Supplementary Information: Quantum Thermodynamics applied for Quantum Refrigerators cooling down a qubit

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I. REALIZATION OF REFRIGERATOR II WITH SUPERCONDUCTING FLUX QUBITS

Our proposal is inspired by experiments [1, 2] where a flux qubit (FQ) and two level system (TLS) interacted with each other and the FQ relaxed quickly when the resonant frequencies of the FQ and TLS matched. Therefore, we may use a FQ as Qubit 2 and a TLS as Qubit 3. We notice, however, that we need more controllability for Qubit 3 than that of a TLS and thus we consider to realize our proposal with two FQs as Qubit 2 and 3. Thus, ω_2 is the order of GHz.

From now, we set $\omega_2/2\pi = 1.0$ GHz. A target qubit (Qubit 1) to be cooled down is assumed to be an electron spin [3] so that we can improve the ESR sensitivity. The coupling strength between the flux qubit and an electron spin can be as large as tens of kHz [4, 5]. The coupling strength between the flux qubits is an order of MHz [6]. The driving strength of the flux qubit can be as large as hundreds of MHz [7]. The energy relaxation time of the flux qubits can be as short as tens of nano seconds [8], while it can be an order of a few micro seconds [9], depending on the circuit design. The parameters listed in Table 1 correspond to $\omega_1/2\pi = 0.1$ GHz, $\lambda/2\pi = 0.1$ GHz, $g_1/2\pi = 50$ kHz, $g_3/2\pi = 0.8 \sim 4$ MHz, $\gamma_2/2\pi = 0.1$ MHz, and $\gamma_3/2\pi = 10$ MHz. Thus, they are realistic even in the current technology.

We show that an energy can be transferred from an electron spin (Qubit 1) to FQ 2 (Qubit 3) via FQ 1 (Qubit 2) by satisfying the resonant condition. We will show the resonant condition below. We assume that the interaction between FQ 1 and 2 and also that between FQ 1 and an electron spin are inductive. Therefore, the components of the total Hamiltonian $H_{\mathrm{C}}(t)$ is given as follows [10, 11]:

$$H_1 = \frac{\omega_1}{2}\sigma_{1z}, \tag{I.1}$$

$$H_2(t) = \frac{\omega_2'}{2}\sigma_{2z} + \frac{\Delta_2}{2}\sigma_{2x} + \lambda\sigma_{2y}\cos(\omega t), \tag{I.2}$$

$$H_3 = \frac{\omega_3'}{2}\sigma_{3z} + \frac{\Delta_3}{2}\sigma_{3x}, \tag{I.3}$$

$$H_{12} = g_1'\sigma_{1x}\sigma_{2z}, \tag{I.4}$$

$$H_{23} = g_3'\sigma_{2z}\sigma_{3z}, \tag{I.5}$$

where ω_1 is now the Zeeman energy of the electron spin, ω_2' (ω_3') is an energy bias of FQ 1 (FQ 2), Δ_2 (Δ_3) the gap energy of FQ 1 (FQ 2), λ the Rabi frequency of FQ 1, g_1' the inductive coupling strength between the electron spin and FQ 1, and g_3' between FQ 1 and FQ 2. We choose $\omega_3' = 0$ by switching off the magnetic field applied to FQ 2.

First, we diagonalize $\frac{\omega_2'}{2}\sigma_{2z} + \frac{\Delta_2}{2}\sigma_{2x}$ by $U = \exp(i\theta\sigma_{2y}/2)$ with $\theta = \tan^{-1}(\Delta_2/\omega_2')$. We obtain

$$H_{\mathrm{C}}'(t) = H_1 + H_2'(t) + H_3 + H_{12}' + H_{23}', \tag{I.6}$$

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where

$$H_2'(t) = \frac{\omega_2}{2}\sigma_{2z} + \lambda\sigma_{2y}\cos(\omega_2 t), \tag{I.7}$$

$$H_{12}' = \frac{g_1'}{\omega_2}\sigma_{1x}\left(\omega_2'\sigma_{2z} - \Delta_2\sigma_{2x}\right), \tag{I.8}$$

$$H_{23}' = \frac{g_3'}{\omega_2}\left(\omega_2'\sigma_{2z} - \Delta_2\sigma_{2x}\right)\sigma_{3z}. \tag{I.9}$$

Here, $\omega_2 = \sqrt{\omega_2'^2 + \Delta_2^2}$. Note that the second term of Eq. (I.8) and the first term in Eq. (I.9) can be neglected owing to the rotating wave approximation (RWA). Then, by some appropriate renaming such as $\sigma_{3z} \leftrightarrow \sigma_{3x}$, we obtain the correspondence to the Hamiltonian in the main text.

We move to the rotating frame defined by the unitary transformation $U_1(t) = \exp(i\omega_2 t\sigma_{2z}/2)\exp(i\omega_2 t\sigma_{3x}/2)$.

$$\begin{aligned}
H_{\mathrm{C}}'' = &\frac{\omega_1}{2}\sigma_{1z} + \lambda\left(\cos(\omega_2 t)\sigma_{2y} + \sin(\omega_2 t)\sigma_{2x}\right)\cos(\omega_2 t) \\
&+ \frac{\Delta_3 - \omega_2}{2}\sigma_{3x} \\
&+ \frac{g_1'}{\omega_2}\sigma_{1x}\left(\omega_2'\sigma_{2z} - \Delta_2\left(\cos(\omega_2 t)\sigma_{2x} - \sin(\omega_2 t)\sigma_{2y}\right)\right) \\
&+ \frac{g_3'}{\omega_2}\left(\omega_2'\sigma_{2z} - \Delta_2\left(\cos(\omega_2 t)\sigma_{2x} - \sin(\omega_2 t)\sigma_{2y}\right)\right) \\
&\times\left(\cos(\omega_2 t)\sigma_{3z} + \sin(\omega_2 t)\sigma_{3y}\right).
\end{aligned}$$

Then, by using the RWA and renaming $g_1'\omega_2'/\omega_2 = g_e$, $-g_3'\Delta_2/(2\omega_2) = g_{\mathrm{FQ}}$, we obtain

$$\begin{aligned}
H_{\mathrm{C}}'' \simeq &\frac{\omega_1}{2}\sigma_{1z} + \frac{\lambda}{2}\sigma_{2y} + \frac{\Delta_3 - \omega_2}{2}\sigma_{3x} \\
&+ g_e\sigma_{1x}\sigma_{2z} + g_{\mathrm{FQ}}\left(\sigma_{2x}\sigma_{3z} - \sigma_{2y}\sigma_{3y}\right).
\end{aligned}$$

After renaming $\sigma_{2x} \to \sigma_{2y}, \sigma_{2y} \to \sigma_{2z}, \sigma_{2z} \to \sigma_{2x}$ and $\sigma_{3x} \to \sigma_{3z}, \sigma_{3y} \to \sigma_{3x}, \sigma_{3z} \to \sigma_{3y}$, we impose the resonant conditions $\Delta_3 - \omega_2 = \omega_1 = \lambda$. Then, we obtain

$$\begin{aligned}
H_{\mathrm{C}}''' = &\frac{\lambda}{2}\sigma_{1z} + \frac{\lambda}{2}\sigma_{2z} + \frac{\lambda}{2}\sigma_{3z} \\
&+ g_e\sigma_{1x}\sigma_{2x} + g_{\mathrm{FQ}}\left(\sigma_{2y}\sigma_{3y} - \sigma_{2z}\sigma_{3x}\right).
\end{aligned}$$

Let us move to another rotating frame defined by the unitary transformation $U_2(t) = \exp\left(\frac{i\lambda t}{2}\sigma_{1z}\right)\exp\left(\frac{i\lambda t}{2}\sigma_{2z}\right)\exp\left(\frac{i\lambda t}{2}\sigma_{3z}\right)$ and we apply the RWA, then we obtain

$$H_{\mathrm{C}}'''' \simeq g_e\left(\sigma_{1-}\sigma_{2+} + \sigma_{1+}\sigma_{2-}\right) + g_{\mathrm{FQ}}\left(\sigma_{2-}\sigma_{3+} + \sigma_{2+}\sigma_{3-}\right), \tag{I.10}$$

where $\sigma_{i\pm} = \frac{\sigma_{ix} \pm i\sigma_{iy}}{2}$. Eq. (I.10) implies that energy can be transferred from the electron spins to FQ 2 via FQ 1.

II. THE WORK FOR INITIALIZING QUBIT 2

We calculate the work W_{ini} necessary for initializing Qubit 2. The procedure of the initialization consists of two steps: (1) the projection measurement with σ_{2z}, (2) the conditional operation $\pm\pi/2$ rotation along the x-axis according to the projective measurement result of ±1. Then, we obtain $|-_y\rangle\langle -_y|$.

We assume that a required work for the projective measurement is negligible compared to the other work. It is not straightforward to quantify the required work for the projective measurements. It strongly depends on how we experimentally realize the projective measurements. Anyway if we include the work of projective measurements, the performance of Refrigerator I becomes worse, while the performance of Refrigerator II is almost the same. Even in such a case the COP of Refrigerator II is still better than that of Refrigerator II, and thus the superiority of the Refrigerator II over Refrigerator I is unchanged.

Then, the work of the $\frac{\pi}{2}$ rotation is regarded as W_{ini}. The time required for the initialization is assumed to be much shorter than any other dynamics, such as relaxation, and thus we only consider $H_2(t) = \frac{\omega_2}{2}\sigma_{2z} + \lambda'\cos\left(\omega_2 t + \frac{\pi}{2}\right)\sigma_{2y}$.

The phase $\frac{\pi}{2}$ is introduced so that we can initialize the state $|0\rangle$ ($|1\rangle$) to $|-_y\rangle$. This Hamiltonian is approximately reduced to $-\frac{\lambda'}{2}\sigma_{2x}$ in the rotating frame whose frequency is ω_2. In this step, Qubit 2 (FQ1) is not necessary to be spin-locked. To emphasize this, we use another parameter λ'.

From now, we only consider Qubit 2 and omit lower indices of the Pauli matrices. If the initial state is assumed as $\rho(0) = |0\rangle\langle 0|$, the dynamics of Qubit 2 is given as

$$\begin{aligned}\rho(t) &= e^{i\frac{\lambda' t}{2}\sigma_x}\,|0\rangle\langle 0|\,e^{-i\frac{\lambda' t}{2}\sigma_x} \\ &= \frac{I - \cos(\lambda' t)\sigma_z - \sin(\lambda' t)\sigma_x}{2}.\end{aligned} \tag{II.1}$$

From the above, the required initialization time is $\frac{\pi}{2\lambda'}$. According to the work rate defined by Eq. (27), W_{ini} is given as,

$$W_{\mathrm{ini}} \simeq \int_0^{\frac{\pi}{2\lambda'}} dt'\,\mathrm{Tr}\left(\rho(t')\left(-\frac{\lambda'\omega_2}{2}\sigma_x\right)\right) = \frac{\omega_2}{2}, \tag{II.2}$$

under the rotating wave approximation. Note that W_{ini} is independent of λ'. We can easily see that W_{ini} for $\rho(0) = |1\rangle\langle 1|$ is also $\frac{\omega_2}{2}$.

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