Supplementary Material: Compact description of quantum phase slip junctions

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I. LOOP-CHARGE QUANTIZATION FOR CIRCUITS INVOLVING ONE AND TWO QPS JUNCTIONS

In the main text, we give the loop-charge Hamiltonians of the two circuits of Fig. 1 and Fig. 3. Here we briefly show how they can be derived. In both cases we will use Fig. 3 of Ref. [1] to build the Lagrangian.

The first circuit (Fig.1) consists of a QPS junction, an inductor and a gate voltage with a capacitance in a loop. The Lagrangian for this system reads

$$L^\circ = E_S \cos\left(2\pi N^\circ\right) + \frac{L}{2}\left(2e\right)^2 \dot{N}^{\circ 2} - \frac{\left(2e\right)^2}{2C_g} N^{\circ 2} - 2eV_g N^\circ \ . \tag{S1}$$

First we have to find the canonically conjugate momentum to the loop-charge N°, φ°,

$$\varphi^\circ = \frac{\partial L^\circ}{\partial \dot{N}^\circ} = L\left(2e\right)^2 \dot{N}^\circ \tag{S2}$$

Thus we can now find the Hamiltonian by means of the Legendre transformation

$$\begin{aligned} H^\circ =& \dot{N}^\circ \varphi^\circ - L^\circ \\ =& \frac{1}{2L}\left(\frac{\varphi^\circ}{2e}\right)^2 - E_S\cos\left(2\pi N^\circ\right) - \frac{C_g}{2}V_g^2 \\ &+ \frac{\left(2e\right)^2}{2C_g}\left(N^{\circ 2} + \frac{2C_g}{(2e)}V_g N^\circ + \left(\frac{C_g}{(2e)}V_g\right)^2\right) \ . \end{aligned} \tag{S3}$$

By setting $E_L = 1/\left(4e^2 L\right)$, $E_C = \left(2e\right)^2/2C_g$ and $N_g = C_g V_g/(2e)$ this leads us exactly to the Hamiltonian of Eq. (6).

The same calculation can be done for the circuit in Fig. 3, consisting of two QPS junction, two inductor and a gate voltage with a capacitance. Again we will follow the recipe of Ref. [1] to construct the Lagrangian

$$\begin{aligned} L^\circ =& E_{S1}\cos\left(2\pi N_1^\circ\right) + E_{S2}\cos\left(2\pi N_2^\circ\right) + \frac{L_1}{2}\left(2e\dot{N}_1^\circ\right)^2 \\ &+ \frac{L_2}{2}\left(2e\dot{N}_2^\circ\right)^2 - \frac{\left(2e\right)^2}{2C_g}\left(N_1^\circ - N_2^\circ\right)^2 - 2eV_g\left(N_1^\circ - N_2^\circ\right) \ . \end{aligned} \tag{S4}$$

The canonically conjugate phase φ_i° will be of the form of φ° in Eq. (S2) with $N^\circ \to N_i^\circ$ and $L \to L_i$. Finally we can construct the Hamiltonian again through

$$\begin{aligned} H^\circ =& \dot{N}_1^\circ \varphi_1^\circ + \dot{N}_2^\circ \varphi_2^\circ - L^\circ \\ =& \sum_{i\in\{1,2\}}\left[\frac{1}{L_i}\left(\frac{\varphi_i^\circ}{2e}\right)^2 - E_{Si}\cos\left(2\pi N_i^\circ\right)\right] \\ &+ \frac{\left(2e\right)^2}{2C_g}\left(N_1^\circ - N_2^\circ + N_g\right) + \text{const.} \end{aligned} \tag{S5}$$

This leads us to the Hamiltonian of Eq. (9).

II. FINITE ELEMENT MODEL AND DIFFERING CHARGE DEFINITIONS

In the main text, we revisit the low-energy physics of QPS junctions. In this section we illustrate some of the conclusion of the main text, by means of a finite element approach, when the wire is modelled by a discrete chain of Josephson junctions, see Fig. S1. We use this model also to explain how different basis choices for the low-energy states of the QPS junction correspond to the measurement or interaction with different charges N and N_q as hinted at in the main text.

For this purpose, we decompose the QPS wire into a finite but large number islands indexed by $j = 1, \ldots, J$, hosting charges and phases $[\widehat{N}_j, \widehat{\varphi}'_j] = i\delta_{jj'}$. Here the last phase $\varphi := \varphi_J$ is kept arbitrary but constant. The islands may be capacitively coupled (either among themselves or to ground) and exchange Cooper pairs formally via JJs. The ends of the resulting JJ array are going to be phase-biased due to a finite φ and choosing $\varphi_0 = 0$ (contact to ground). The total Hamiltonian is given as $H = T + V$, where we cast the entirety of the capacitive energy into the kinetic energy term

$$T = \frac{\left(2e\right)^2}{2}\sum_{j,k=1}\widehat{N}_j\left(C^{-1}\right)_{jk}\widehat{N}_k \ , \tag{S6}$$

and all Josephson energies into a so-called potential energy term

$$V = -E_J\sum_{j=1}^{J}\cos\left(\widehat{\varphi}_j - \widehat{\varphi}_{j-1}\right) \ . \tag{S7}$$

While this model is obviously precise if the QPS junction is actually made of a Josephson junction array, we believe that it should retain some validity in a continuous limit, i.e., for a thin wire.

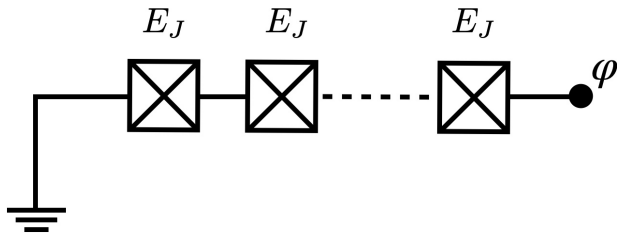

Figure S1. Josephson junction array. Note that the individual islands may be in general connected by a highly complex capacitive network. For our considerations the details of the capacitive coupling is irrelevant, which is why we do not explicitly show it here. Also note that if we use the JJ array model to simulate a bulk superconducting wire, then the circuit dynamics will likely be dominated by the capacitances to ground.

A. Derive low-energy Hamiltonian in Eq. (12)

As for the influence of the capacitive contribution to the circuit dynamics, we notice the following. With all Josephson energies the same, and at sufficiently low-energies, the potential energy (the Josephson-energy contributions of the JJ array) have well-defined local minima. From this perspective, the capacitances merely provide a mechanism whereby the system can perform quantum jumps between different local minima. These are the quantum phase slips. For our purposes, we do not need to concern ourselves with the details of the capacitance matrix, so long as they provide this quantum tunneling mechanism. Ultimately the capacitive energy will provide a means to estimate the energy associated to jumps between different minima, which has already been done for JJ arrays, see Refs. [2–4].

Let us therefore evaluate the energy minima of the potential energy. For this purpose, we may consider all phase variables as classical parameters $\widehat{\varphi}_j \to \varphi_j$. We can deploy the minimization scheme used in Refs. [5, 6]. For fixed $\varphi := \varphi_J$, and large J, the minimization for the remaining φ_j (with $j \neq 0, J$) results in

$$\varphi^{(0,f)}_{j=1\ldots J-1} = \frac{j}{J}\left(\varphi + 2\pi f\right) \mod 2\pi \ . \tag{S8}$$

In order to obtain the energy at the minima, we approximate the potential energy V (S7) for small phase differences $\varphi_j - \varphi_{j-1} \ll 1$ as

$$V \approx \text{const.} + \frac{E_J}{2}\sum_{j=0}^{J}\left(\varphi_j - \varphi_{j-1}\right)^2 \ . \tag{S9}$$

Importantly, due to the potential energy being periodic in all phases, the minima can be defined up to modulo 2π. This corresponds to the compact representation of the local minima, see Fig. 5b in the main text. Plugging in (S8) into the approximated potential energy will yield

$$V \approx \frac{E_J}{2}\frac{1}{J}\left(\varphi + 2\pi f\right)^2 \tag{S10}$$

The first quadratic term corresponds to the inductive energy appearing in the main text in Eq. (12). The Josephson junction array has an effective array inductance $E_L \equiv E_J/J$. In the continuous limit we find $E_L \to 1/(4e^2 lL)$, with the array (wire) length $L = \Delta x J$, where Δx is the distance between two junctions. The quantum phase slips between different minima f are in the main text described by the operators $e^{\pm i\widehat{S}} \equiv \sum_f |f \pm 1\rangle_{\varphi_J}\langle f|_{\varphi_J}$. The feature usually neglected is the φ-dependence of the f-basis. Namely, we can think of the minima solutions as wave functions centered around the minima. This gives rise to the φ-dependent QPS operators.

B. Elimination of local Berry connection

In the main text, we argued that locally, we can neglect the phase dependence, if the Berry connection term $-iU\partial_\varphi U^\dagger$ vanishes. This is approximately true when the Josephson energies are sufficiently large with respect to the capacitive energies. Then, the wave functions of the local minima resemble the local ground states of the harmonic oscillator chain. Applying the derivative ∂_φ maps the local ground states to excited states, such that in the low-energy sector, where we keep only the local ground states, we indeed find $-iU\partial_\varphi U^\dagger \approx 0$. This is however only approximately true. We see that if we included the full Hilbert space with the excited states, the φ-dependence of the basis will start to play a role. This will be examined in more detail in future work.

C. Local versus non-local charge definition

As we discussed in the main text, we believe that neglecting charge quantization in QPS junctions is intimately connected to neglecting the phase dependence of the QPS basis $|f\rangle$. Here we want to elucidate this point a bit more explicitly. This discussion is inspired by the findings of Ref. [6].

As already stated above, we can think of the minima given in Eq. (S8) as the reference points around which the wave function $|f\rangle$ is centered. Since the minima in Eq. (S8) depend on the total phase difference φ between the two contacts linked by the QPS junction, the basis of $\widehat{f}$ is likewise φ-dependent, $|f\rangle_\varphi$. As pointed out in the main text, this phase dependence of the minima can be removed by a unitary transformation, $U(\varphi) = \sum_f |\widetilde{f}\rangle\langle f|_\varphi$.

In the space of the individual phases φ_j, this unitary

is simply a translation operator,

$$U = \prod_{j=1}^{J-1} e^{-i\frac{j}{J}\varphi\widehat{N}_j} \ . \tag{S11}$$

This realization allows us now to formulate the two different charges. Originally, φ is only defined on the rightmost contact. Its conjugate charge, here denoted as N_q is guaranteed to have integer eigenvalues in the finite element model given by the Hamiltonian in Eqs. (S6) and (S7), because it is assigned to a sharply defined region, see also the arguments in Ref. [6]. In fact, N_q is the charge that should enter the Hamiltonian, if we embed the QPS junction into a larger circuit, e.g., shunting it with a Josephson junction element or simply a capacitive element, as in Fig. 1b. The charge emerging from the unitary transformation on the other hand is given by

$$\widehat{N} = \widehat{N}_q + iU^\dagger\partial_\varphi U = \widehat{N}_q + \sum_{j=1}^{J-1}\frac{j}{J}\widehat{N}_j \ . \tag{S12}$$

We now see very explicitly, that while each local charge within the array, $\widehat{N}_j$, has again integer eigenvalues, the transformed charge $\widehat{N}$ does not, due to the prefactor $j/J < 1$ (obviously, a rational number, which, in the continuum limit $J \to \infty$ approaches an irrational value). As a matter of fact, the transformed charge $\widehat{N}$ can be understood as a "fuzzy" measurement of the charge, with a support falling off linearly along the JJ-array, see Fig 5c in the main text. Importantly, as we argue in the main text, when removing the high-energy state, we can locally flatten the basis. That is, due to the low-energy approximation, the originally quantized charge N_q is removed from the dynamics, and the fuzzy charge N takes its place. Nonetheless, the compactness of the phase φ remains important, as it fixes the total number of available low-energy states (that is, the number of local minima).

III. SOLUTION FOR TWO QPS JUNCTIONS

In principle the compactness condition in the main text (that is, the periodic constraints in Eqs. (13) and (15), for circuits including one or more QPS junctions) in conjunction with the standard node-flux are sufficient to compute the correct energy eigenspectra. The (superfluous) larger Hilbert space is carried around for the full calculation of the eigenvectors and eigenenergies of the Schrödinger equation, and can be reduced by imposing the periodic constraints. However, our new treatment also allows to calculate right from the start with the reduced Hilbert space, allowing for significant reduction of the computational effort.

As an example we will discuss here a system consisting of two QPS-junctions connected in series with a gate voltage in between (Fig. 3), with the Hamiltonian given in the main text in Eq. (14). This Hamiltonian can be readily solved numerically by discretizing φ-space and imposing the boundary condition Eq. (15) on the discrete lattice. There is a particularly simple regime, for $E_{Si} > E_{Li}$ $(i = 1, 2)$ where we can approximate the effect of the two QPS junctions as effective Josephson junction with a Josephson energy given in Eq. (7). Thus we find a new effective Hamiltonian of the form

$$\begin{aligned} H_{\text{eff}} = -E_{J,\text{eff}_1}\cos\left(\widehat{\varphi}\right) - E_{J,\text{eff}_2}\cos\left(\widehat{\varphi} - \phi_{\text{ext}}\right) \\ +E_C\left(\widehat{N} + N_g\right)^2 \ . \end{aligned} \tag{S13}$$

This is straightforwardly mapped onto a regular charge qubit Hamiltonian, whose solutions are known, via the Mathieu functions [7]. These solutions give rise to the energy spectrum for the compact node-flux approach, see third panel in Fig. 7. The corresponding standard node-flux and loop-charge results (first and second panels) can be obtained by replacing N_g, respectively, ϕ_{ext} by Bloch vectors k, and plotting the resulting energy band widths.

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