Supplementary Information

This supplementary document describes in detail the analytical expressions for the pitchfork bifurcation point and the saddle-node bifurcation point after mass perturbation. Their frequency shifts are utilized as sensitivity outputs when performing single-mass, dual-mass, or even multi-mass detection.

S1: Analytical expression of pitchfork bifurcation point after mass perturbation

$$\frac{8\omega_1\omega_2\omega_3\begin{pmatrix}\omega_1\kappa_{c11}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_1^2\right)\left(\left(\Delta m_2\omega_2+\left(\Omega_{PF}-\omega_1\right)\right)\left(2\left(\Omega_{PF}-\omega_1\right)\right)-4\mu_2\mu_3\right)\\+\omega_3\kappa_{c3}\kappa_{c7}\left(\left(2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_3^2\right)\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)\left(\Delta m_2\omega_2+\left(\Omega_{PF}-\omega_1\right)\right)-4\mu_1\mu_2\right)\end{pmatrix}}{\begin{matrix}2\omega_3\omega_1\kappa_{c11}\kappa_{c3}\kappa_{c7}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)\left(2\left(\Omega_{PF}-\omega_1\right)\right)+4\mu_1\mu_3\right)\\+\omega_1^2\kappa_{c11}^2\kappa_{c8}^2\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_1^2\right)+\omega_3^2\kappa_{c3}^2\kappa_{c7}^2\left(\left(2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_3^2\right)\end{matrix}}=$$

$$4\sqrt{\frac{\omega_3^2\begin{pmatrix}4\omega_1^2\omega_2^2\begin{pmatrix}\omega_1\kappa_{c11}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_1^2\right)\left(\left(\Delta m_2\omega_2+\left(\Omega_{PF}-\omega_1\right)\right)\left(2\left(\Omega_{PF}-\omega_1\right)\right)-4\mu_2\mu_3\right)\\+\omega_3\kappa_{c3}\kappa_{c7}\left(\left(2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_3^2\right)\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)\left(\Delta m_2\omega_2+\left(\Omega_{PF}-\omega_1\right)\right)-4\mu_1\mu_2\right)\end{pmatrix}^2\\-\left(\left(2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_3^2\right)\left(f_1^2\kappa_{c7}^2-4\omega_1^2\omega_2^2\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_1^2\right)\left(\left(\Delta m_2\omega_2+\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_2^2\right)\right)\\\begin{pmatrix}-2\omega_3\omega_1\kappa_{c11}\kappa_{c3}\kappa_{c7}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)\left(2\left(\Omega_{PF}-\omega_1\right)\right)+4\mu_1\mu_3\right)\\+\omega_1^2\left(-\kappa_{c11}^2\right)\kappa_{c8}^2\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_1^2\right)-\omega_3^2\kappa_{c3}^2\kappa_{c7}^2\left(\left(2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_3^2\right)\end{pmatrix}\end{pmatrix}}{\begin{pmatrix}2\omega_3\omega_1\kappa_{c11}\kappa_{c3}\kappa_{c7}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)\left(2\left(\Omega_{PF}-\omega_1\right)\right)+4\mu_1\mu_3\right)\\+\omega_1^2\kappa_{c11}^2\kappa_{c8}^2\left(\left(\Delta m_1\omega_1+2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_1^2\right)+\omega_3^2\kappa_{c3}^2\kappa_{c7}^2\left(\left(2\left(\Omega_{PF}-\omega_1\right)\right)^2+4\mu_3^2\right)\end{pmatrix}^2}}$$

(S,1)

S2: Analytical expression of saddle-node bifurcation point after mass perturbation

$$f_1^2\kappa_{c7}^2\left(\left(\Delta m_3\omega_3+2\left(\Omega_{SN}-\omega_1\right)\right)^2+4\mu_3^2\right)=$$

$$\frac{16\omega_1^2\omega_2^2\begin{pmatrix}\omega_1\kappa_{c11}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{SN}-\omega_1\right)\right)^2+4\mu_1^2\right)\left(\mu_3\left(\Delta m_2\omega_2+\left(\Omega_{SN}-\omega_1\right)\right)+\mu_2\left(\Delta m_3\omega_3+2\left(\Omega_{SN}-\omega_1\right)\right)\right)\\+\omega_3\kappa_{c3}\kappa_{c7}\left(\left(\Delta m_3\omega_3+2\left(\Omega_{SN}-\omega_1\right)\right)^2+4\mu_3^2\right)\left(\mu_2\left(\Delta m_1\omega_1+2\left(\Omega_{SN}-\omega_1\right)\right)+\mu_1\left(\Delta m_2\omega_2+\left(\Omega_{SN}-\omega_1\right)\right)\right)\end{pmatrix}^2}{\begin{matrix}2\omega_3\omega_1\kappa_{c11}\kappa_{c3}\kappa_{c7}\kappa_{c8}\left(\left(\Delta m_1\omega_1+2\left(\Omega_{SN}-\omega_1\right)\right)\left(\Delta m_3\omega_3+2\left(\Omega_{SN}-\omega_1\right)\right)+4\mu_1\mu_3\right)\\+\omega_1^2\kappa_{c11}^2\kappa_{c8}^2\left(\left(\Delta m_1\omega_1+2\left(\Omega_{SN}-\omega_1\right)\right)^2+4\mu_1^2\right)+\omega_3^2\kappa_{c3}^2\kappa_{c7}^2\left(\left(\Delta m_3\omega_3+2\left(\Omega_{SN}-\omega_1\right)\right)^2+4\mu_3^2\right)\end{matrix}}$$

(S,2)