Additional information

Appendix

Global stability of the disease-free equilibrium point $E_0 = (T_0, T_0^*, v_0)$ for the within-host dynamics (fast subsystem) given by system (4) with $g(y) = 0$.

Proof. Let $c(d+m) > kp(\frac{\lambda}{m})$, i.e. $R_{0v} < 1$ and $S_0 = \frac{p}{d+m}$, then the critical point E_0 is globally asymptotically stable. In order to prove the asymptotic stability of E_0, consider the Lyapunov function

$$\mathscr{U}(T, T^*, v) = T_0\left(\frac{T}{T_0} - \ln\frac{T}{T_0}\right) + T^* + \frac{v}{S_0} = T - T_0\ln(T) + T_0\ln(T_0) + T^* + \frac{v}{S_0}.$$

Then from the last expression

$$\begin{aligned}
\frac{d\mathscr{U}}{dt} &= T' - \frac{T}{T_0}T' + T^{*'} + \frac{1}{S_0}v' \\
&= \lambda - kTv - mT - \frac{T_0}{T}(\lambda - kTv - mT) + kTv - (d+m)T^* + \\
&\quad \frac{1}{S_0}(pT^* - cv) \\
&= \lambda - mT - \frac{T_0}{T}\lambda + kT_0 v + mT_0 - (d+m)T^* \\
&\quad -\frac{1}{S_0}(cv) + \frac{1}{S_0}pT^* \\
&= \lambda + mT_0 - \frac{T_0}{T}\lambda - mT + kT_0 v - (d+\delta_2)T^* \\
&\quad -\frac{1}{S_0}(cv) + \frac{1}{S_0}pT^*
\end{aligned}$$

Substituting $mT_0 = \lambda$ in the second term, $m = \lambda/T_0$ in the forth term and $S_0 = p/(d+m)$ in the last one we get

$$\begin{aligned}
\frac{d\mathscr{U}}{dt} &= \lambda\left(2 - \frac{T_0}{T} - \frac{T}{T_0}\right) + kT_0 v - (d+m)T^* \\
&\quad -\frac{1}{S_0}(cv) + (d+m)T^* \\
&= \lambda\left(2 - \frac{T_0}{T} - \frac{T}{T_0}\right) + kT_0 v - \frac{1}{S_0}(cv)
\end{aligned}$$

A further simplification yields

$$\frac{d\mathscr{U}}{dt} = \lambda\left(2 - \frac{T_0}{T} - \frac{T}{T_0}\right) + \left(kT_0 - \frac{c}{S_0}\right)v < 0$$

The last inequality follows from the hypothesis and the inequality of the geometric and arithmetic means. □

Here we give the full expression of the within-host reproduction function $R_v(y)$ that appears in expression $v^*(y)$

$$R_v(y) = \frac{2T_0}{T_0(R_{0v}+1) + \frac{(d+m)ry}{mp} - \sqrt{\left(T_0(R_{0v}+1) + \frac{(d+m)ry}{mp}\right)^2 - \frac{4T_0^2}{R_{0v}}}}$$