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Research Article

Keywords: Eccentricity matrix, Eccentricity spectral radius, Uniform hypertree, Diameter

Posted Date: February 7th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-2546099/v1

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Additional Declarations: No competing interests reported.
The $\varepsilon$-spectral radii of $k$-uniform hypertrees

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Abstract

The $\varepsilon$-spectral radius of a connected hypergraph is the largest eigenvalue of its eccentricity matrix. Let $T_{m,d}$ be the set of $k$-uniform hypertrees with size $m$ and diameter $d$. In this paper, we show that the eccentricity matrix of a $k$-uniform hypertree is irreducible, and we characterize the unique hypertrees with the minimum $\varepsilon$-spectral radius among $\bigcup_{d \neq 3} T_{m,d}$.

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AMS Classification: 05C50, 05C65

1 Introduction

A hypergraph $G$ consists of a vertex set $V(G)$ and an edge set $E(G)$, where each edge in $E(G)$ is a subset of $V(G)$, with cardinality greater than or equal to 2, see [2]. Obviously, hypergraphs are natural generalizations of undirected graphs in which ‘edges’ may consist of more than 2 vertices. For an integer $k \geq 2$, the hypergraph $G$ is $k$-uniform if all edges have the same cardinality $k$. A (simple) graph is a 2-uniform hypergraph.

For $u, v \in V(G)$, a walk from $u$ to $v$ in $G$ is defined as a sequence of vertices and edges intersect $(v_0, e_1, v_1, \ldots, e_{p-1}, e_p, v_p)$ with $v_0 = u$ and $v_p = v$ such that edge $e_i$ contains vertices

*Supported by Natural Science Foundation of Guangdong Province (No.2021A1515010028), Guangdong Basic and Applied Basic Research Foundation (No. 2021A1515012047) and Science and Technology Program of Guangzhou (No. 202002030401).
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$v_{i-1}$ and $v_i$ for $i = 1, 2, \ldots, p$, where the value $p$ is the length of this walk. If all $v_i$ and $e_i$ are distinct, then we say that this walk is a path, denoted by $P_{uv}$. If $p \geq 2$, and all $e_i$ and $v_i$ are distinct except $v_0 = v_p$, then we say this walk is a cycle.

If there is a path from $u$ to $v$ for any $u, v \in V(G)$, then we say that $G$ is connected. A hypertree is a connected hypergraph without cycles. A hyper-caterpillar is a hypertree consisting of a path and possibly pendant edges at vertices of this path. Note that a $k$-uniform hypertree with $n$ vertices always has $\frac{n}{k-1}$ edges. For a vertex $v \in V(G)$, the degree $d_G(v)$ of $v$ is the number of edges containing $v$ of $G$. A vertex of degree one of $G$ is called a pendant vertex of $G$. An edge $e = \{w_1, \ldots, w_k\} \in E(G)$ is called a pendant edge at $w_1$ if $d_G(w_1) \geq 2$, $d_G(w_i) = 1$ for $2 \leq i \leq k$.

Let $G$ be a connected hypergraph on $n$ vertices. For $u, v \in V(G)$, the distance between $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest path connecting them in $G$. The diameter $diam(G)$ of $G$ is defined as $diam(G) = \max\{d_G(u, v) | u, v \in V(G)\}$. The distance matrix of $G$ is defined as $D(G) = (d_G(u, v))_{u, v \in V(G)}$. The distance matrix is very useful in different fields which include the design of communication networks, graph embedding theory and molecular stability. The distance spectral properties of ordinary graphs (2-uniform hypergraphs) and hypergraphs have been studied extensively, see [1,7–9].

For $u \in V(G)$, the eccentricity $e_G(u)$ is given by $e_G(u) = \max\{d_G(u, v) | v \in V(G)\}$. The eccentricity matrix of the graph $G$, denoted by $\epsilon(G)$, is defined as [17]

$$\epsilon(G)_{uv} = \begin{cases} d_G(u, v), & \text{if } d_G(u, v) = \min\{e_G(u), e_G(v)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the eccentricity matrix $\epsilon(G)$ of $G$ is obtainable from the distance matrix $D(G)$ by retaining the eccentricities in each row and each column and setting the rest elements in the corresponding row and column to be zero. Therefore, the eccentricity matrix can be viewed as an extreme of distance-like matrix. Since the matrix $\epsilon(G)$ is a real symmetric matrix, all its eigenvalues, called $\epsilon$-eigenvalues of $G$, are real, which can be arranged as $\xi_1(G) \geq \xi_2(G) \geq \cdots \geq \xi_n(G)$. As usual, $\xi_1(G)$ is called the $\epsilon$-spectral radius of $G$, denoted also by $\rho_\epsilon(G)$. The $\epsilon$-polynomial of $G$ is defined as $\varphi_\epsilon(G, \lambda) = \det(\lambda I - \epsilon(G))$, where $I$ is the identity matrix.

Very recently, the $\epsilon$-eigenvalues and especially on $\epsilon$-spectral radius have been studied extensively. Wang et al. [17] showed that the eccentricity matrix of trees is irreducible and investigated the relations between the eigenvalues of the adjacency and eccentricity matrices. Wei, He and Li [20] determined the trees with given diameter or fixed order minimizing the $\epsilon$-spectral radius. Wang et al. [18] and Mahato et al. [11] respectively determined the lower and upper bounds for the $\epsilon$-spectral radius of graphs, and identified the corresponding extremal graphs. He and Lu [4] determined the maximum $\epsilon$-spectral radius of $n$-vertex trees with fixed odd diameter and characterized the corresponding extremal trees. Yang and Wang [21] characterized the lower bound of the $\epsilon$-spectral radius of a digraph and the corresponding extremal digraphs. The energy and the inertia of eccentricity matrices were also studied, one may be referred to [10–12, 14, 19].

For $k \geq 3$, denote by $T_{m,d}$ the set of $k$-uniform hypertrees with size $m$ and diameter $d$. This article is organized as follows. In Section 3, we show that the eccentricity matrix of
a $k$-uniform hypertree is irreducible. In section 4, we characterize the extremal hypertrees with the minimum $\epsilon$-spectral radius among $\bigcup_{d \neq 3} \mathcal{T}_{m,d}$.

2 Preliminaries

The following results will be used in the proofs of our main results.

**Lemma 2.1** ([13]) Let $M$ be an $s \times s$ symmetric matrix and $N$ be its $t \times t$ principal submatrix with $t < s$. Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ are the eigenvalues of $M$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_t$ are the eigenvalues of $N$, then $\lambda_i \geq \mu_i \geq \lambda_{s-t+i}$ for $1 \leq i \leq t$.

**Lemma 2.2** ([13]) Let $M$ and $N$ be two nonnegative irreducible matrices with same order. If $(N)_{ij} \leq (M)_{ij}$ for each $i,j$, then $\rho(N) \leq \rho(M)$ with equality if and only if $M = N$, where $\rho(N)$ and $\rho(M)$ denote the spectral radius of $N$ and $M$, respectively.

Let $J_{n \times n}$ or simply $J_n$ denote the $n \times n$ matrix with all entries equal to 1.

**Lemma 2.3** Let $T$ be a $k$-uniform hypertree with order $n$ and diameter $d \geq 2$. Then $\rho_e(T) \geq d(k-1)$ and $\xi_n(T) \leq -d(k-1)$.

**Proof.** If $k = 2$, then $T$ is a simple tree and the result holds, see [11,18].

Suppose that $k \geq 3$. Let $P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ be a diametrical path of $T$, where $e_i = \{w^1_i, \ldots, w^k_i\}$ with $w^1_i = v_{i-1}$ and $w^k_i = v_i$. Then $d_T(w^1_i, w^d_i) = d = e_T(w^1_i) = e_T(w^d_i)$ for any $w^1_i \in V(e_1) \setminus \{v_1\}$ and any $w^d_i \in V(e_d) \setminus \{v_{d-1}\}$. By the definition of eccentricity matrix, we have that $e(T)_{w^1_iw^d_i} = e(T)_{w^d_iw^1_i} = d$. Hence,

$$M = \begin{pmatrix} 0 & dJ_{k-1} \\ dJ_{k-1} & 0 \end{pmatrix}$$

is a $(2k-2) \times (2k-2)$ principal submatrix of $e(T)$. By direct calculation, we have that $\lambda_1(M) = d(k-1)$, $\lambda_2(M) = \ldots = \lambda_{2k-3}(M) = 0$ and $\lambda_{2k-2}(M) = -d(k-1)$. By Lemma 2.1, we have $\rho_e(T) \geq \lambda_1(M) = d(k-1)$ and $\xi_n(T) \leq \lambda_{2k-2} = -d(k-1)$. \hfill \Box

3 The eccentricity matrix of uniform hypertrees

Recall that an $n \times n$ matrix $M$ is said to be reducible if there exists a permutation matrix $P$ such that

$$M = P^T \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} P$$

where $A$ is a $r \times r$ submatrix with $1 \leq r < n$. If no such permutation matrix $P$ exists, then $M$ is said to be irreducible. Clearly, if $M$ is symmetric, then $M$ is irreducible means that it is not permutation similar to a block diagonal matrix.
Theorem 3.1 Let $T$ be a $k$-uniform hypertree of order $n$. Then $\epsilon(T)$ is irreducible.

Proof. Let $d_{uw} = d_T(u, v)$. It is trivial for $n = k$. Suppose that $n \geq k + 1$. Suppose to the contrary that $\epsilon(T)$ is reducible. Then there exists a partition $V(T) = V_1 \cup V_2$ with $|V_1| = r$ and $|V_2| = n - r$ such that $\epsilon(T)$ has the following form

$$\epsilon(T) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix},$$

where $X \in M_r, Y \in M_{n-r}$ and $1 \leq r \leq n$.

Let $P_{uv} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ be the diametrical path of $T$ with $v_0 = u$ and $v_d = v$. Note that the distance between $u$ and $v$ is the largest. Hence, the edges $e_1$ and $e_d$, which containing $u$ and $v$ respectively, are pendant edges. Without loss of generality, let $u, v \in V_1$. For any $w \in V(T)$, there exists a vertex $w' \in V(T)$ such that $d_{uw'} = e_T(w)$, and then $(\epsilon(T))_{uw'} > 0$. Thus there exists a pair of vertices $w, z \in V_2$ such that $(\epsilon(T))_{wz} = \max\{(\epsilon(T))_{w_1w_2} \mid w_1, w_2 \in V_2\} > 0$. Let $P_{wz} = (w, e'_1, w_1, \ldots, w_{l-1}, e'_l, z)$. Then $t \leq d$. Thus, corresponding to the rows and columns indexed by $u, v, w, z$,

$$M = \begin{pmatrix} u & v & w & z \\ u & 0 & d_{uw} & 0 & 0 \\ v & d_{uv} & 0 & 0 & 0 \\ w & 0 & 0 & 0 & d_{wz} \\ z & 0 & 0 & d_{wz} & 0 \end{pmatrix}$$

is a submatrix of $\epsilon(T)$.

By the definition of $\epsilon(T)$, we have

$$d_{uw} \geq \max\{d_{wu}, d_{uw}, d_{zu}, d_{zv}\}$$

and

$$d_{wz} \geq \max\{d_{wu}, d_{uw}, d_{zu}, d_{zv}\},$$

implying that at most one of $w$ and $z$ is an internal vertex of the path $P_{uw}$. Now we consider the following two cases, separately.

Case 1. Only one of $w$ and $z$, say $z$, is an internal vertex of the path $P_{uw}$.

If $V(P_{wz}) \cap V(P_{uw}) = \{z\}$, then $d_{wu} = d_{wz} + d_{zu} > d_{wz}$, a contradiction. If at least a edge of $P_{wz}$ lie, say on the path $P_{uw}$, then $d_{wu} = d_{wz} + d_{zu} > d_{wz}$ for $z \in \{v_{i-1}, v_i\}$ with $2 \leq i \leq d - 1$, or $d_{wu} = d_{wz} + (d_{zu} - 1) > d_{wz}$ otherwise, a contradiction again.

Case 2. Neither $w$ nor $z$ is an internal vertex of the path $P_{uw}$.

Suppose that $V(P_{wz}) \cap V(P_{uw}) \neq \emptyset$. If $V(P_{wz}) \cap V(P_{uw}) = \{x\}$, then $e_T(x) = \max\{d_{ux}, d_{ux}\}$. Without loss of generality, let $d_{ux} = e_T(x)$. Then $d_{ux} \geq \max\{d_{ux}, d_{ux}, d_{ux}\}$. Thus, we have $d_{wz} = d_{wx} + d_{xz} \leq d_{ux} + d_{xu} = d_{wu}$, which implies that $d_{wz} \leq d_{wu}$, a contradiction. Otherwise, there exists at least an edge $e \in V(P_{wz}) \cap V(P_{uw})$. Let $x \in e$. Then, by a similar discussion as earlier, we have $d_{wz} \leq d_{wu}$, a contradiction again.
Suppose now that \( V(P_{uw}) \cap V(P_{wz}) = \emptyset \). Then there exists a path \( P_{xy} \) connecting \( P_{uw} \) and \( P_{wz} \), where \( x \in V(P_{uw}) \) and \( y \in V(P_{wz}) \). Note that \( d_{uw} < d \). Otherwise, \( \max\{d_{uw}, d_{uz}, d_{uw}, d_{uz}\} > d \). Hence \( e_T(x) = \max\{d_{ux}, d_{ux}\} \geq \max\{d_{xx}, d_{ux}\} \). Without loss of generality, let \( e_T(x) = d_{ux} \). Then \( d_{wu} = d_{wy} + d_{yz} + d_{xz} \geq d_{wy} + d_{yx} + d_{xz} = d_{wy} + 2d_{yx} + d_{yz} = d_{uw} + 2d_{yx} > d_{uw} \), where the last inequality follows because \( d_{yx} \geq 1 \). Thus \( d_{uw} > d_{uw} \), a contradiction.

Combining the two cases, we see that the matrix \( \epsilon(T) \) is irreducible. \( \square \)

Let \( T \) be a \( k \)-uniform hypertree of order \( n \). From the above theorem, \( \epsilon(T) \) is irreducible. By Perron-Frobenius theorem, \( \rho(T) \) is simple and there is a unique unit positive eigenvector corresponding to \( \rho(T) \), which is called the eccentricity Perron vector of \( T \), denoted by \( x(T) \).

A column vector \( x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in R^n \) is considered as a function defined on \( V(T) \) which maps vertex \( v_i \) to \( x(v_i) \), i.e., \( x(v_i) = x_{v_i} \) for \( i = 1, 2, \ldots, n \).

## 4 Extremal uniform hypertrees with the minimum \( \epsilon \)-spectral radius

Let \( G = (V, E) \) be a \( k \)-uniform hypergraph with \( u, v \in V(G) \) and \( e_1, \ldots, e_r \in E(G) \) such that \( u \notin e_i \) and \( v \in e_i \) for \( 1 \leq i \leq r \). Let \( e'_i = (e_i \setminus \{v\}) \cup \{u\} \) for \( 1 \leq i \leq r \). Let \( G' = (V, E') \) be the hypergraph with \( E' = (E \setminus \{e_1, e_2, \ldots, e_r\}) \cup \{e'_1, e'_2, \ldots, e'_r\} \). Then we say that \( G' \) is obtained from \( G \) by moving edges \( e_1, \ldots, e_r \) from \( v \) to \( u \).

**Lemma 4.1** Let \( d \geq 4 \) and \( T \in \mathcal{T}_{m,d} \) with a diametrical path \( P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d) \). Let \( e_i = \{w^i_1, \ldots, w^i_k\} \) in which \( v_{i-1} = w^i_1 \) and \( v_i = w^i_k \) and \( T^i_j \) be the connected component of \( T - E(P_{d+1}) \) containing \( w^i_j \) for \( 2 \leq i \leq d - 1 \) and \( 1 \leq j \leq k \). Let \( e' = \{u'_1, \ldots, u'_k\} \) be a pendant edge of \( T^i_j \) with \( d_T(u'_1) \geq 2 \) and \( \tilde{T} \in \mathcal{T}_{m,d} \) be obtained from \( T \) by moving an edge \( e' \) from \( u'_1 \) to \( w^i_j \). If \( d_{T^i_j}(u'_1, w^i_j) \geq 1 \), then \( \rho(T) \leq \rho(T) \).

**Proof.** Let \( V_1 = \{u'_2, u'_3, \ldots, u'_k\} \), \( V_2 = V(T^i_j) \setminus V_1 \) and \( V_3 = V(G) \setminus V(T^i_j) \). Obviously, \( w^i_j \in V_2 \) and \( V(G) = V_1 \cup V_2 \cup V_3 \). As we pass from \( T \) to \( \tilde{T} \), the eccentricity of every vertex of \( V(G) \setminus V_1 \) is unchanged and the distance between two vertices in \( V(G) \setminus V_1 \) is unchanged too. Thus, for each \( \{w, w'\} \subseteq V_2 \cup V_3 \), we have \( \epsilon(T)_{ww} = \epsilon(\tilde{T})_{ww} \).

Let \( \{w, w'\} \subseteq V_1 \). Note that \( d_T(w, w') = 1 < 4 \leq e_T(w) = e_T(w') \) and \( d_{\tilde{T}}(w, w') = 1 < 3 \leq e_{\tilde{T}}(w) = e_{\tilde{T}}(w') \). By the definition of eccentricity matrix, we have \( \epsilon(T)_{ww'} = 0 = \epsilon(\tilde{T})_{ww'} \).

Suppose that \( w \in V_1 \) and \( w' \in V_2 \). Then \( e_{\tilde{T}}(w) < e_T(w) \) and \( e_{\tilde{T}}(w') < e_T(w') \). If \( d_T(w, w') < \min\{e_T(w), e_T(w')\} \), then \( \max\{d_{T}(w, w^i_j), d_{T}(w', w^i_j)\} > \left\lfloor \frac{d}{2} \right\rfloor \). Thus

\[
 d_{\tilde{T}}(w, w') \leq d_{\tilde{T}}(w, w^i_j) + d_{\tilde{T}}(w', w^i_j) = 1 + d_{\tilde{T}}(w', w^i_j) < 1 + \left\lfloor \frac{d}{2} \right\rfloor \leq e_{\tilde{T}}(w) \leq e_{\tilde{T}}(w'),
\]

which implies that \( \epsilon(T)_{ww'} = 0 = \epsilon(\tilde{T})_{ww'} \).
If \( d_T(w, w') = \min\{e_T(w), e_T(w')\} \), then \( \epsilon(T)_{ww'} \leq d_T(w, w') = d_T(w', w_j) + 1 = d_T(w', w_j) + 1 < d_T(w', w_j) + d_T(w, w_j) = d_T(w, w') = \epsilon(T)_{ww'} \).

Suppose now that \( w \in V_1 \) and \( w' \in V_\delta \). Let \( q = d_T(w_j, w'_j) \). Then \( q \geq 1 \). Obviously, \( d_T(w, w') = d_T(w, w') - q \) and \( e_T(w) = e_T(w) - q \). If \( d_T(w, w') = \min\{e_T(w), e_T(w')\} \), then
\[
\epsilon(T)_{ww'} = d_T(w, w') = d_T(w, w') - q < d_T(w, w') = \epsilon(T)_{ww'}.
\]

Suppose that \( d_T(w, w') < \min\{e_T(w), e_T(w')\} \), then
\[
d_T(w, w') = d_T(w, w') - q < d_T(w, w') < e_T(w') = e_T(w')
\]
and
\[
d_T(w, w') = d_T(w, w') - q < e_T(w) - q = e_T(w).
\]

Thus, \( d_T(w, w') < \min\{e_T(w), e_T(w')\} \) and \( \epsilon(T)_{ww'} = 0 = \epsilon(T)_{ww'} \).

Hence, \( \epsilon(T) < \epsilon(T) \). Combining with Lemma 2.2 and Theorem 3.1, we have \( \rho_c(T) < \rho_c(T) \). \( \square \)

4.1. The minimum \( \epsilon \)-spectral radius of uniform hypertrees with odd diameter

If \( d \geq 5 \) is odd, let \( T_{m,d} \) be the \( k \)-uniform hypertree obtained from a diametrical path \( P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d) \) by attaching \( a \) and \( c \) pendant edges to \( v_{d-1} \) and \( v_{d+1} \) respectively, and attaching \( b \) pendant edges to some vertices of \( V(e_{d+1}) \setminus \{v_{d-1}, v_{d+1}\} \), where \( a + b + c = m - d, a \geq c \geq 0 \) and \( b \geq 0 \).

**Lemma 4.2** Let \( T \) be a \( k \)-uniform hyper-caterpillar and \( P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d) \) be a diametrical path of \( T \), where \( d \geq 5 \) is odd. For \( 1 \leq i \leq d \), let \( e_i = \{w_i^1, \ldots, w_i^k\} \) with \( v_{i-1} = w_i^1 \) and \( v_i = w_i^k \). Suppose that there exists a pendant edge \( e \in E(T) \setminus E(P_{d+1}) \) containing \( w_i^j \) for some \( i \in \{2, \ldots, \frac{d-1}{2}\} \) and \( 1 \leq j \leq k - 1 \). Moving \( e \) from \( w_i^j \) to \( v_{d-1} \) yields a hypertree \( \tilde{T} \). Then \( \rho_c(\tilde{T}) < \rho_c(T) \).

**Proof.** Let \( e = \{u_1, u_2, \ldots, u_k\} \) with \( u_1 = w_i^j \). As we pass from \( T \) to \( \tilde{T} \), the eccentricity of every vertex of \( V(T) \setminus \{u_2, \ldots, u_k\} \) is unchanged and the distance between two vertices in \( V(T) \setminus \{u_2, \ldots, u_k\} \) is unchanged too. Thus, for each \( \{w, w'\} \subseteq V(T) \setminus \{u_2, \ldots, u_k\} \), we have \( \epsilon(T)_{ww'} = \epsilon(\tilde{T})_{ww'} \).

Let \( e^1, \ldots, e^t \) be the pendant edges containing \( v_{d-1} \) with \( e^1 = e_d \). Let \( w = u_t \) with \( 2 \leq t \leq k \). If \( w' \in (V(e^1) \cup \ldots \cup V(e^t)) \setminus \{v_{d-1}\} \), then \( d_T(w, w') = e_T(w) \leq e_T(w') \) and \( d_{\tilde{T}}(w, w') = e_{\tilde{T}}(w) < e_{\tilde{T}}(w') \). Recall that \( 2 \leq i \leq \frac{d-1}{2} \). Thus
\[
\epsilon(\tilde{T})_{ww'} = d_{\tilde{T}}(w, w') = d - \frac{d-1}{2} + 1 < d - (i - 1) + 1 = d_T(w, w') = \epsilon(T)_{ww'}.
\]

Otherwise, \( d_T(w, w') < \min\{e_T(w), e_T(w')\} \) and \( d_{\tilde{T}}(w, w') < \min\{e_{\tilde{T}}(w), e_{\tilde{T}}(w')\} \). By the definition of eccentricity matrix, we have \( \epsilon(T)_{ww'} = 0 = \epsilon(\tilde{T})_{ww'} \).
Therefore, $\rho_c(\tilde{T}) < \rho_c(T)$ by Lemma 2.2 and Theorem 3.1.

By making frequent use of Lemmas 4.1 and 4.2, we obtain the following result.

**Lemma 4.3** Among $\mathcal{T}_{m,d}$ with odd $d \geq 5$, the minimum $\epsilon$-spectral radius is achieved by some hypertree $T^{a,b,c}_{m,d}$, where $a + b + c = m - d$, $a \geq c \geq 0$ and $b \geq 0$.

**Theorem 4.1** Among $\mathcal{T}_{m,d}$ with odd $d \geq 5$, the minimum $\epsilon$-spectral radius is achieved uniquely by the hypertree $T^{[\frac{m-d}{2}],0,[\frac{m-d}{2}]}_{m,d}$.

**Proof.** Let $T$ be a hypertree with minimum $\epsilon$-spectral radius among $\mathcal{T}_{m,d}$ with odd $d \geq 5$. By Lemma 4.3, $T \cong T^{a,b,c}_{m,d}$ with $a + b + c = m - d$, $a \geq c \geq 0$ and $b \geq 0$. Let $P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ be a diametrical path of $T$, where $e_i = \{w_i^1, \ldots, w_i^k\}$ with $v_{i-1} = w_i^1$ and $v_i = w_i^k$ for $1 \leq i \leq d$. Let $U_1$ be the set of pendant edges containing $v_{d-1}^{d+1}$ with $|U_1| = a$, $U_2$ be the set of pendant edges containing $v_{d+1}^2$ with $|U_2| = c$ and $U_3 = E(T) \setminus (E(P_{d+1}) \cup U_1 \cup U_2)$.

Let $x$ be a Perron eigenvector corresponding to $\rho := \rho_c(T^{a,b,c}_{m,d})$, whose coordinate with respect to vertex $v$ is $x_v$. Since

$$\rho x_v^{\frac{d+1}{2}} = \rho x_v^{\frac{d+1}{2}} = \frac{d+1}{2} \sum_{u \in V(e_1) \setminus \{v_1\}} x_u + \frac{d+1}{2} \sum_{w \in V(e_d) \setminus \{v_{d-1}\}} x_w \quad \text{for } 3 \leq j \leq k-1,$$

$$\rho x_{v_{i-1}} = \rho x_{w_i^j} = (d-i) \sum_{u \in V(e_d) \setminus \{v_{d-1}\}} x_u \quad \text{for } 1 \leq i \leq \frac{d-1}{2} \text{ and } 2 \leq j \leq k-1,$$

$$\rho x_x = \frac{d+3}{2} \sum_{u \in V(e_d) \setminus \{v_{d-1}\}} x_u \quad \text{for any } x \in V(U_1) \setminus \{v_{d+1}^{d+1}\},$$

we have that

$$x_v^{\frac{d+1}{2}} = x_v^{\frac{d+1}{2}} \quad \text{for } 3 \leq j \leq k-1,$$

$$x_{v_{i-1}} = x_{w_i^j} \quad \text{for } 1 \leq i \leq \frac{d-1}{2} \text{ and } 2 \leq j \leq k-1,$$

$$x_x = x_{x'} \quad \text{for } \{x, x'\} \subseteq V(U_1) \setminus \{v_{d+1}^{d+1}\}.$$

Similarly, we get

$$x_z = x_{z'} \quad \text{for } \{z, z'\} \subseteq V(U_3) \setminus \{w_2^{d+1}, \ldots, w_{k-1}^{d+1}\},$$

$$x_{v_i} = x_{w_i^j} \quad \text{for } \frac{d+3}{2} \leq i \leq d \text{ and } 2 \leq j \leq k-1,$$

$$x_y = x_y' \quad \text{for } \{y, y'\} \subseteq V(U_2) \setminus \{v_{d+1}^{d+1}\}.$$

Thus

$$\rho x_{v_0} = \frac{d+1}{2} x_{v_{d+1}^d} + (k-1) \frac{d+3}{2} x_{v_{d+3}} + \ldots + (k-1) d x_{v_{d}} + (k-2) \frac{d+1}{2} x_{w_{d+1}^{d+1}}$$
\[
\begin{align*}
\rho x_{v_1} &= (k - 1)(d - 1)x_{v_d}, \\
\rho x_{v_{d-1}} &= (k - 1)\frac{d + 1}{2}x_{v_d}, \\
\rho x_{v_d} &= (k - 1)dx_{v_0} + \ldots + (k - 1)\frac{d + 3}{2}x_{v_{d-3}} + \frac{d + 1}{2}x_{v_{d-1}} + (k - 2)\frac{d + 1}{2}x_{v_{d+1}} \\
\rho x_{v_{w_2}} &= (k - 1)\frac{d + 1}{2}x_{v_0} + (k - 1)\frac{d + 1}{2}x_{v_d}, \\
\rho x_{x} &= (k - 1)\frac{d + 3}{2}x_{v_d}, \\
\rho x_{y} &= (k - 1)\frac{d + 3}{2}x_{v_0}, \\
\rho x_{z} &= (k - 1)\frac{d + 3}{2}x_{v_0} + (k - 1)\frac{d + 3}{2}x_{v_d}.
\end{align*}
\]

Let \( \alpha(d) := (d - 1)^2 + \ldots + \left(\frac{d+3}{2}\right)^2 + \left(\frac{d+1}{2}\right)^2 = \frac{d(d-1)(7d-5)}{24} \). Hence

\[
\begin{align*}
\rho^2 x_{v_0} &= (k - 1)\left(\frac{d + 1}{2}\right)^2x_{v_0} + (k - 1)^2\left(\frac{d + 3}{2}\right)^2x_{v_0} + \ldots + (k - 1)^2(d - 1)^2x_{v_0} + (k - 1)d\rho x_{v_d} \\
&\quad + (k - 2)(k - 1)\left(\frac{d + 1}{2}\right)^2(x_{v_0} + x_{v_d}) + (k - 1)^2\left(\frac{d + 3}{2}\right)^2x_{v_0} + (k - 1)^2b\left(\frac{d + 3}{2}\right)^2(x_{v_0} + x_{v_d}) \\
&\quad = (k - 1)^2\alpha(d)x_{v_0} + (k - 1)^2c\left(\frac{d + 3}{2}\right)^2x_{v_0} + (k - 1)^2b\left(\frac{d + 3}{2}\right)^2x_{v_0} \\
&\quad + (k - 1)d\rho x_{v_d} + (k - 1)^2b\left(\frac{d + 3}{2}\right)^2x_{v_d} + (k - 2)(k - 1)\left(\frac{d + 1}{2}\right)^2x_{v_d}
\end{align*}
\]

and

\[
\begin{align*}
\rho^2 x_{v_d} &= (k - 1)d\rho x_{v_0} + (k - 1)^2(d - 1)^2x_{v_d} + \ldots + (k - 1)^2\left(\frac{d + 3}{2}\right)^2x_{v_d} + (k - 1)^2\left(\frac{d + 1}{2}\right)^2x_{v_d} \\
&\quad + (k - 2)(k - 1)\left(\frac{d + 1}{2}\right)^2(x_{v_0} + x_{v_d}) + (k - 1)^2\alpha(d)\left(\frac{d + 3}{2}\right)^2x_{v_0} + (k - 1)^2b\left(\frac{d + 3}{2}\right)^2(x_{v_0} + x_{v_d}) \\
&\quad = (k - 1)^2\alpha(d)x_{v_d} + (k - 1)^2a\left(\frac{d + 3}{2}\right)^2x_{v_d} + (k - 1)^2b\left(\frac{d + 3}{2}\right)^2x_{v_d} \\
&\quad + (k - 1)d\rho x_{v_0} + (k - 1)^2b\left(\frac{d + 3}{2}\right)^2x_{v_0} + (k - 2)(k - 1)\left(\frac{d + 1}{2}\right)^2x_{v_0},
\end{align*}
\]

That is

\[
\left(\rho^2 - (k - 1)^2(\alpha(d) + ai\left(\frac{d + 3}{2}\right)^2 + bi\left(\frac{d + 3}{2}\right)^2)\right)x_{v_0} - (k - 1)\left(d\rho + (k - 1)b\left(\frac{d + 3}{2}\right)^2 + (k - 2)\left(\frac{d + 1}{2}\right)^2\right)x_{v_d} = 0
\]
Claim 2.

Suppose first that $m - d - b$ is odd. Then by Claim 1, we have that $a = c + 1$. Suppose that $b \geq 1$. Let $\rho = \rho_e(T^{a, b, c}_{m, d})$ and $\hat{\rho} = \rho_e(T^{a-1, b, c+1}_{m, d})$. Then

$$F_{a, b, c}(\rho) = F_{a, b, c}(\hat{\rho}) - F_{a-1, b, c+1}(\hat{\rho})$$

$$= \left( \rho^2 - (k - 1)^2(\alpha'(d) + (d + 3)^2) \right) \cdot \left( \rho^2 - (k - 1)^2(\alpha'(d) + a(d + 3)^2) \right)$$

$$= \left( \rho^2 - (k - 1)^2(\alpha'(d) + (d + 3)^2) \right) \cdot \left( \rho^2 - (k - 1)^2(\alpha'(d) + (a - 1)(d + 1)^2) \right)$$

$$= (k - 1)^4(d + 3)^4(-a + c + 1)$$

$$\leq (k - 1)^4(d + 3)^4(-2 + 1)$$

$$< 0.$$ 

Thus $\rho_e(T^{a-1, b, c+1}_{m, d}) = \hat{\rho} < \rho = \rho_e(T^{a, b, c}_{m, d})$, which contradicts to the choice of $T^{a, b, c}_{m, d}$.

Claim 2. $b = 0$

Suppose first that $m - d - b$ is odd. Then by Claim 1, we have that $a = c + 1$. Suppose that $b \geq 1$. Let $\rho = \rho_e(T^{c+1, b, c}_{m, d})$ and $\hat{\rho} = \rho_e(T^{c+1, b-1, c+1}_{m, d})$. Then

$$F_{c+1, b-1, c+1}(\lambda) = \left( \lambda^2 - (k - 1)^2(\alpha(d) + (c + 1)(d + 3)^2 + (b - 1)(d + 3)^2) \right)^2$$

$$- (k - 1)^2 \left( d\lambda + (k - 1)(b - 1)(d + 3)^2 + (k - 2)(d + 1)^2 \right)^2$$

$$= g(\lambda)h(\lambda),$$

where $g(\lambda) = \lambda^2 - (k - 1)d\lambda - (k - 1)^2(\alpha(d) + (c + 1)(d + 3)^2 + 2(b - 1)(d + 3)^2) - (k - 1)(k - 2)(d + 1)^2$ and $h(\lambda) = \lambda^2 + (k - 1)d\lambda - (k - 1)^2(\alpha(d) + (c + 1)(d + 3)^2) + (k - 1)(k - 2)(d + 1)^2$. 

and

$$\left( \rho^2 - (k - 1)^2(\alpha(d) + a(d + 3)^2) \right) \cdot \left( \rho^2 - (k - 1)^2(\alpha(d) + (d + 3)^2) \right) = 0.$$
Assume $\rho'$ is the largest root of $h(\lambda) = 0$. Then

$$\rho'^2 = -(k-1)d\rho' + (k-1)^2(\alpha(d) + (c+1)(\frac{d+3}{2})^2 - (k-1)(k-2)(\frac{d+1}{2})^2).$$

By a direct calculation, one has

$$g(\rho') = \rho'^2 - (k-1)d\rho' - (k-1)^2(\alpha(d) + (c+1)(\frac{d+3}{2})^2 + 2(b-1)(\frac{d+3}{2})^2 - (k-1)(k-2)(\frac{d+1}{2})^2)
\leq -2(k-1)d\rho' - 2(k-1)(k-2)(\frac{d+1}{2})^2 - 2(k-1)^2(b-1)(\frac{d+3}{2})^2 < 0,$$

which implies the largest root of $F_{c+1,b-1,c+1}(\lambda) = 0$ is the largest root of $g(\lambda) = 0$. Thus

$$\hat{\rho}^2 - (k-1)^2 \left(\alpha(d) + c(\frac{d+3}{2})^2 + b(\frac{d+3}{2})^2\right) = (k-1) \left(d\hat{\rho} + (k-1)(b-1)(\frac{d+3}{2})^2 + (k-2)(\frac{d+1}{2})^2\right).$$

Therefore,

$$F_{c+1,b,c}(\hat{\rho}) = \left(\hat{\rho}^2 - (k-1)^2(\alpha(d) + c(\frac{d+3}{2})^2 + b(\frac{d+3}{2})^2)\right)
\times \left(\hat{\rho}^2 - (k-1)^2(\alpha(d) + (c+1)(\frac{d+3}{2})^2 + b(\frac{d+3}{2})^2)\right)
- (k-1)^2 \left(d\hat{\rho} + (k-1)b(\frac{d+3}{2})^2 + (k-2)(\frac{d+1}{2})^2\right)^2
= (k-1)^2 \left(d\hat{\rho} + (k-1)(b-1)(\frac{d+3}{2})^2 + (k-2)(\frac{d+1}{2})^2\right)
\times \left(d\hat{\rho} + (k-1)(b-2)(\frac{d+3}{2})^2 + (k-2)(\frac{d+1}{2})^2\right)
\times \left(d\hat{\rho} + (k-1)\hat{\rho}(\frac{d+3}{2})^2 + (k-2)(\frac{d+1}{2})^2\right)^2
< 0. $$

Thus $\rho(T_{m,d}^{c+1,b-1,c+1}) = \hat{\rho} < \rho = \rho(T_{m,d}^{c+1,b,c})$, a contradiction. Therefore $b = 0$.

We assume that $m - d - b$ is even. Then by Claim 1, we have that $a = c$. Note that

$$F_{c,b,c}(\lambda) = \left(\lambda^2 - (k-1)^2(\alpha(d) + c(\frac{d+3}{2})^2 + b(\frac{d+3}{2})^2)\right)^2
- (k-1)^2 \left(d\lambda + (k-1)b(\frac{d+3}{2})^2 + (k-2)(\frac{d+1}{2})^2\right)^2
= \left(\lambda^2 - (k-1)d\lambda - (k-1)^2(\alpha(d) + c(\frac{d+3}{2})^2 + 2b(\frac{d+3}{2})^2 - (k-1)(k-2)(\frac{d+1}{2})^2)\right)
\times \left(\lambda^2 + (k-1)d\lambda - (k-1)^2(\alpha(d) + c(\frac{d+3}{2})^2 + (k-1)(k-2)(\frac{d+1}{2})^2)\right).$$
Taking a similar discussion as earlier, we may get that the largest root of $F_{c,b,c}(\lambda) = 0$, is the largest root of
\[
\lambda^2 - (k - 1)d\lambda - (k - 1)^2(\alpha(d) + c\left(\frac{d + 3}{2}\right)^2 + 2b\left(\frac{d + 3}{2}\right)^2) - (k - 1)(k - 2)(\frac{d + 1}{2})^2 = 0.
\]
Recall that $b = m - d - 2c$. Then $c \leq \frac{m - d}{2}$. By direct calculation, we have that
\[
\rho_c(T_{m,d}^{c,b,c}) = \frac{d(k - 1)}{2} + \frac{1}{2}\sqrt{d^2(k - 1)^2 + 4((k - 1)^2(\alpha(d) + c\left(\frac{d + 3}{2}\right)^2 + 2b\left(\frac{d + 3}{2}\right)^2) + (k - 1)(k - 2)(\frac{d + 1}{2})^2)}
\]
\[
= \frac{d(k - 1)}{2} + \frac{1}{2}\sqrt{d^2(k - 1)^2 + 4((k - 1)^2(\alpha(d) + c\left(\frac{d + 3}{2}\right)^2 + 2(m - d - 2c)(\frac{d + 1}{2})^2) + (k - 1)(k - 2)(\frac{d + 1}{2})^2)}
\]
\[
= \frac{d(k - 1)}{2} + \frac{1}{2}\sqrt{(k - 1)^2(d^2 + 4\alpha(d) + 8(m - d)(\frac{d + 3}{2})^2) + 4(k - 1)(k - 2)(\frac{d + 1}{2})^2 - 6(m - d)(k - 1)^2(\frac{d + 3}{2})^2}
\]
with equality if and only if $c = \frac{m - d}{2}$, i.e., $b = 0$.

Together with Claims 1 and 2, the $\epsilon$-spectral radius of graph $T_{m,d}^{a,b,c}$ achieves the minimum value when $a = \lceil \frac{m - d}{2} \rceil$, $c = \lceil \frac{m - d}{2} \rceil$ and $b = 0$, as required. \qed

**Lemma 4.4** For odd $d \geq 7$, one has $\rho_c(T_{m,d-2}^{a+1,b,c+1}) < \rho_c(T_{m,d}^{a,b,c})$.

**Proof.** Let $T_1 = T_{m,d-2}^{a+1,b,c+1}$ and $T = T_{m,d}^{a,b,c}$. Let $P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ be a diametrical path of $T$ and $e_i = \{w_i, w_i^1, \ldots, w_i^k\}$ in which $v_{i-1} = w_i^1$ and $v_i = w_i^k$ for $1 \leq i \leq d$. In fact, $T_1$ is obtained from $T$ by moving $e_d$ from $v_{d-1}$ to $v_d$ and $e_d^{-1}$ from $v_d$ to $v_{d-1}$. For the hypertree $T$, let $U_1$ be the set of pendant edges containing $v_{d-1}$ with $|U_1| = a$, $U_2$ be the set of pendient edges containing $v_{d+1}$ with $|U_2| = c$ and $U_3 = E(T) \setminus (E(P_{d+1}) \cup U_1 \cup U_2).

If $\{w, w'\} \not\in (V(e_1) \cup V(e_d)) \setminus \{v_1, v_{d-1}\}$, then $d_T(w, w') < \min\{\epsilon_T(w), \epsilon_T(w')\}$ and $d_{T_1}(w, w') < \min\{\epsilon_{T_1}(w), \epsilon_{T_1}(w')\}$. Therefore, $\epsilon(T)_{ww'} = 0 = \epsilon(T_1)_{ww'}$.

Let $w' \in V(e_1) \setminus \{v_1\}$. If $w \in V(e_1) \cup \ldots \cup V(e_{d-1}) \cup V(U_1)$, then $d_T(w, w') < \frac{d+1}{2} \leq \min\{\epsilon_T(w), \epsilon_T(w')\}$ and $d_{T_1}(w, w') < \frac{d-1}{2} \leq \min\{\epsilon_{T_1}(w), \epsilon_{T_1}(w')\}$. Thus $\epsilon(T)_{ww'} = 0 = \epsilon(T_1)_{ww'}$. Suppose that $w \in \left(V(e_{d+1}) \cup V(U_2) \cup V(U_3) \cup V(e_{d+1})\right) \setminus \{v_{d-1}\}$, then $\epsilon_T(w) = d_T(w, w') = d_{T_1}(w, w') + 1 = \epsilon_{T_1}(w) + 1$. Thus $\epsilon(T)_{ww'} = \epsilon(T_1)_{ww'} + 1$. Suppose now that $w \in \left(V(e_{d+2}) \cup \ldots \cup V(e_d)\right) \setminus \{v_d\}$, then $\epsilon_T(w) = d_T(w, w') = d_{T_1}(w, w') + 2 = \epsilon_{T_1}(w) + 2$. Thus $\epsilon(T)_{ww'} = \epsilon(T_1)_{ww'} + 2$.

Similarly, we have $\epsilon(T_1)_{ww'} \leq \epsilon(T)_{ww'}$ if $w' \in V(e_d) \setminus \{v_{d-1}\}$.

By Lemma 2.2 and Theorem 3.1, we get $\rho_c(T_{m,d-2}^{a+1,b,c+1}) < \rho_c(T_{m,d}^{a,b,c})$. \qed

By using Lemmas 4.3, 4.4 and Theorem 4.1, we obtain the following result.
Theorem 4.2 \( T_{m, 5}^{[\frac{m-5}{2}, \frac{m-5}{2}]} \) is the unique hypertree with the minimum \( \epsilon \)-spectral radius among all the hypertrees in \( \bigcup_{d \geq 5} T_{m, d} \) for odd \( d \).

Let \( H^{n_1, \ldots, n_k} \) be the \( k \)-uniform hypertree obtained from the edge \( e = \{v_1, \ldots, v_k\} \) by attaching \( n_i \) pendant edges to \( v_i \) for \( 1 \leq i \leq k \), where \( n_1 + \ldots + n_k = m - 1 \). Without loss of generality, we suppose that \( n_i \neq 0 \) for \( 1 \leq i \leq l \) and \( n_{l+1} = \ldots = n_k = 0 \), then \( H^{n_1, \ldots, n_k} \) can be rewritten as \( H^{n_1, \ldots, n_l} \). If \( \text{diam}(T) = 3 \), then \( T \cong H^{n_1, \ldots, n_l} \) with \( 2 \leq l \leq k \).

Proposition 4.3 Let \( T \) be a \( k \)-uniform hypertree with diameter 3. Then \( \text{Rank}(\epsilon(T)) = 2l \).

Proof. Let \( T_1, \ldots, T_t \) be all components of \( T - E(e) \) containing \( v_1, \ldots, v_t \), respectively. Let \((\epsilon(T))_v \) denote the row indexed by vertex \( v \), and \((\epsilon(T))_{S \times S} \) the submatrix corresponding to the rows and columns indexed by vertices in \( S \), where \( S \subseteq V(T) \). For \( 1 \leq i \leq l \),

\[
(\epsilon(H^{n_1, \ldots, n_l}))_{v_i} = \left( \frac{k}{n_1(k-1)} \frac{n_1(k-1)}{2} \frac{n_i(k-1)}{2} \frac{n_{i-1}(k-1)}{2} \frac{n_{i-1}(k-1)}{2}, \ldots, \frac{n_1(k-1)}{2} \frac{n_1(k-1)}{2}, \ldots, \frac{n_1(k-1)}{2} \frac{n_1(k-1)}{2} \right).
\]

For \( l + 1 \leq i \leq k \),

\[
(\epsilon(H^{n_1, \ldots, n_l}))_{v_i} = \left( \frac{k}{n_1(k-1)} \frac{n_1(k-1)}{2} \frac{n_i(k-1)}{2}, \ldots, \frac{n_1(k-1)}{2} \frac{n_1(k-1)}{2}, \ldots, \frac{n_1(k-1)}{2} \frac{n_1(k-1)}{2} \right) = \frac{1}{l-1}((\epsilon(H^{n_1, \ldots, n_l}))_{v_1} + \ldots + (\epsilon(H^{n_1, \ldots, n_l}))_{v_l}).
\]

So \( (\epsilon(H^{n_1, \ldots, n_l}))_{v_i} \) is linear combination of \( (\epsilon(H^{n_1, \ldots, n_l}))_{v_1}, \ldots, (\epsilon(H^{n_1, \ldots, n_l}))_{v_l} \). If \( \{w, w'\} \subseteq V(T_j) \setminus \{v_j\} \) for \( 1 \leq j \leq l \), then

\[
(\epsilon(H^{n_1, \ldots, n_l}))_w = (\epsilon(H^{n_1, \ldots, n_l}))_{w'} = \left( \frac{j-1}{n_1(k-1)} \frac{n_1(k-1)}{2} \frac{n_j(k-1)}{n_{j-1}(k-1)}, \ldots, \frac{n_1(k-1)}{2} \frac{n_1(k-1)}{2}, \ldots, \frac{n_1(k-1)}{2} \frac{n_1(k-1)}{2} \right).
\]

Thus we get the \( \text{Rank}(\epsilon(H^{n_1, \ldots, n_l})) \leq 2l \).

Let \( S = \{v_1, \ldots, v_t, w_1, \ldots, w_t\} \), where \( w_j \in v(T_j) \) for \( 1 \leq j \leq l \). Then \( \text{Rank}(\epsilon(H^{n_1, \ldots, n_l})_{S \times S}) = 2l \). Hence, we prove the result \( \text{Rank}(\epsilon(T)) = \text{Rank}(\epsilon(H^{n_1, \ldots, n_l})) = 2l \). \( \square \)

According to proposition 4.3 and conditions as \( l \geq 2 \), then \( \text{Rank}(\epsilon(T)) = 2l \geq 4 \). Hence, as \( l \) gets bigger, it is difficult to find which uniform hypertree achieves the minimum eccentricity spectral radius among all uniform hypertrees on \( m \) edges with diameter 3.

4.2. The minimum \( \epsilon \)-spectral radius of uniform hypertrees with even diameter

For a \( k \)-uniform hypertree \( T \) of size \( m \), if all edges contain a common vertex \( u \), such that \( |V(e) \setminus \{u\}| = k - 1 \) for \( 1 \leq i \leq m \), then we call \( T \) is a \((k\text{-uniform})\) hyperstar (with center \( u \)), denoted by \( S_{m,k} \).
If $d \geq 4$ is even, let $D_{m,d}^{a,b,c}$ be the $k$-uniform hypertree obtained from a diametrical path $P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ by attaching $b$ pendant edges to $v_{d-1}^2$, $a$ pendant edges to some vertices of $V(e_d^2) \setminus \{v_d^2\}$ and $c$ pendant edges to some vertices of $V(e_{d+1}^2) \setminus \{v_d^2\}$, respectively, where $a + b + c = m - d$, $a \geq c \geq 0$ and $b \geq 0$.

**Theorem 4.4** Let $T$ be a hypertree among $\mathcal{T}_{m,2}$, then $\epsilon$-spectral radius of $T$ is $(k - 1)(m - 1) + \sqrt{(k - 1)^2(m - 1)^2 + (k - 1)m}$.

**Proof.** Obviously, $\mathcal{T}_{m,2} = \{S_{m,k}\}$. Let $u$ be the center of $S_{m,k}$. Let $V_i = V(e_i) \setminus \{u\}$ with $1 \leq i \leq m$, then $V(S_{m,k}) = \{u\} \cup V_1 \cup \ldots \cup V_m$ be a partition of the vertex set of $S_{m,k}$. Corresponding to the partition, the eccentricity matrix of $S_{m,k}$ is equal to

$$
\epsilon(S_{m,k}) = \begin{pmatrix}
u & V_1 & V_2 & \ldots & V_m \\
0 & 1^T_{k-1} & 1^T_{k-1} & \ldots & 1^T_{k-1} \\
1_{k-1} & 0 & 2J_{k-1} & \ldots & 2J_{k-1} \\
1_{k-1} & 2J_{k-1} & 0 & 2J_{k-1} & \ldots & 2J_{k-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1_{k-1} & 2J_{k-1} & \ldots & 2J_{k-1} & 0 & \ldots & 0 
\end{pmatrix}
$$

By direct calculation, the $\epsilon$-polynomial of $S_{m,k}$ is

$$
\varphi_\epsilon(S_{m,k}, \lambda) = \lambda^{m(k-2)}(\lambda^2 - 2(k-1)(1-m)\lambda - (k-1)m)
$$

Thus, $\rho_\epsilon(S_{m,k}) = (k - 1)(m - 1) + \sqrt{(k - 1)^2(m - 1)^2 + (k - 1)m}$. $\square$

By a similar proof of Lemma 4.2, we have

**Lemma 4.5** Let $T$ be a $k$-uniform hyper-caterpillar and $P_{d+1} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ be a diametrical path of $T$, where $d \geq 4$ is even. For $1 \leq i \leq d$, let $e_i = \{w_i^1, \ldots, w_i^k\}$ with $v_{i-1} = w_i^1$ and $v_i = w_i^k$. Suppose that there exists a pendant edge $e \in E(T) \setminus E(P_{d+1})$ containing $w_j^i$ for some $i \in \{2, \ldots, \frac{d-4}{2}\}$ and $1 \leq j \leq k - 1$. Moving $e$ from $w_j^i$ to $v_{d-2}$ yields a hypertree $\tilde{T}$. Then $\rho_\epsilon(\tilde{T}) < \rho_\epsilon(T)$.

By frequently using of Lemmas 4.1 and 4.5, we obtain the following result.

**Lemma 4.6** Among $\mathcal{T}_{m,d}$ with even $d \geq 4$, the minimum $\epsilon$-spectral radius is achieved by some hypertree $D_{m,d}^{a,b,c}$, where $a + b + c = m - d$ and $a \geq c \geq 0$, $b \geq 0$.

**Theorem 4.5** The minimum $\epsilon$-spectral radius is achieved uniquely by hypertree $D_{m,4}^{0,m-4,0}$ among all the hypetrees in $\mathcal{T}_{m,4}$.

**Proof.** Let $T$ be a hypertree with minimum $\epsilon$-spectral radius among $\mathcal{T}_{m,4}$. By Lemma 4.6, $T \cong D_{m,d}^{a,b,c}$ with $a + b + c = m - d$, $a \geq c \geq 0$ and $b \geq 0$. Let $P_6 = (v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4)$ be a diametrical path of $T$, where $e_i = \{w_i^1, \ldots, w_i^k\}$ with $v_{i-1} = w_i^1$ and $v_i = w_i^k$ for $1 \leq i \leq 4$. 
Let $U_1$ be the set of pendant edges containing $w_j^2$ for $1 \leq j \leq k-1$, $U_2$ be the set of pendant edges containing $v_2$, $U_3$ be the set of pendant edges containing $w_j^3$ for $2 \leq j \leq k$, respectively. Note that $e_1 \in U_1$, $e_4 \in U_3$. Then $|U_1| = a + 1$, $|U_2| = b$ and $|U_3| = c + 1$.

Let $x$ be a Perron eigenvector corresponding to $\rho := \rho_e(D_{m,4}^{a,b,c})$, whose component with respect to vertex $v$ is $x_v$. Since
\[
\rho x_u = 2x_u + 3 \sum_{v \in V(U_1 \setminus \{v_u\})} x_v + 3 \sum_{v \in V(U_2 \setminus \{v_u\})} x_v + 4 \sum_{v \in V(U_3 \setminus \{v_u\})} x_v
\]
for any $u \in V(U_1) \setminus \{w_1^2, \ldots, w_{k-1}^2\}$, we have
\[
x_{u_1} = x_{u_1'} \quad \text{for any} \quad u_1, u_1' \in V(U_1) \setminus \{w_1^2, \ldots, w_{k-1}^2\}.
\]
Similarly, we have
\[
x_{u_2} = x_{u_2'} \quad \text{for any} \quad u_2, u_2' \in V(U_2) \setminus \{v_2\},
\]
\[
x_{u_3} = x_{u_3'} \quad \text{for any} \quad u_3, u_3' \in V(U_3) \setminus \{w_2^3, \ldots, w_k^3\},
\]
\[
x_{u_1} = x_{u_1^2} = x_{u_2^2} = \cdots = x_{u_{k-1}^2},
\]
\[
x_{u_3} = x_{u_3^3} = x_{u_3^3} = \cdots = x_{u_3^3}.
\]

Thus
\[
\rho x_{u_1} = 3(c + 1)(k - 1)x_{u_3}
\]
\[
\rho x_{u_2} = 2(a + 1)(k - 1)x_{u_1} + 2(c + 1)(k - 1)x_{u_3}
\]
\[
\rho x_{u_3} = 3(a + 1)(k - 1)x_{u_1}
\]
\[
\rho x_{u_1} = 2x_{u_2} + 3(k - 1)x_{u_3} + 3b(k - 1)x_{u_2} + 4(c + 1)(k - 1)x_{u_3}
\]
\[
\rho x_{u_2} = 3(a + 1)(k - 1)x_{u_1} + 3(c + 1)(k - 1)x_{u_3}
\]
\[
\rho x_{u_3} = 3(k - 1)x_{u_1} + 2x_{u_2} + 3b(k - 1)x_{u_2} + 4(a + 1)(k - 1)x_{u_1}
\]

By eliminating $x_{u_1}$, $x_{u_2}$, $x_{u_3}$ and $x_{u_2}$ from the above system, we obtain
\[
(\rho^2 - (k - 1)(a + 1)(4 + 9(b + 1)(k - 1)))x_{u_1} = (k - 1)(c + 1)(4\rho + 4 + 9b(k - 1))x_{u_3}
\]
and
\[
(k - 1)(a + 1)(4\rho + 4 + 9b(k - 1))x_{u_1} = (\rho^2 - (k - 1)(c + 1)(4 + 9(b + 1)(k - 1)))x_{u_3}
\]

Since $x_{u_1} > 0$ and $x_{u_3} > 0$, $\rho$ is the largest root of $H_{a,b,c}(\lambda) = 0$, where
\[
H_{a,b,c}(\lambda) = \begin{vmatrix}
\lambda^2 - (k - 1)(a + 1)(4 + 9(b + 1)(k - 1)) & -(k - 1)(c + 1)(4\lambda + 4 + 9b(k - 1)) \\
-(k - 1)(a + 1)(4\lambda + 4 + 9b(k - 1)) & \lambda^2 - (k - 1)(c + 1)(4 + 9(b + 1)(k - 1))
\end{vmatrix}
\]
\[
= (\lambda^2 - (k - 1)(a + 1)(4 + 9(b + 1)(k - 1)))(\lambda^2 - (k - 1)(c + 1)(4 + 9(b + 1)(k - 1)))
- (k - 1)^2(a + 1)(c + 1)(4\lambda + 4 + 9b(k - 1))^2.
\]

Suppose that $c \geq 1$. Let $\rho_1$ be the $\epsilon$-spectral radius of graph $D_{m,4}^{a+1,b,c-1}$, then $H_{a+1,b,c-1}(\rho_1) = 0$. Hence,
\[
H_{a,b,c}(\rho_1) = H_{a,b,c}(\rho_1) - H_{a+1,b,c-1}(\rho_1)
\]
\[
= (k-1)^2(c-a-1)(16\rho_1^2 + 72b(k-1) + 32\rho_1 - (k-1)^2(162b + 81) - 72(k-1))
\]
\[
= (k-1)^2(c-a-1)(4\rho_1 - 9(k-1))(4\rho_1 + (k-1)(18b + 9) + 8).
\]

By Lemma 2.3, we get \( \rho_1 \geq 4(k-1) \), then \((4\rho_1 - 9(k-1))(4\rho_1 + (k-1)(18b + 9) + 8) > 0\).

Recall that \(a \geq c\). Thus \(H_{a,b,c}(\rho_1) < 0\), which implies \(\rho_1 < \rho\). This is a contradiction with the choice of \(D_{m,4}^{a,b,c}\). Thus \(c = 0\).

Now we prove that \(a = 0\). Suppose that \(a \geq 1\). Let \(\rho_2\) be the \(\epsilon\)-spectral radius of graph \(D_{m,4}^{0,a+b,0}\), then \(H_{0,a+b,0}(\rho_2) = 0\). Thus

\[
H_{a,b,0}(\rho_2) = H_{a,b,0}(\rho_2) - H_{0,a+b,0}(\rho_2) = -a(k-1)h(\rho_2),
\]

where \(h(\rho_2) = ((9b + 7)(k-1) + 4)\rho_2^2 + (k-1)(36(k-1)(2b - 1) + 32)\rho_2 + (k-1)^2(81(k-1)(1 + a) - 36)\). By Lemma 2.3, we get \(\rho_2 \geq 4(k-1)\). Since \(-(k-1)\frac{36(k-1)(2b-1)+32}{(18b+14)(k-1)+8} < 4(k-1)\), \(h(\rho_2)\) is monotonically increasing for \(\rho_2 \geq 4(k-1)\). Then \(h(\rho_2) \geq h(4(k-1)) = 156(k-1)^2 + 49(k-1)^3 + 81(k-1)^3a + 351(k-1)^3b > 0\). Thus \(H_{a,b,0}(\rho_2) = -a(k-1)h(\rho_2) < 0\), which implies that \(\rho_2 < \rho\), a contradiction.

Therefore \(T \cong D_{m,4}^{0,m-4,0}\).

\[\square\]

By a similar proof of Theorem 4.1, we have

**Theorem 4.6** Among \(T_{m,d}\) with even \(d \geq 6\), the minimum \(\epsilon\)-spectral radius is achieved uniquely by hypertree \(D_{m,d}^{\lceil \frac{m-d}{2} \rceil,0,\lfloor \frac{m-d}{2} \rfloor}\).

By a similar proof of Lemma 4.4, we have

**Lemma 4.7** For even \(d \geq 6\), one has \(\rho_\epsilon(D_{m,d-2}^{a+1,b,c+1}) < \rho_\epsilon(D_{m,d}^{a,b,c})\).

**Lemma 4.8** \(\rho_\epsilon(D_{m,4}^{0,m-4,0}) < \rho_\epsilon(S_{m,k})\).

**Proof.** Obviously, \(m \geq 4\). Let \(\rho_1 = \rho_\epsilon(S_{m,k})\) and \(\rho_2 = \rho_\epsilon(D_{m,4}^{0,m-4,0})\). Then by theorem 4.4, \(\rho_1 = (k-1)(m-1) + \sqrt{(k-1)^2(m-1)^2 + (k-1)m}\). By theorem 4.5, \(\rho_2\) is the largest root of \(H_{0,m-4,0}(\lambda) = 0\), where

\[
H_{0,m-4,0}(\lambda) = \begin{vmatrix}
\lambda^2 - (k-1)(4 + 9(m-3)(k-1)) & -(k-1)(4\lambda + 4 + 9(m-4)(k-1)) \\
-(k-1)(4\lambda + 4 + 9(m-4)(k-1)) & \lambda^2 - (k-1)(4 + 9(m-3)(k-1))
\end{vmatrix}
\]
\[
= (\lambda^2 - (k-1)(4 + 9(m-3)(k-1)))^2 - (k-1)^2(4\lambda + 4 + 9(m-4)(k-1))^2
\]
\[
= (\lambda^2 - 4(k-1)\lambda - 8(k-1) - 9(k-1)^2(2m-7)) \cdot (\lambda^2 + 4(k-1)\lambda - 9(k-1)^2).
\]

Assume that \(\rho_0\) is the largest root of \(\lambda^2 + 4(k-1)\lambda - 9(k-1)^2 = 0\). Then \(\rho_0^2 = -4(k-1)\rho_0 + 9(k-1)^2\). By a direct calculation, one has

\[
\rho_0^2 - 4(k-1)\rho_0 - 8(k-1) - 9(k-1)^2(2m-7) = -8(k-1)\rho_0 - 8(k-1) - 9(k-1)^2(2m-6) < 0.
\]
Thus, \( \rho_2 \) is the largest root of \( \lambda^2 - 4(k-1)\lambda - 8(k-1) - 9(k-1)^2(2m-7) = 0 \), that is, 
\[
\rho_2 = 2(k-1) + \sqrt{(k-1)^2(18m-59) + 8(k-1)}.
\]

The well known inequality for \( x_1, x_2 > 0 \) and \( 0 < \alpha < 1 \): \( (x_1 + x_2)^\alpha < x_1^\alpha + x_2^\alpha \). Hence,
\[
\frac{\rho_2 - \rho_1}{2(k-1)} + \sqrt{(k-1)^2(18m-59) + 8(k-1) - (k-1)(m-1) - \sqrt{(k-1)^2(m-1)^2 + (k-1)m}}
= -\frac{(k-1)(m-3) - \sqrt{(k-1)^2(m-1)^2 + (k-1)m + \sqrt{(k-1)^2(18m-59) + 8(k-1)}}}{\sqrt{(k-1)^2(m-3)^2 + (m-1)^2} + (k-1)m + \sqrt{(k-1)^2(18m-59) + 8(k-1)}}
\]
\[
< -\frac{(k-1)^2(m-3) + (m-1)^2}{(k-1)m + \sqrt{(k-1)^2(18m-59) + 8(k-1)}}
\]
\[
= -\frac{(k-1)^2(2m^2 - 8m + 10) + (k-1)m + \sqrt{(k-1)^2(18m-59) + 8(k-1)}}{\sqrt{(k-1)^2(18m-59) + 8(k-1)}}.
\]

For \( m \geq 10 \), we have that \( (k-1)^2(18m-59) + 8(k-1) - (k-1)^2(2m^2 - 8m + 10) - (k-1)m = (k-1)^2(-2m^2 + 26m - 69) + (k-1)(8 - m) < (k-1)^2(-2m^2 + 26m - 69) < 0 \). Thus, for \( m \geq 10 \), we have that \( \rho_2 - \rho_1 < 0 \).

For \( 4 \leq m \leq 9 \), by Table 1, it is easy to check that \( \rho_2 - \rho_1 < 0 \) for \( k \geq 3 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \rho_2 - \rho_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( \sqrt{8k+13(k-1)^2-8} - \sqrt{4k+9(k-1)^2-4} - (k-1) )</td>
</tr>
<tr>
<td>5</td>
<td>( \sqrt{8k+31(k-1)^2-8} - \sqrt{5k+16(k-1)^2-5} - 2(k-1) )</td>
</tr>
<tr>
<td>6</td>
<td>( \sqrt{8k+49(k-1)^2-8} - \sqrt{6k+25(k-1)^2-6} - 3(k-1) )</td>
</tr>
<tr>
<td>7</td>
<td>( \sqrt{8k+67(k-1)^2-8} - \sqrt{7k+36(k-1)^2-7} - 4(k-1) )</td>
</tr>
<tr>
<td>8</td>
<td>( \sqrt{8k+85(k-1)^2-8} - \sqrt{8k+49(k-1)^2-8} - 5(k-1) )</td>
</tr>
<tr>
<td>9</td>
<td>( \sqrt{8k+103(k-1)^2-8} - \sqrt{9k+64(k-1)^2-9} - 6(k-1) )</td>
</tr>
</tbody>
</table>

Thus \( \rho_e(D_{m,4}^{0,m-4,0}) < \rho_e(S_m,k) \).

By Lemmas 4.6, 4.7, 4.8 and Theorem 4.5, we obtain the following result.

**Theorem 4.7** \( D_{m,4}^{0,m-4,0} \) is the unique hypertree with minimum \( \epsilon \)-spectral radius among all the hypertrees in \( T_{m,d} \) for even \( d \).

### 4.3. The minimum \( \epsilon \)-spectral radius of uniform hypertrees

**Theorem 4.8** Let \( T \in \bigcup_{d \neq 3} T_{m,d} \).

(i) If \( m = 5 \) and \( m = 6 \) with \( k \geq 4 \), then \( \rho_e(T) \geq \rho_e(D_{m,4}^{0,m-4,0}) \) with equality if and only if \( T \cong D_{m,4}^{0,m-4,0} \).

(ii) If \( m \geq 7 \) and \( m = 6 \) with \( k = 3 \), then \( \rho_e(T) \geq \rho_e(T_{m,5}^{[m-5],0,[m-5]}) \) with equality if and only if \( T \cong T_{m,5}^{[m-5],0,[m-5]} \).
Proof. By Theorems 4.2 and 4.7, it suffices to compare the $\epsilon$-spectral radius of $D_{m,4}^{0,4}$ and $T_{m,5}^{m-2,0,\lfloor m/2 \rfloor}$.

Let $\rho_1 = \rho_\epsilon(D_{m,d}^{0,m-4,0})$ and $\rho_2 = \rho_\epsilon(T_{m,5}^{m-2,0,\lfloor m/2 \rfloor})$, then $\rho_1 = 2(k-1)+\sqrt{(k-1)^2(18m-59)+8(k-1)}$ and $\rho_2$ is the largest root of $F_{m-2,0,\lfloor m/2 \rfloor}^{m-2,0,\lfloor m/2 \rfloor}(\lambda) = 0$.

If $m = 5$, then
\[
F_{0,0,0}(\lambda) = (\lambda^2 - 25(k-1)^2)^2 - (k-1)^2(5\lambda + 9(k-2))^2
\]
\[
= (\lambda^2 - 5(k-1)\lambda - 9(k-1)(k-2) - 25(k-1)^2)
\]
\[
\times (\lambda^2 + 5(k-1)\lambda + 9(k-1)(k-2) - 25(k-1)^2).
\]
Thus $\rho_2 = \frac{5(k-1)}{2} + \sqrt{31.25(k-1)^2 + 9(k-1)(k-2)}$. Obviously,
\[
\rho_1 = 2(k-1) + \sqrt{31(k-1)^2 + 8(k-1)} < \rho_2.
\]

Suppose that $m = 6$. Then $\rho_2$ is the largest root of $F_{1,0,0}(\lambda) = 0$, where
\[
F_{1,0,0}(\lambda) = (\lambda^2 - 25(k-1)^2) \cdot (\lambda^2 - 41(k-1)^2) - (k-1)^2 \cdot (5\lambda + 9(k-2))^2.
\]

Recall that $k \geq 3$. Let $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \rho_2$ be the roots of $g(\lambda) = 0$. Note that
\[
\lim_{\lambda \to -\infty} F_{1,0,0}(\lambda) = \lim_{\lambda \to +\infty} F_{1,0,0}(\lambda) = +\infty,
\]
\[
F_{1,0,0}(-5(k-1)) = -(k-1)^2 \cdot (-16k + 7)^2 < 0,
\]
\[
F_{1,0,0}(0) = (k-1)^2 \cdot (1025(k-1)^2 - 81(k-2)^2) > 0,
\]
\[
F_{1,0,0}(5(k-1)) = -(k-1)^2 \cdot (34k - 43)^2 < 0.
\]

Then $F_{1,0,0}(\lambda) > 0$ for $\lambda \in (-\infty, \gamma_1) \cup (\gamma_2, \gamma_3) \cup (\rho_2, +\infty)$ and $F_{1,0,0}(\lambda) < 0$ for $\lambda \in (\gamma_1, \gamma_2) \cup (\gamma_3, \rho_2)$, where $\gamma_1 \in (-\infty, -5(k-1))$, $\gamma_2 \in (-5(k-1), 0)$, $\gamma_3 \in (0, 5(k-1))$ and $\rho_2 \in (5(k-1), +\infty)$.

Note that $\rho_1^2 = 4(k-1)\rho_1 + 8(k-1) + 45(k-1)^2$. Hence, we get
\[
F_{1,0,0}(\rho_1) = (\rho_1^2 - 25(k-1)^2) \cdot (\rho_1^2 - 41(k-1)^2) - (5\rho_1 + 9(k-1)(k-2))^2
\]
\[
= (4(k-1)\rho_1 + 8(k-1) + 20(k-1)^2) \cdot (4(k-1)\rho_1 + 8(k-1) + 4(k-1)^2)
\]
\[
- (k-1)^2 \cdot (5\rho_1 + 9(k-2))^2
\]
\[
= (k-1)^2 \cdot (-9\rho_1^2 + 6k + 148)\rho_1 - k^2 + 356k - 372).
\]
Since $\frac{6k+148}{18} < 9(k-1)$, we have $F_{1,0,0}(\rho_1)$ is monotonically decrease for $\lambda > 9(k-1)$. By direct calculation, we have $\rho_1 = 2(k-1) + \sqrt{49(k-1)^2 + 8(k-1)} \in (9(k-1), 10(k-1))$. Note that
\[
F_{1,0,0}(9(k-1)) = (k-1)^2 \cdot (-676k^2 + 3092k - 2433),
\]
\[
F_{1,0,0}(10(k-1)) = (k-1)^2 \cdot (-841k^2 + 3576k - 2752).
Then, $F_{1,0,0}(10(k-1)) = 1628 > 0$ for $k = 3$, and $F_{1,0,0}(9(k-1)) < 0$ for $k \geq 4$. Thus, $F_{1,0,0}(\rho_1) > 0$ for $k = 3$ and $F_{1,0,0}(\rho_1) < 0$ for $k \geq 4$. Since $\rho_1 > 5(k-1) > \gamma_3$, we have $\rho_2 < \rho_1$ for $k = 3$ and $\rho_2 > \rho_1$ for $k \geq 4$.

Suppose now that $m \geq 7$. If $m$ is odd, then by the proof of Theorem 4.1, we have $\rho_2 = \frac{5(k-1)}{2} + \frac{1}{2} \sqrt{(k-1)^2(32m+1) - 36(k-1)}$. Thus,

$$\rho_1 - \rho_2 = \sqrt{(k-1)^2(18m - 59) + 8(k - 1)} - \frac{1}{2} \sqrt{(k-1)^2(32m+1) - 36(k-1)} - \frac{(k-1)}{2}$$

$$> (k - 1) \cdot \left(\sqrt{18m - 59} - \sqrt{8m + \frac{1}{4}} - \frac{1}{2}\right)$$

$$= (k - 1) \cdot f(m),$$

where $f(m) = \sqrt{18m - 59} - \sqrt{8m + \frac{1}{4}} - \frac{1}{2}$. Since for $m \geq 7$,

$$f'(m) = \frac{9\sqrt{8m + \frac{1}{4}} - 4\sqrt{18m - 59}}{\sqrt{18m - 59} \cdot \sqrt{8m + \frac{1}{4}}} > 0.$$

Then, we have $f(m)$ is monotonically increasing for $m \geq 7$. Thus $f(m) \geq f(7) = 0.1854$, that is, $\rho_1 > \rho_2$ for $m$ is odd.

Suppose that $m$ is even. Then $\rho_2$ is the largest root of $F_{\frac{m-4}{2}, \frac{m-4}{2}}(\lambda) = 0$. Let $\rho_3$ be the largest root of $F_{\frac{m-4}{2}, \frac{m-4}{2}}(\lambda) = 0$, that is $\rho_3 = \frac{5(k-1)}{2} + \frac{1}{2} \sqrt{(k-1)^2(32m+3) - 36(k-1)}$. Then

$$F_{\frac{m-4}{2}, \frac{m-4}{2}}(\rho_2) = F_{\frac{m-4}{2}, \frac{m-4}{2}}(\rho_2) - F_{\frac{m-4}{2}, \frac{m-4}{2}}(\rho_2)$$

$$= (\rho_2^2 - (k - 1)^2(8m - 7))^2 - (\rho_2^2 - (k - 1)^2(8m - 23)) \cdot (\rho_2^2 - (k - 1)^2(8m - 7))$$

$$< 0.$$

Thus $\rho_2 < \rho_3$.

By direct calculation, one has

$$\rho_1 - \rho_3 = \sqrt{(k-1)^2(18m - 59) + 8(k - 1)} - \frac{1}{2} \sqrt{(k-1)^2(32m+33) - 36(k-1)} - \frac{(k-1)}{2}$$

$$> (k - 1) \cdot \left(\sqrt{18m - 59} - \sqrt{8m + \frac{33}{4}} - \frac{1}{2}\right).$$

By a similar discussion as above, we have that $\rho_1 > \rho_3$. Thus $\rho_1 > \rho_3 > \rho_2$. 

\[\Box\]

References


