	0.974345	0.974914
0.1	0.975472	0.976019	0.976556	0.977082	0.977598	0.978103	0.978599	0.979085	0.979561	0.980027
0.2	0.980485	0.980933	0.981371	0.981801	0.982223	0.982635	0.983039	0.983435	0.983823	0.984202
0.3	0.984574	0.984938	0.985294	0.985643	0.985984	0.986318	0.986645	0.986965	0.987278	0.987585
0.4	0.987885	0.988178	0.988465	0.988746	0.989020	0.989289	0.989551	0.989808	0.990059	0.990305
0.5	0.990545	0.990780	0.991009	0.991234	0.991453	0.991667	0.991877	0.992081	0.992281	0.992477
0.6	0.992668	0.992854	0.993036	0.993214	0.993388	0.993558	0.993724	0.993886	0.994044	0.994198
0.7	0.994349	0.994496	0.994640	0.994780	0.994917	0.995050	0.995181	0.995308	0.995432	0.995553
0.8	0.995671	0.995787	0.995899	0.996009	0.996116	0.996220	0.996322	0.996421	0.996518	0.996612
0.9	0.996704	0.996794	0.996881	0.996967	0.997050	0.997131	0.997210	0.997286	0.997361	0.997434
1.0	0.997506	0.997575	0.997642	0.997708	0.997772	0.997835	0.997896	0.997955	0.998012	0.998069
1.1	0.998123	0.998177	0.998228	0.998279	0.998328	0.998376	0.998422	0.998468	0.998512	0.998554
1.2	0.998596	0.998637	0.998676	0.998715	0.998752	0.998789	0.998824	0.998858	0.998892	0.998924
1.3	0.998956	0.998987	0.999017	0.999046	0.999074	0.999102	0.999128	0.999154	0.999180	0.999204
1.4	0.999228	0.999251	0.999274	0.999296	0.999317	0.999338	0.999358	0.999377	0.999396	0.999414
1.5	0.999432	0.999450	0.999467	0.999483	0.999499	0.999514	0.999529	0.999544	0.999558	0.999572
1.6	0.999585	0.999598	0.999610	0.999622	0.999634	0.999646	0.999657	0.999668	0.999678	0.999688
1.7	0.999698	0.999708	0.999717	0.999726	0.999735	0.999743	0.999751	0.999759	0.999767	0.999774
1.8	0.999782	0.999789	0.999795	0.999802	0.999808	0.999815	0.999821	0.999827	0.999832	0.999838
1.9	0.999843	0.999848	0.999853	0.999858	0.999863	0.999867	0.999871	0.999876	0.999880	0.999884
2.0	0.999888	0.999891	0.999895	0.999898	0.999902	0.999905	0.999908	0.999911	0.999914	0.999917
2.1	0.999920	0.999923	0.999925	0.999928	0.999930	0.999933	0.999935	0.999937	0.999939	0.999941
2.2	0.999943	0.999945	0.999947	0.999949	0.999951	0.999952	0.999954	0.999956	0.999957	0.999959
2.3	0.999960	0.999961	0.999963	0.999964	0.999965	0.999967	0.999968	0.999969	0.999970	0.999971
2.4	0.999972	0.999973	0.999974	0.999975	0.999976	0.999977	0.999977	0.999978	0.999979	0.999980