Occupancy Models with Autocorrelated Detection Heterogeneity – Supplementary Material

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1 Simulation Study Supplemental Information

1.1 Additional Simulation Details

We simulated data for $S = 2$ species at $n = 100$ locations on an equispaced 10×10 grid over $[0, 1] \times [0, 1]$ for $T = 5, 10, 20$ seasons with $J = 1, 3, 5$ surveys per season. Let x and y denote the coordinates of the grid centroids. We generated a spatio-temporal variable v_1 using a n-dimensional random walk with a spatial exponential covariance function with variance of 0.1 and scale parameter of 0.8. A temporal variable v_2 was generated such that for time t, it took the value $\sin(2\pi t/T)$. The occupancy design matrix, $\mathbf{X}$, used to simulate and fit the data for both species had columns for x, y, v_1, and v_2. The occupancy random effect was generated assuming $q = 100$ Moran's I basis functions and MCAR precision matrix $\Sigma^{-1} \otimes (\mathbf{D} - 0.9\mathbf{A})$. The true occupancy probabilities were simulated using true values of the regression coefficients of $\boldsymbol{\alpha}^{(1)} = (-0.75, 0.5, 0.25, -0.8, 0.75)$ and $\boldsymbol{\alpha}^{(1)} = (-0.5, 0.25, -0.25, -0.75, 0.5)$.

We also generated covariates to be used for detectability. Let w_1 be a spatio-temporal random variable generated using a multivariate random walk with squared exponential covariance function using variance 0.2 and scale parameter of $\sqrt{1.4}$. A temporal variable w_2 was generated by taking the log of the enumerated values for the season. So the detection design matrix had columns for the x-coordinate, w_1, and w_2. Note that in the multi-survey per season setting, the detection covariates were assumed to be the same fore very survey within the season. The true values for the detection regression coefficients were $\boldsymbol{\beta}^{(1)} = (-0.1, -0.5, 0.8, -0.1)$ and $\boldsymbol{\beta}^{(2)} = (0, -0.25, 0.5, 0.1)$.

1.2 Simulation Additional Results

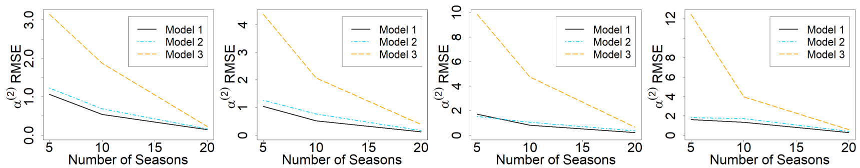

Fig. 1: Root mean squared errors of the occupancy regression coefficients for species 2 under all three models and number of seasons.

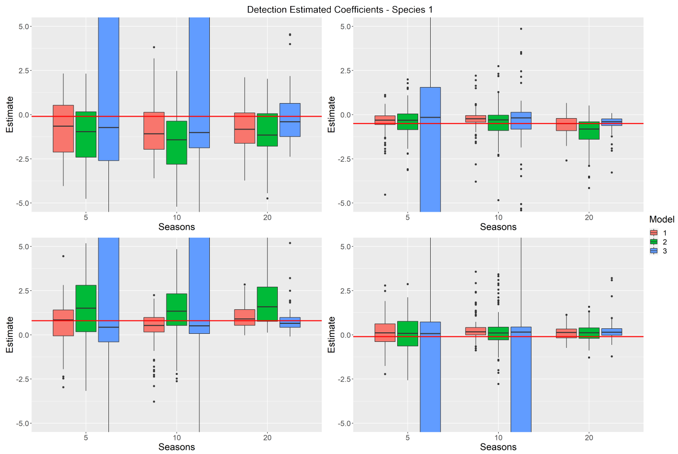

Fig. 2: $H = 100$ posterior means of each species 1 detection regression coefficient for each model and number of seasons when $J = 1$. We note that boxplots for model 3 extend past the limit of the y-axis, but this view makes it easier to assess models 1 and 2.

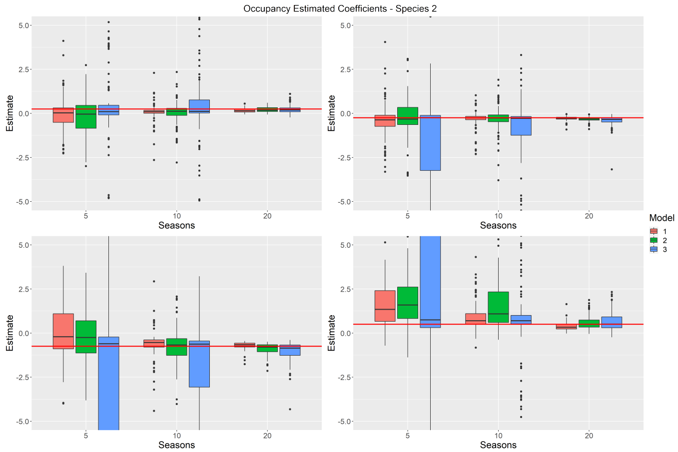

Fig. 3: $H = 100$ posterior means of each species 2 occupancy regression coefficient for each model and number of seasons when $J = 1$.

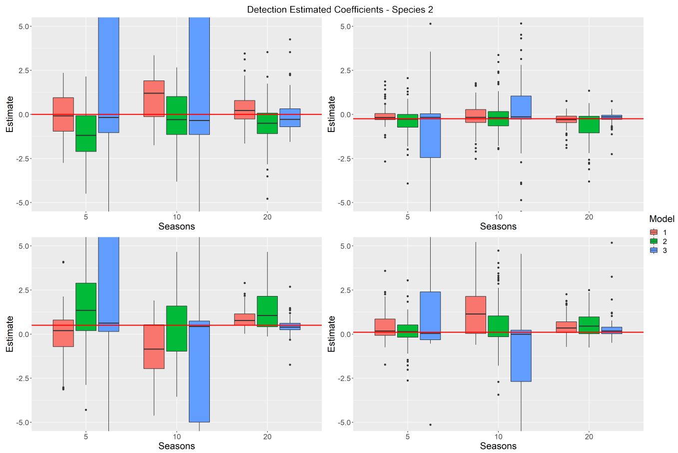

Fig. 4: $H = 100$ posterior means of each species 2 detection regression coefficient for each model and number of seasons when $J = 1$.

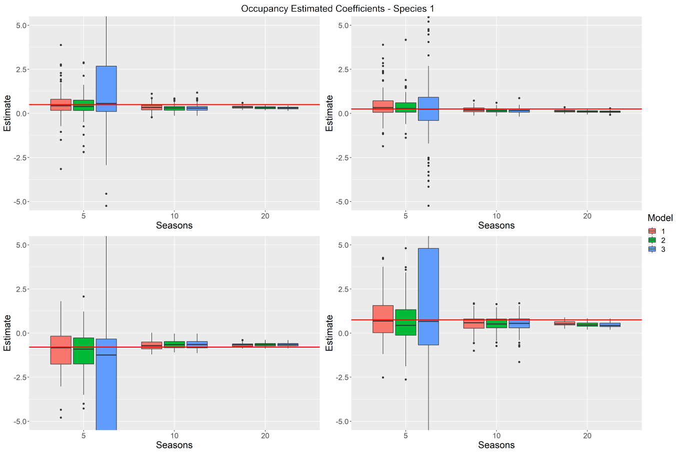

Fig. 5: $H = 100$ posterior means of each species 2 occupancy regression coefficient for each model and number of seasons when $J = 3$ surveys are taken per season.

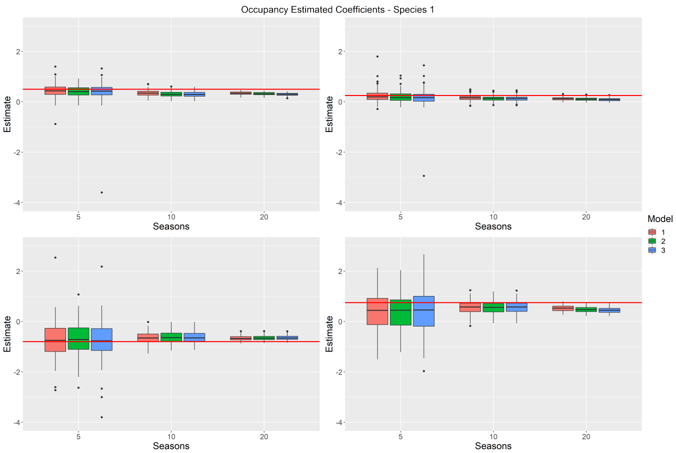

Fig. 6: $H = 100$ posterior means of each species 2 occupancy regression coefficient for each model and number of seasons when $J = 5$ surveys are taken per season.

2 Application Supplemental Results

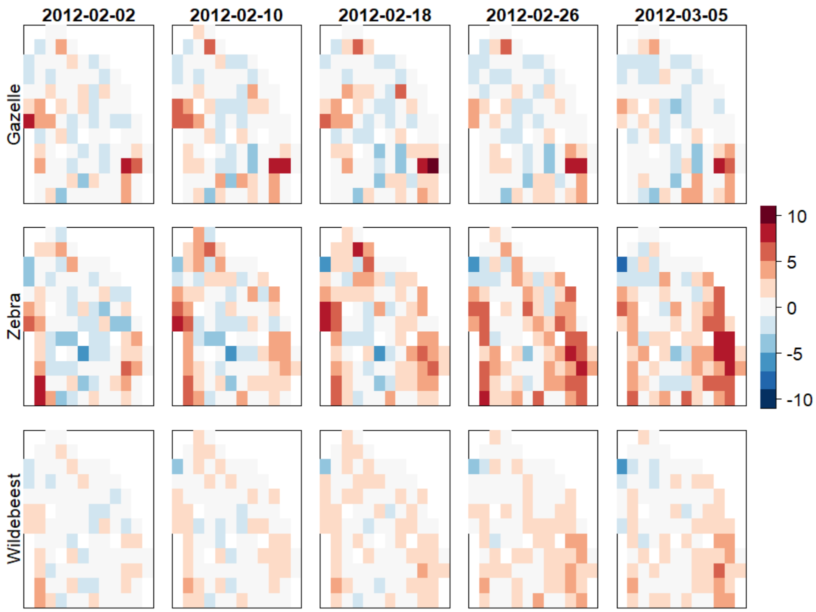

Fig. 7: Estimated values of the multivariate temporal random effect, $\omega_{it}^{(s)}$ for five consecutive seasons for each species.

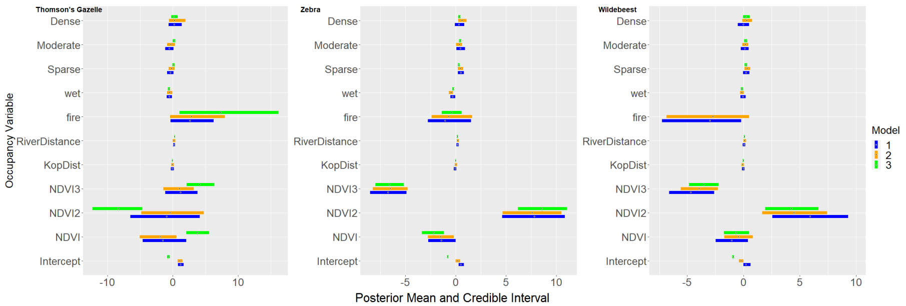

Fig. 8: Occupancy regression coefficient credible intervals for each species under each of the three models. Note that the coefficient for fire for wildebeest was less than -5 so it doesn't appear on the chart with the given x-limits

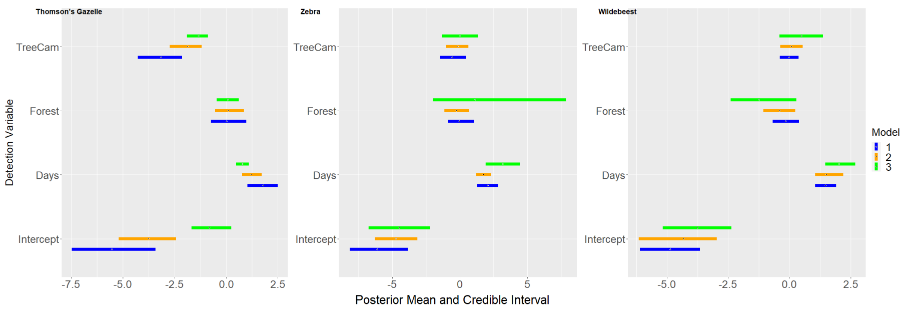

Fig. 9: Detection regression coefficient credible intervals for each species under each of the three models.

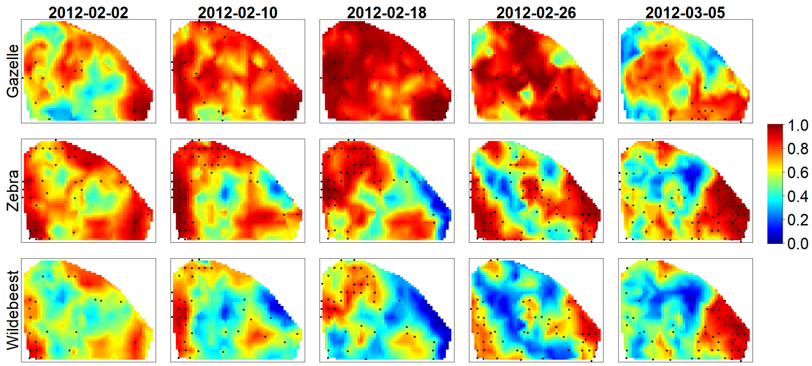

Fig. 10: Estimated occupancy probabilities for five consecutive seasons under model $\mathcal{M}_2$ that has a detection temporal random effect that is independent across species and has a spatially homogeneous temporal autocorrelation parameter in detection.

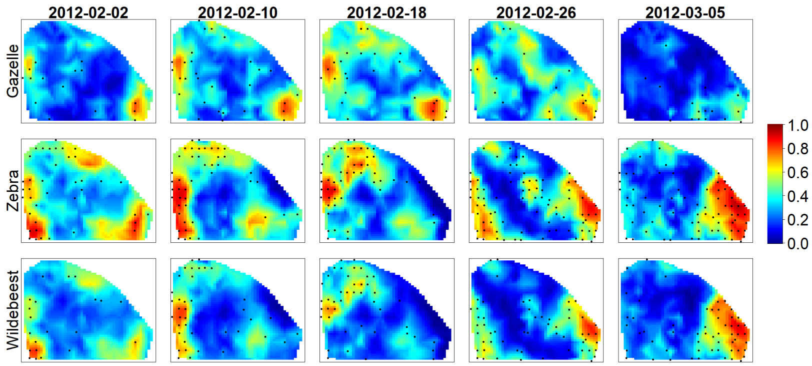

Fig. 11: Estimated occupancy probabilities for five consecutive seasons under model $\mathcal{M}_3$ that only has covariates in detection.

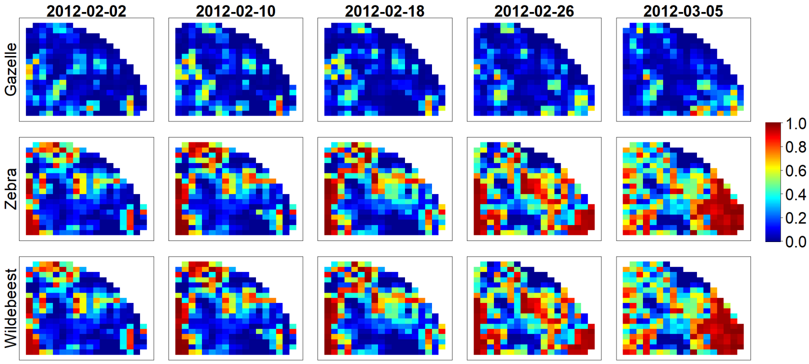

Fig. 12: Estimated detection probabilities for five consecutive seasons under model $\mathcal{M}_2$.

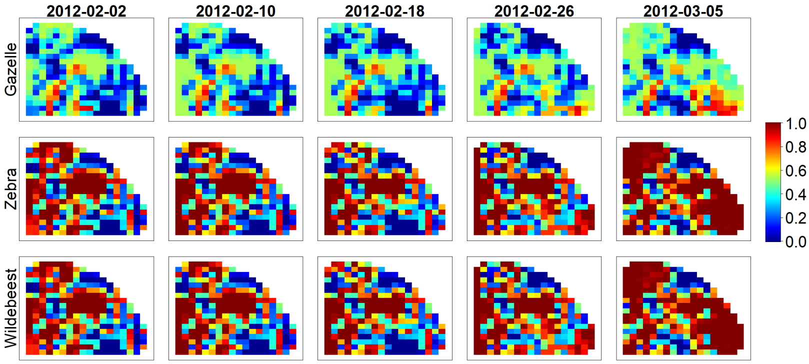

Fig. 13: Estimated detection probabilities for five consecutive seasons under model $\mathcal{M}_3$.

3 Full Conditional Distributions

The posterior distribution can be explored with a Gibbs sampling algorithm since the full conditionals for each unknown quantity are known. These full conditionals are given below, where the notation $[V \mid \cdot]$ is used to denote the distribution of a random variable V conditional on all other random variables and on the data. Note that the derivations have been modified accordingly from Hepler and Erhardt (2021).

- For sites $i = 1, ..., n$, time $t = 1, ..., T$, and species $s = 1, ..., S$, if $\sum_{j=1}^{J} Y_{itj}^{(s)} > 0$ then $Z_{it}^{(s)} = 1$ with probability 1. If $\sum_{j=1}^{J} Y_{itj}^{(s)} = 0$, then

$$[Z_{it}^{(s)} \mid \boldsymbol{\alpha}^{(s)}, \boldsymbol{\gamma}_t^{(s)}, \boldsymbol{\beta}^{(s)}, \omega_{it}^{(s)}] = \text{Bernoulli}(\bar{\Phi}),$$

where

$$\bar{\Phi} = \frac{\Phi\left(\mathbf{X}_{it}^{(s)}\boldsymbol{\alpha}^{(s)} + \mathbf{K}_{it}^{(s)}\boldsymbol{\gamma}_t^{(s)}\right)\prod_{j=1}^{J}\left(1 - \Phi(\mathbf{W}_{itj}\boldsymbol{\beta}^{(s)} + \omega_{it}^{(s)})\right)}{\Phi\left(\mathbf{X}_{it}^{(s)}\boldsymbol{\alpha}^{(s)} + \mathbf{K}_{it}^{(s)}\boldsymbol{\gamma}_t^{(s)}\right)\prod_{j=1}^{J}\left(1 - \Phi\left(\mathbf{W}_{itj}\boldsymbol{\beta}^{(s)} + \omega_{it}^{(s)}\right)\right) + 1 - \Phi(\mathbf{X}_{it}^{(s)}\boldsymbol{\alpha}^{(s)} + \mathbf{K}_{it}^{(s)}\boldsymbol{\gamma}_t^{(s)})}$$

In the above, $\mathbf{K}_{it}^{(s)}$ is the ith row of the matrix of basis functions $\mathbf{K}_t^{(s)}$.

- For sites $i = 1, ..., n$, time $t = 1, ..., T$, and species $s = 1, ..., S$, the latent occupancy process has full conditional distributions

$$[\tilde{Z}_{it}^{(s)} \mid \cdot] = \begin{cases} TN_{(-\infty,0)}\left(\mathbf{X}_{it}^{(s)}\boldsymbol{\alpha}^{(s)} + \mathbf{K}_{it}^{(s)}\boldsymbol{\gamma}_t^{(s)}, 1\right) & \text{if } Z_{it}^{(s)} = 0 \\ TN_{(0,\infty)}\left(\mathbf{X}_{it}^{(s)}\boldsymbol{\alpha}^{(s)} + \mathbf{K}_{it}^{(s)}\boldsymbol{\gamma}_t^{(s)}, 1\right) & \text{if } Z_{it}^{(s)} = 1, \end{cases}$$

where $TN_{(a,b)}$ denotes the truncated normal distribution, truncated to (a, b).

- Provided $T > 2$, for each $t = 1, ..., T$, we update the Sq-dimensional vector $\boldsymbol{\gamma}_t$ jointly.
  - For $t = 1$, the full conditional is $[\boldsymbol{\gamma}_1 \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\Sigma}^{-1} = \mathbf{I}_{Sq} + \mathbf{K}_1^{*'}\mathbf{Q}\mathbf{K}_1^* + \mathbf{M}'\mathbf{K}_2^{*'}\mathbf{Q}\mathbf{K}_2^*\mathbf{M}$$

$$\bar{\mu} = \bar{\Sigma} \times \left\{\mathbf{K}_1^{*'}(\tilde{\mathbf{Z}}_1 - \mathbf{X}_1^*\boldsymbol{\alpha}) + \mathbf{M}'\left(\mathbf{K}_2^{*'}\mathbf{Q}\mathbf{K}_2^*\right)\boldsymbol{\gamma}_2\right\}$$

  - For $t = 2, ..., T-1$, the full conditional is $[\boldsymbol{\gamma}_t \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\Sigma}^{-1} = \mathbf{I}_{Sq} + \mathbf{K}_t^{*'}\mathbf{Q}\mathbf{K}_t^* + \mathbf{M}'\mathbf{K}_{t+1}^*\mathbf{Q}\mathbf{K}_{t+1}^*\mathbf{M}$$

$$\bar{\mu} = \bar{\Sigma} \times \left\{\mathbf{K}_t^{*'}(\tilde{\mathbf{Z}}_t - \mathbf{X}_t^*\boldsymbol{\alpha}) + \mathbf{M}'\left(\mathbf{K}_{t+1}^{*'}\mathbf{Q}\mathbf{K}_{t+1}^*\right)\boldsymbol{\gamma}_{t+1} + \left(\mathbf{K}_t^{*'}\mathbf{Q}\mathbf{K}_t^*\right)\mathbf{M}\boldsymbol{\gamma}_{t-1}\right\}$$

  - For $t = T$, the full conditional is $[\boldsymbol{\gamma}_T \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\Sigma}^{-1} = \mathbf{I}_{Sq} + \mathbf{K}_T^{*'}\mathbf{Q}\mathbf{K}_T^*$$

$$\bar{\mu} = \bar{\Sigma} \times \left\{\mathbf{K}_T^{*'}(\tilde{\mathbf{Z}}_T - \mathbf{X}_T^*\boldsymbol{\alpha}) + \left(\mathbf{K}_T^{*'}\mathbf{Q}\mathbf{K}_T^*\right)\mathbf{M}\boldsymbol{\gamma}_{T-1}\right\}$$

Note that since the prior on $\boldsymbol{\gamma}_t$ is intrinsic MCAR, a centering constraint $1/q\sum_{i=1}^{q}\gamma_{it}^{(s)} = 0$ must be enforced for all s, t to ensure the posterior is proper and to be able to identify an intercept parameter.

- For each $s = 1, ..., S$, we update the vector of occupancy fixed effects for species s jointly. Observe that

$$\tilde{\mathbf{Z}}^{(s)} = \mathbf{X}^{(s)}\boldsymbol{\alpha}^{(s)} + \mathbf{K}^{(s)}\boldsymbol{\gamma}^{(s)} + \boldsymbol{\epsilon}^{(s)},$$

where $\mathbf{X}^{(s)}$ denotes the full design matrix for species s, that is, $\mathbf{X}^{(s)} = (\mathbf{X}_1^{(s)'}, ..., \mathbf{X}_T^{(s)'})'$ so that the design matrices for each time period are row-concatenated, $\mathbf{K}^{(s)} = \text{diag}(\mathbf{K}_t^{(s)})$ is the block diagonal matrix of basis functions for each time period, and $\boldsymbol{\gamma}^{(s)} = (\boldsymbol{\gamma}_1^{(s)'}, ..., \boldsymbol{\gamma}_T^{(s)'})'$ is the vector of spatial random effects for each time period. The full conditional is $[\boldsymbol{\alpha}^{(s)} \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where depending on the choice of prior distribution,

$$\bar{\Sigma}^{-1} = \begin{cases} \left(\mathbf{X}^{(s)'}\mathbf{X}^{(s)}\right) & \text{if } \pi(\boldsymbol{\alpha}^{(s)}) = 1 \\ \left(\mathbf{X}^{(s)'}\mathbf{X}^{(s)} + \tau_a\mathbf{I}\right) & \text{if } \boldsymbol{\alpha}^{(s)} \sim N(0, \tau_a), \end{cases}$$

for precision τ_a. Then,

$$\bar{\mu} = \bar{\Sigma} \times \left\{\mathbf{X}^{(s)'}\left(\tilde{\mathbf{Z}}^{(s)} - \mathbf{K}^{(s)}\boldsymbol{\gamma}^{(s)}\right)\right\}.$$

- Our model assumes the occupancy propagator matrix is of the form $\mathbf{M} = \text{diag}(\rho_1, ..., \rho_S) \otimes \mathbf{I}_q$. Let $\boldsymbol{\rho} \equiv (\rho_1, ..., \rho_S)'$. We update the vector $\boldsymbol{\rho}$ with a single Gibbs update. Let $D(\boldsymbol{\gamma}_{t-1})$ be the $qS \times S$ block diagonal matrix with $q \times 1$-dimensional blocks $\boldsymbol{\gamma}_{t-1}^{(s)}$ for $s = 1, ..., S$. Under this specification, the full conditional distribution of $\boldsymbol{\rho}$ is multivariate normal (or truncated multivariate normal if the prior on $\boldsymbol{\rho}$ is chosen to have support smaller than the whole real line). That is, $[\boldsymbol{\rho} \mid \cdot] = N\left(\bar{\mu}, \bar{\Sigma}\right)$, where

$$\bar{\Sigma}^{-1} = \sum_{t=2}^{T} D(\boldsymbol{\gamma}_{t-1})'\mathbf{K}_t^{*'}\mathbf{Q}\mathbf{K}_t^{*}D(\boldsymbol{\gamma}_{t-1})$$

$$\bar{\mu} = \bar{\Sigma} \times \left(\sum_{t=2}^{T} D(\boldsymbol{\gamma}_{t-1})'\mathbf{K}_t^{*'}\mathbf{Q}\mathbf{K}_t^{*}\boldsymbol{\gamma}_t\right)$$

- A Gibbs update can be performed for Σ assuming $\mathbf{X}_t^{(s)} = \mathbf{X}_t$ for all $s = 1, ..., S$, which implies $\mathbf{K}_t^{(s)} = \mathbf{K}_t$ for all $s = 1, ..., S$ and $t = 1, ..., T$, and thus $\mathbf{X}_t^{*} = \text{I}_S \otimes \mathbf{X}_t$ and $\mathbf{K}_t^{*'}\mathbf{Q}\mathbf{K}_t^{*} = \Sigma^{-1} \otimes \mathbf{K}_t'(\mathbf{D} - \mathbf{A})\mathbf{K}_t$. Assuming an inverse Wishart prior distribution for Σ with parameters ν and Ψ, the full conditional distribution of Σ is inverse Wishart with parameters $\nu + qT$ and $\Psi + \sum_{t=1}^{T}\mathbf{B}_t$, where $\mathbf{B}_t$ is the $S \times S$ matrix with $(i,j)th$ entry $(B_t)_{i,j} =$

trace $\left(\mathbf{C}_{i,j}^{(t)}\mathbf{K}_t'(\mathbf{D}-\mathbf{A})\mathbf{K}_t\right)$, where for $t=1$, $\mathbf{C}^{(1)}$ is the block matrix with (i,j)th block

$$\mathbf{C}_{i,j}^{(1)} = (\boldsymbol{\gamma}_1^{(i)})(\boldsymbol{\gamma}_1^{(j)})'$$

for $i,j=1,...,S$, and for $t=2,...,T$ we have the block matrix with (i,j)th block

$$\mathbf{C}_{i,j}^{(t)} = (\boldsymbol{\gamma}_t^{(i)} - \rho_i\boldsymbol{\gamma}_{t-1}^{(i)})(\boldsymbol{\gamma}_t^{(j)} - \rho_j\boldsymbol{\gamma}_{t-1}^{(j)})'.$$

- For each site, time, species, and survey $j=1,...,J$, the full conditional distributions for the latent continuous detection process are

$$[\tilde{Y}_{itj}^{(s)} \mid \cdot] = \begin{cases} TN_{(-\infty,0)}\left(\mathbf{W}_{it}\boldsymbol{\beta}^{(s)} + \omega_{it}^{(s)}, 1\right) & \text{if } Y_{itk}^{(s)} = 0 \text{ and } Z_{it}^{(s)} = 1 \\ TN_{(0,\infty)}\left(\mathbf{W}_{it}\boldsymbol{\beta}^{(s)} + \omega_{it}^{(s)}, 1\right) & \text{if } Y_{itk}^{(s)} = 1 \\ \delta_0 & \text{if } Z_{it}^{(s)} = 0, \end{cases}$$

where δ_0 is the dirac delta function that indicates a point mass at zero.

- For each $s=1,...,S$, we update the vector of detection regression coefficients $\boldsymbol{\beta}^{(s)}$ jointly. Let $\mathbf{W}^{(s)}$ denote the row-concatenated detection design matrix for species s at all locations, times, and surveys, that is, $\mathbf{W}^{(s)} = (\mathbf{W}_1^{(s)'},...,\mathbf{W}_T^{(s)'})'$ where $\mathbf{W}_t^{(s)} = (\mathbf{W}_{t1}^{(s)'},...,\mathbf{W}_{tJ}^{(s)'})'$ is the concatenation of design matrices at all J surveys during time t. Define $\boldsymbol{\omega}^{(s)} = (\tilde{\boldsymbol{\omega}}_1^{(s)'},...,\tilde{\boldsymbol{\omega}}_T^{(s)'})'$ where $\tilde{\boldsymbol{\omega}}_t^{(s)} = \boldsymbol{\omega}_t^{(s)} \otimes \mathbf{1}_J$ repeats the values of the random effect for each survey within the given season. For a given vector or matrix Ω, let Ω^* denote the sub-vector/matrix created by subsetting only the rows of Ω for which the corresponding $Z_{it}^{(s)} = 1$. Then, the full conditional distribution of $\boldsymbol{\beta}^{(s)}$ is $[\boldsymbol{\beta}^{(s)} \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where depending on the choice of prior distribution,

$$\bar{\Sigma}^{-1} = \begin{cases} \left(\mathbf{W}^{(s)*'}\mathbf{W}^{(s)*}\right) & \text{if } \pi(\boldsymbol{\beta}^{(s)}) = 1 \\ \left(\mathbf{W}^{(s)*'}\mathbf{W}^{(s)*} + \tau_b\mathbf{I}\right) & \text{if } \boldsymbol{\beta}^{(s)} \sim N(0,\tau_b) \end{cases}$$

$$\bar{\mu} = \bar{\Sigma} \times \left\{\mathbf{W}^{(s)*'}(\tilde{\mathbf{Y}}^{(s)*} - \boldsymbol{\omega}^{(s)*})\right\}.$$

- Note that we define $\omega_{it}^{(s)}$ at all combinations of i,t even if that site-season combination is not observed. Also note that the full conditional of $\omega_{it}^{(s)}$ only depends on observations which were observed and had $Z_{it}^{(s)} = 1$ since if the species was not actually there, then detection is undefined. Define I_{itj} to be an indicator that a survey was conducted at site i during season t. Observe that the latent detection process for all site season combinations (observed and unobserved) is then a mixture distribution such that $[\tilde{Y}_{itj}^{(s)} \mid Z_{it}^{(s)}, \omega_{it}^{(s)}, \boldsymbol{\beta}^{(s)}] = Z_{it}^{(s)}\mathrm{I}_{itj}\phi\left(\tilde{Y}_{itj}^{(s)} - \mathbf{W}_{itj}^{(s)'}\boldsymbol{\beta}^{(s)} - \omega_{it}^{(s)}\right) + (1 - Z_{it}^{(s)}\mathrm{I}_{itj})\delta_0$, where $\phi(\cdot)$ is the standard normal probability density function. We will update the n-dimensional vector $\boldsymbol{\omega}_t^{(s)}$ for each t,s.

- For $t = 1$, observe

$$\begin{aligned}
\left[\boldsymbol{\omega}_1^{(s)} \mid \cdot\right] &\propto \left\{ Z_{it}^{(s)} \mathrm{I}_{itj} \prod_{j=1}^{J_{it}} [\tilde{Y}_{itj}^{(s)} \mid \cdot] + (1 - Z_{it}^{(s)} \mathrm{I}_{itj}) \delta_0 \right\} \\
&\quad \times [\boldsymbol{\omega}_1^{(s)} \mid \boldsymbol{\omega}_1^{(-s)}][\boldsymbol{\omega}_2^{(s)} \mid \boldsymbol{\omega}_1^{(s)}, \boldsymbol{\omega}_1^{(-s)}, \boldsymbol{\omega}_2^{(-s)}] \\
&\propto \exp\left(-\frac{1}{2} \sum_j \mathbf{Z}_1 \mathrm{I}_{1j} (\tilde{\mathbf{Y}}_{1j}^{(s)} - \mathbf{W}_{1j} \boldsymbol{\beta}^{(s)} - \boldsymbol{\omega}_1^{(s)})' (\tilde{\mathbf{Y}}_{1j}^{(s)} - \mathbf{W}_{1j} \boldsymbol{\beta}^{(s)} - \boldsymbol{\omega}_1^{(s)}) \right) \\
&\quad \times [\boldsymbol{\omega}_1^{(s)} \mid \boldsymbol{\omega}_1^{(-s)}][\boldsymbol{\omega}_2^{(s)} \mid \boldsymbol{\omega}_1^{(s)}, \boldsymbol{\omega}_1^{(-s)}, \boldsymbol{\omega}_2^{(-s)}]
\end{aligned}$$

Note that from the conditional property of the multivariate normal distribution, $[\boldsymbol{\omega}_1^{(s)} \mid \boldsymbol{\omega}_1^{(-s)}]$ is the pdf of a $N(\boldsymbol{\omega}_1^{(-s)} \Sigma_{-s,-s}^{-1} \Sigma_{-s,s}, (\Sigma_{s,s} - \Sigma_{s,-s} \Sigma_{-s,-s}^{-1} \Sigma_{-s,s}) \mathrm{I}_n)$, and $[\boldsymbol{\omega}_2^{(s)} \mid \boldsymbol{\omega}_1^{(s)}, \boldsymbol{\omega}_1^{(-s)}, \boldsymbol{\omega}_2^{(-s)}]$ is also a normal pdf with mean $\tilde{\rho}^{(s)} \boldsymbol{\omega}_1^{(s)} + (\boldsymbol{\omega}_2^{(-s)} - (\mathbf{1}_n \otimes \tilde{\boldsymbol{\rho}}^{(-s)}) \boldsymbol{\omega}_1^{(-s)}) \Sigma_{-s,-s}^{-1} \Sigma_{-s,s}$ and covariance $(\Sigma_{s,s} - \Sigma_{s,-s} \Sigma_{-s,-s}^{-1} \Sigma_{-s,s}) \mathrm{I}_n$. In the above, we use $-s$ as a subscript or superscript to indicate the rows/columns that exclude s. Also, Σ refers to Σ_ω, but the subscript is dropped here for convenience. Let $\Sigma_{s|-s} = \Sigma_{s,s} - \Sigma_{s,-s} \Sigma_{-s,-s}^{-1} \Sigma_{-s,s}$ Simplifying yields that when $t = 1$, the full conditional distribution of $\boldsymbol{\omega}_1^{(s)}$ is $[\boldsymbol{\omega}_1^{(s)} \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\Sigma}^{-1} = \mathrm{diag}(\mathbf{Z}_1 \sum_j \mathrm{I}_{1j}) + (1 + \tilde{\rho}^{(s)2}) \Sigma_{s|-s}^{-1} \mathrm{I}_n$$

and

$$\begin{aligned}
\bar{\mu} = \bar{\Sigma} \Bigg\{ & \mathbf{Z}_1 \sum_j \mathrm{I}_{1j} \left(\tilde{\mathbf{Y}}_{1j}^{(s)} - \mathbf{W}_{1j} \boldsymbol{\beta}^{(s)} \right) + \Sigma_{s|-s}^{-1} (\Sigma'_{-s,s} \Sigma_{-s,-s}^{-1} \otimes \mathrm{I}_n) \boldsymbol{\omega}_1^{(-s)} \\
& + \tilde{\rho}^{(s)} \Sigma_{s|-s}^{-1} \left[\boldsymbol{\omega}_2^{(s)} - (\Sigma'_{-s,s} \Sigma_{-s,-s}^{-1} \otimes \mathrm{I}_n) \left(\boldsymbol{\omega}_2^{(-s)} - \tilde{\rho}^{(-s)} \boldsymbol{\omega}_1^{(-s)} \right) \right] \Bigg\}
\end{aligned}$$

- Similarly, for $t = 2, ..., T-1$ we get that the full conditional distribution of $\boldsymbol{\omega}_t^{(s)}$ is $[\boldsymbol{\omega}_t^{(s)} \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\Sigma}^{-1} = \mathrm{diag}(\mathbf{Z}_t \sum_j \mathrm{I}_{tj}) + (1 + \tilde{\rho}^{(s)2}) \Sigma_{s|-s}^{-1} \mathrm{I}_n$$

and

$$\begin{aligned}
\bar{\mu} = \bar{\Sigma} \Bigg\{ & \mathbf{Z}_t \sum_j \mathrm{I}_{tj} \left(\tilde{\mathbf{Y}}_{tj}^{(s)} - \mathbf{W}_{tj} \boldsymbol{\beta}^{(s)} \right) \\
& + \Sigma_{s|-s}^{-1} \left(\tilde{\rho}^{(s)} \boldsymbol{\omega}_{t-1}^{(s)} + (\Sigma'_{-s,s} \Sigma_{-s,-s}^{-1} \otimes \mathrm{I}_n) \left(\boldsymbol{\omega}_t^{(-s)} - \tilde{\rho}^{(-s)} \boldsymbol{\omega}_{t-1}^{(-s)}) \right) \right)
\end{aligned}$$

$$+ \tilde{\rho}^{(s)}\Sigma_{s|-s}^{-1}\left[\boldsymbol{\omega}_{t+1}^{(s)} - (\Sigma'_{-s,s}\Sigma_{-s,-s}^{-1} \otimes \mathrm{I}_n)\left(\boldsymbol{\omega}_2^{(-s)} - \tilde{\rho}^{(-s)}\boldsymbol{\omega}_t^{(-s)}\right)\right]\Bigg\}$$

- For $t = T$ the full conditional distribution of $\boldsymbol{\omega}_T^{(s)}$ is $[\boldsymbol{\omega}_T^{(s)} \mid \cdot] = N(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\Sigma}^{-1} = \mathrm{diag}(\mathbf{Z}_T \sum_j \mathrm{I}_{Tj}) + \Sigma_{s|-s}^{-1}\mathrm{I}_n$$

and

$$\bar{\mu} = \bar{\Sigma}\Bigg\{\mathbf{Z}_t \sum_j \mathrm{I}_{tj}\left(\tilde{\mathbf{Y}}_{tj}^{(s)} - \mathbf{W}_{tj}\boldsymbol{\beta}^{(s)}\right)$$
$$+ \Sigma_{s|-s}^{-1}\left(\tilde{\rho}^{(s)}\boldsymbol{\omega}_{T-1}^{(s)} + (\Sigma'_{-s,s}\Sigma_{-s,-s}^{-1} \otimes \mathrm{I}_n)\left(\boldsymbol{\omega}_T^{(-s)} - \tilde{\rho}^{(-s)}\boldsymbol{\omega}_{T-1}^{(-s)})\right)\right)\Bigg\}$$

- Our model assumes the detection propagator matrix is of the form $\tilde{\boldsymbol{\rho}}_i = \mathrm{diag}(\tilde{\rho}_i^{(1)}, ..., \tilde{\rho}_i^{(S)})$. We update this S-dimensional vector jointly for each i. Let $D(\boldsymbol{\omega}_{i,t-1}) = \mathrm{diag}(\omega_{i,t-1}^{(1)}, ..., \omega_{i,t-1}^{(S)})$. be the $S \times S$ diagonal matrix. The full conditional distribution of the vectorized diagonal elements in $\tilde{\boldsymbol{\rho}}_i$ is truncated multivariate normal $[\tilde{\boldsymbol{\rho}} \mid \cdot] = N\left(\bar{\mu}, \bar{\Sigma}\right)$, where

$$\bar{\Sigma}^{-1} = \sum_{t=2}^{T} D(\boldsymbol{\omega}_{i,t-1})'\Sigma_{\omega}^{-1}D(\boldsymbol{\omega}_{i,t-1})$$

$$\bar{\mu} = \bar{\Sigma} \times \left(\sum_{t=2}^{T} D(\boldsymbol{\omega}_{i,t-1})'\Sigma_{\omega}^{-1}\boldsymbol{\omega}_{i,t}\right)$$

- Assume the prior for Σ_ω is inverse Wishart with parameters ν and Ψ. The full conditional distribution of Σ_ω is inverse Wishart with parameters $\nu + nT$ and $\Psi + \sum_{t=1}^{T}\mathbf{B}_t$, where $\mathbf{B}_t$ is the $S \times S$ matrix with $(i,j)th$ entry $(B_t)_{i,j} = \mathrm{trace}\left(\mathbf{C}_{i,j}^{(t)}\right)$, where for $t = 1$, $\mathbf{C}^{(1)}$ is the block matrix with (i,j)th block

$$\mathbf{C}_{i,j}^{(1)} = (\boldsymbol{\omega}_1^{(i)})(\boldsymbol{\omega}_1^{(j)})'$$

for $i, j = 1, ..., S$, and for $t = 2, ..., T$ we have the block matrix with (i,j)th block

$$\mathbf{C}_{i,j}^{(t)} = (\boldsymbol{\omega}_t^{(i)} - \tilde{\boldsymbol{\rho}}^{(i)}\boldsymbol{\omega}_{t-1}^{(i)})(\boldsymbol{\omega}_t^{(j)} - \tilde{\boldsymbol{\rho}}^{(j)}\boldsymbol{\omega}_{t-1}^{(j)})',$$

where $\boldsymbol{\omega}_t^{(i)} = (\omega_{1t}^{(i)}, ..., \omega_{nt}^{(i)})$ and $\tilde{\boldsymbol{\rho}}^{(i)} = (\tilde{\rho}_1^{(i)}, ..., \tilde{\rho}_n^{(i)})$.

References

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