

Limit theorems of occupation times of high-dimensional normalized binary contact path processes

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Abstract: The binary contact path process (BCPP) introduced in Griffeath (1983) describes the spread of an epidemic on a graph and is an auxiliary model in the study of improving upper bounds of the critical value of the contact process. In this paper, we are concerned with limit theorems of the occupation time of a normalized version of the BCPP (NBCPP) on a lattice. We first show that the law of large numbers of the occupation time process is driven by the identity function when the dimension of the lattice is at least 3 and the infection rate of the model is sufficiently large conditioned on the initial state of the NBCPP being distributed with a particular invariant distribution. Then we show that the centred occupation time process of the NBCPP converges in finite dimensional distributions to a Brownian motion when the dimension of the lattice and the infection rate of the model are sufficiently large and the initial state of the NBCPP is distributed with the same invariant distribution as that in the law of large numbers.

Keywords: binary contact path process, occupation time, central limit theorem.

1 Introduction

In this paper we are concerned with the normalized binary contact path process (NBCPP). For later use, we first introduce some notations. For $d \geq 1$, the d -dimensional lattice is denoted by $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when they are neighbors. The origin of $\mathbb{Z}^d$ is denoted by O . Now we recall the definition of the binary contact path process (BCPP) introduced in [4] by Griffeath. The binary contact path process $\{\phi_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ is a continuous-time Markov process with state space $\mathbb{X} = \{0, 1, 2, \dots\}^{\mathbb{Z}^d}$ and evolves as follows. For any $x \in \mathbb{Z}^d$, $x \sim y$ and $t \geq 0$,

$$\phi_t(x) \rightarrow \begin{cases} 0 & \text{at rate } \frac{1}{1+2\lambda d}, \\ \phi_t(x) + \phi_t(y) & \text{at rate } \frac{\lambda}{1+2\lambda d}, \end{cases}$$

where λ is a positive constant. As a result, the generator Ω of $\{\phi_t\}_{t \geq 0}$ is given by

$$\Omega f(\phi) = \frac{1}{1+2\lambda d} \sum_{x \in \mathbb{Z}^d} (f(\phi^{x,-}) - f(\phi)) + \frac{\lambda}{1+2\lambda d} \sum_{x \in \mathbb{Z}^d} \sum_{y \sim x} (f(\phi^{x,y}) - f(\phi))$$

for any f from $\mathbb{X}$ to $\mathbb{R}$ depending on finite coordinates and $x \in \mathbb{Z}^d$, where

$$\phi^{x,-}(z) = \begin{cases} \phi(z) & \text{if } z \neq x, \\ 0 & \text{if } z = x \end{cases}$$

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and

$$\phi^{x,y}(z) = \begin{cases} \phi(z) & \text{if } z \neq x, \\ \phi(x) + \phi(y) & \text{if } z = x \end{cases}$$

for all $z \in \mathbb{Z}^d$.

Intuitively, $\{\phi_t\}_{t \geq 0}$ describes the spread of an epidemic on $\mathbb{Z}^d$. The integer value a vertex taking is the seriousness of the illness on this vertex. A vertex taking value 0 is healthy and one taking positive value is infected. An infected vertex becomes healthy at rate $\frac{1}{1+2\lambda d}$. A vertex x is infected by a given neighbor y at rate $\frac{\lambda}{1+2\lambda d}$. When the infection occurs, the seriousness of the illness on x is added with that on y .

The BCPP $\{\phi_t\}_{t \geq 0}$ is introduced to improve upper bounds of critical values of high-dimensional contact processes (CP) according to the fact that the contact process $\{\xi_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ can be equivalently defined as

$$\xi_t(x) = \begin{cases} 0 & \text{if } \phi_{(1+2\lambda d)t}(x) = 0, \\ 1 & \text{if } \phi_{(1+2\lambda d)t}(x) > 0 \end{cases}$$

for any $x \in \mathbb{Z}^d$. For a detailed survey of the contact process, see Chapter 6 of [5] and Part II of [6]. The critical value λ_c of the contact process is defined as

$$\lambda_c = \sup \left\{ \lambda : P \left(\sum_{x \in \mathbb{Z}^d} \xi_t^\lambda(x) = 0 \text{ for some } t > 0 \mid \sum_x \xi_0^\lambda(x) = 1 \right) = 1 \right\}.$$

According to the above coupling relationship between BCPP and CP, it is shown in [4] that $\lambda_c(d) \leq \frac{1}{2d(2\gamma_d-1)}$ for $d \geq 3$, where γ_d is the probability that the simple random walk on $\mathbb{Z}^d$ starting at O never return to O again. Especially, $\lambda_c(3) \leq 0.523$ as a corollary of the above upper bound. In [7], a modified version of BCPP is introduced and then a further improved upper bound of $\lambda_c(d)$ is given for $d \geq 3$. It is shown in [7] that $\lambda_c(d) \leq \frac{2-\gamma_d}{2d\gamma_d}$ and consequently $\lambda_c(3) \leq 0.340$.

The BCPP belongs to a family of continuous-time Markov processes called linear systems defined in Chapter 9 of [5], since there are a series of linear transformations $\{\mathcal{A}_k : k \geq 1\}$ on $\mathbb{X}$ such that $\phi_t = \mathcal{A}_k \phi_{t-}$ for some $k \geq 1$ at each jump moment t . As a result, for each $m \geq 1$, Kolmogorov-Chapman equations of

$$\left\{ \mathbb{E} \left(\prod_{i=1}^m \phi_t(x_i) \right) : x_1, \dots, x_m \in \mathbb{Z}^d \right\}$$

are given by a series of linear ordinary differential equations. For mathematical details, see Theorems 9.1.27 and 9.3.1 of [5].

For technical reasons which we will recall in Section 2, it is convenient to investigate a normalized version $\{\eta_t\}_{t \geq 0}$ of the BCPP defined by

$$\eta_t = \exp \left\{ \frac{1-2d\lambda}{2d\lambda} t \right\} \phi_{\frac{1+2d\lambda}{2d\lambda} t}.$$

The state space of the normalized binary contact path process (NBCPP) $\{\eta_t\}_{t \geq 0}$ is $\mathbb{Y} = [0, +\infty)^{\mathbb{Z}^d}$ and the generator $\mathcal{L}$ of $\{\eta_t\}_{t \geq 0}$ is given by

$$\begin{aligned} \mathcal{L}f(\eta) &= \frac{1}{2\lambda d} \sum_x (f(\eta^{x,-}) - f(\eta)) + \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \sum_{y \sim x} (f(\eta^{x,y}) - f(\eta)) \\ &\quad + \left(\frac{1}{2\lambda d} - 1 \right) \sum_{x \in \mathbb{Z}^d} f'_x(\eta) \eta(x), \end{aligned}$$

where f'_x is the partial derivative of f with respect to the coordinate $\eta(x)$.

Inspired by counterpart results given in [2] and [1] for voter models and branching random walks, in this paper we study the law of large numbers and the central limit theorem of the occupation time process $\left\{ \int_0^t \eta_u(O) du \right\}_{t \geq 0}$ of NBCPP. According to a calculation of the variance, it is natural to guess that the above central limit should be with three different forms in respective cases where $d \geq 5$, $d = 4$ and $d = 3$, as Theorems 1 of [2] and 1.1 of [1]. However, in this paper we only give a rigorous result for part of the first case, i.e., CLT of the occupation time for sufficiently large d , according to the fact that we can not bound the fourth moment of $\eta_t(O)$ uniformly for $t \geq 0$ when the dimension d is low. For our main result and mathematical details in the proof, see Sections 2 and 3.

2 Main results

In this section we give our main result. For later use, we first introduce some notions and definitions. We denote by $\{S_n\}_{n \geq 0}$ the discrete-time simple random walk on $\mathbb{Z}^d$, i.e.,

$$P(S_{n+1} = y | S_n = x) = \frac{1}{2d}$$

for any $n \geq 0$, $x \in \mathbb{Z}^d$ and $y \sim x$. We denote by $\{Y_t\}_{t \geq 0}$ the continuous-time simple random walk on $\mathbb{Z}^d$ with generator L given by

$$Lh(x) = \frac{1}{2d} \sum_{y \sim x} (h(y) - h(x))$$

for any bounded h from $\mathbb{Z}^d$ to $\mathbb{R}$ and $x \in \mathbb{Z}^d$. As defined in Section 1, we use γ_d to denote the probability that $\{S_n\}_{n \geq 1}$ never return to O again conditioned on $S_0 = O$, i.e.,

$$\gamma_d = P(S_n \neq O \text{ for all } n \geq 1 | S_0 = O) = P(Y_t \neq O \text{ for any } t \geq 0 | Y_0 = x)$$

for any $x \sim O$. For any $t \geq 0$, we use $p_t(\cdot, \cdot)$ to denote transition probabilities of Y_t , i.e.,

$$p_t(x, y) = P(Y_t = y | Y_0 = x)$$

for any $x, y \in \mathbb{Z}^d$. For any $x \in \mathbb{Z}^d$, we define

$$\Phi(x) = P(S_n = x \text{ for some } n \geq 0 | S_0 = O).$$

We use $\vec{1}$ to denote the configuration in $\mathbb{Y}$ where all vertices take value 1. For any $\eta \in \mathbb{Y}$, we use $\mathbb{E}_\eta$ to denote the expectation operator of $\{\eta_t\}_{t \geq 0}$ conditioned on $\eta_0 = \eta$. Furthermore, for any probability measure μ on $\mathbb{Y}$, we use $\mathbb{E}_\mu$ to denote the expectation operator of $\{\eta_t\}_{t \geq 0}$ conditioned on η_0 being distributed with μ .

Now we recall some basic properties of NBCPP $\{\eta_t\}_{t \geq 0}$ proved in Chapter 9 of [5] and Section 2 of [8].

Proposition 2.1. (Liggett, [5]) For any $\eta \in \mathbb{Y}$, $t \geq 0$ and $x \in \mathbb{Z}^d$,

$$\mathbb{E}_\eta \eta_t(x) = \sum_{y \in \mathbb{Z}^d} p_t(x, y) \eta(y).$$

Furthermore, $\mathbb{E}_{\vec{1}} \eta_t(x) = 1$ for any $x \in \mathbb{Z}^d$ and $t \geq 0$.

Proposition 2.2. (Liggett, [5]) There exist a series of functions $\{q_t\}_{t \geq 0}$ from $(\mathbb{Z}^d)^2$ to $\mathbb{R}$ and a series of functions $\{\hat{q}_t\}_{t \geq 0}$ from $(\mathbb{Z}^d)^4$ to $\mathbb{R}$ such that

$$\mathbb{E}_{\bar{1}}(\eta_t(O)\eta_t(x)) = \mathbb{E}_{\bar{1}}(\eta_t(y)\eta_t(x+y)) = \sum_{z \in \mathbb{Z}^d} q_t(x, z)$$

and

$$\mathbb{E}_{\eta}(\eta_t(x)\eta_t(y)) = \sum_{z, w \in \mathbb{Z}^d} \hat{q}_t((x, y), (z, w)) \eta(z)\eta(w)$$

for any $t \geq 0$, $x, y \in \mathbb{Z}^d$ and $\eta \in \mathbb{Y}$. Furthermore,

$$q_t(O, w) = \sum_{(y, z): y-z=w} \hat{q}_t((x, x), (y, z))$$

for any $x, w \in \mathbb{Z}^d$.

Proposition 2.1 is an application of Theorem 9.1.27 of [5] and Proposition 2.2 is an application of Theorem 9.3.1 of [5]. We write q_t and $\hat{q}_t$ as $q_t^{\lambda, d}$ and $\hat{q}_t^{\lambda, d}$ respectively when we need to distinguish the dimension d and infection rate λ .

Proposition 2.3. (Liggett, [5]) When $d \geq 3$ and $\lambda > \frac{1}{2d(2\gamma_d-1)}$, we have following properties.

1) For any $x, y, z, w \in \mathbb{Z}^d$,

$$\lim_{t \rightarrow +\infty} q_t^{\lambda, d}(x, y) = \lim_{t \rightarrow +\infty} \hat{q}_t^{\lambda, d}(x, y, z, w) = 0.$$

2) Conditioned on $\eta_0 = \bar{1}$, η_t converges weakly as $t \rightarrow +\infty$ to a probability measure $\nu_{\lambda, d}$ on $\mathbb{Y}$.

3) For any $x, y \in \mathbb{Z}^d$,

$$\lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{1}}((\eta_t(x))^2) = \sup_{t \geq 0} \mathbb{E}_{\bar{1}}((\eta_t(x))^2) = \mathbb{E}_{\nu_{\lambda, d}}((\eta_0(O))^2) = 1 + \frac{1}{h_{\lambda, d}}$$

and

$$\lim_{t \rightarrow +\infty} \text{Cov}_{\bar{1}}(\eta_t(x), \eta_t(y)) = \text{Cov}_{\nu_{\lambda, d}}(\eta_0(x), \eta_0(y)) = \frac{\Phi(x-y)}{h_{\lambda, d}},$$

where $h_{\lambda, d} = \frac{2\lambda d(2\gamma_d-1)-1}{1+2d\lambda}$.

Proposition 2.3 is an application of Corollary 2.8.20 and Theorem 9.3.17 of [5].

Proposition 2.4. ([8]) There exist an integer $d_0 \geq 5$ and a real number $\lambda_0 > 0$ satisfying following properties.

(1) If $d \geq d_0$ and $\lambda \geq \lambda_0$, then $\lambda > \frac{1}{2d(2\gamma_d-1)}$.

(2) For any $d \geq d_0$, $\lambda \geq \lambda_0$ and $x \in \mathbb{Z}^d$,

$$\mathbb{E}_{\nu_{\lambda, d}}((\eta_0(O))^4) \leq \liminf_{t \rightarrow +\infty} \mathbb{E}_{\bar{1}}^{\lambda, d}((\eta_t(x))^4) < +\infty.$$

(3) For any $d \geq d_0$ and $\lambda \geq \lambda_0$,

$$\lim_{M \rightarrow +\infty} \sup_{\substack{(x, y, z) \in (\mathbb{Z}^d)^3: \\ \|x-y\|_1 \wedge \|x-z\|_1 \geq M}} \text{Cov}_{\nu_{\lambda, d}}((\eta_0(x))^2, \eta_0(y)\eta_0(z)) = 0,$$

where $\|\cdot\|_1$ is the l_1 -norm on $\mathbb{R}^d$ and $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$.

Parts one and two of Proposition 2.4 follow from Proposition 2.1 of [8] and Fatou's lemma. Part three of Proposition 2.4 follows from the analysis leading to Proposition 2.2 of [8] and Fatou's lemma.

We omit proofs of Propositions 2.1 to 2.4 in this paper since they are only repeats of analyses given in Chapter 9 of [5] and Sections 3, 4 of [8]. Now we give our main result. Our first main result is about the law of large numbers of the occupation time process $\{\int_0^t \eta_u(O) du\}_{t \geq 0}$.

Theorem 2.5. *If $d \geq 3$, $\lambda > \frac{1}{2d(2\gamma_d-1)}$ and the initial state η_0 of the NBCPP on $\mathbb{Z}^d$ is distributed with $\nu_{\lambda,d}$, then*

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \int_0^{tN} \eta_u(O) du = t$$

in L^2 for any $t > 0$.

Our second main result is about the central limit theorem of the occupation time process. For any $t \geq 0$ and $N \geq 1$, we define

$$X_t^N = \frac{1}{\sqrt{N}} \int_0^{tN} (\eta_u(O) - 1) du.$$

We write X_t^N as $X_{t,\lambda,d}^N$ when we need to distinguish d and λ . We have the following theorem.

Theorem 2.6. *Let d_0 and λ_0 be defined as in Proposition 2.4. For any $d \geq d_0$, $\lambda > \lambda_0$, integer $m \geq 1$ and $t_1, t_2, \dots, t_m \geq 0$, conditioned on the initial state η_0 of the NBCPP on $\mathbb{Z}^d$ being distributed with $\nu_{\lambda,d}$, $(X_{t_1,\lambda,d}^N, X_{t_2,\lambda,d}^N, \dots, X_{t_m,\lambda,d}^N)$ converges weakly to $\sqrt{C_1(\lambda, d)}(B_{t_1}, B_{t_2}, \dots, B_{t_m})$ as $N \rightarrow +\infty$, where $\{B_t\}_{t \geq 0}$ is a standard Brownian motion and*

$$C_1(\lambda, d) = \frac{2 \int_0^{+\infty} \int_0^{+\infty} p_{r+\theta}(O, O) dr d\theta}{h_{\lambda,d} \int_0^{+\infty} p_\theta(O, O) d\theta}.$$

Remark 2.1. *Theorem 2.6 is consistent with a calculation of variance. According to Propositions 2.1 to 2.3 and the Markov property of $\{\eta_t\}_{t \geq 0}$,*

$$\lim_{N \rightarrow +\infty} \text{Var}_{\nu_{\lambda,d}}(X_t^N) = \frac{2t}{h_{\lambda,d}} \int_0^{+\infty} \sum_x \Phi(x) p_r(O, x) dr.$$

According to the strong Markov property of the simple random walk,

$$\int_0^{+\infty} p_\theta(x, O) d\theta = \Phi(x) \int_0^{+\infty} p_\theta(O, O) d\theta$$

and hence

$$\begin{aligned} \int_0^{+\infty} \sum_x \Phi(x) p_\theta(O, x) d\theta &= \frac{\int_0^{+\infty} \int_0^{+\infty} \sum_x p_r(O, x) p_\theta(x, O) dr d\theta}{\int_0^{+\infty} p_\theta(O, O) d\theta} \\ &= \frac{\int_0^{+\infty} \int_0^{+\infty} p_{r+\theta}(O, O) dr d\theta}{\int_0^{+\infty} p_\theta(O, O) d\theta}. \end{aligned}$$

Theorem 2.6 shows that the central limit theorem of the NBCPP is an analogue of that of voter models and branching random walks given in [2] and [1] when the dimension d is sufficiently large. According to Proposition 2.3, calculations of variances as in Remark

2.1 imply that central limit theorems of occupation times of NBCPP on $\mathbb{Z}^3$, $\mathbb{Z}^4$ and $\mathbb{Z}^d$ for $d \geq 5$ should respectively be analogues of that of voter models and branching random walks in each cases. However, our current proof of Theorem 2.6 relies heavily on the fact that $\sup_{t \geq 0} \mathbb{E}_{\Gamma}^{\lambda, d} \left((\eta_t(O))^4 \right) < +\infty$ when d and λ are sufficiently large, which we have not managed to prove yet for small d . That is why we currently only discuss the NBCPP on $\mathbb{Z}^d$ with d sufficiently large.

Proofs of Theorems 2.5 and 2.6 are given in Section 3. The proof of Theorem 2.5 is relatively easy, where it is shown that

$$\lim_{N \rightarrow +\infty} \mathbb{E}_{\nu_{\lambda, d}} \left(\frac{1}{N} \int_0^{tN} \eta_u(O) du \right) = t$$

and

$$\lim_{N \rightarrow +\infty} \text{Var}_{\nu_{\lambda, d}} \left(\frac{1}{N} \int_0^{tN} \eta_u(O) du \right) = 0$$

according to Propositions 2.1 and 2.3. The proof of Theorem 2.6 follows the strategy introduced in [1] to prove central limit theorems of occupation times of branching random walks. The core idea of the strategy is to decompose $\int_0^t (\eta_u(O) - 1) du$ as a martingale M_t plus a remainder R_t such that the quadratic variation process $\frac{1}{N} \langle M \rangle_{tN}$ of $\frac{1}{\sqrt{N}} M_{tN}$ converges to $C_1 t$ in L^2 and $\frac{1}{\sqrt{N}} R_{tN}$ converges to 0 in probability as $N \rightarrow +\infty$. To give the above decomposition, a resolvent function of the simple random walk $\{Y_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ is utilized. There are two main technical difficulties in the execution of the above strategy for the NBCPP. The first one is to prove that $\text{Var} \left(\frac{1}{N} \langle M \rangle_{tN} \right)$ converges to 0 as $N \rightarrow +\infty$, which ensures that the limit of $\frac{1}{N} \langle M \rangle_{tN}$ is deterministic. The second one is to show that $\mathbb{E} \left(\frac{1}{N} \sup_{0 \leq s \leq tN} (M_s - M_{s-})^2 \right)$ converges to 0 as $N \rightarrow +\infty$, which ensures that the limit process of $\{\frac{1}{\sqrt{N}} M_{tN}\}_{t \geq 0}$ is continuous. To overcome the first difficulty, we relate the calculation of $\text{Var}(\langle M \rangle_{tN})$ to a random walk on $\mathbb{Z}^{2d}$ according to the Kolmogorov-Chapman Equation of linear systems introduced in [5]. To overcome the second difficulty, we bound $P \left(\frac{1}{N} \sup_{0 \leq s \leq tN} (M_s - M_{s-})^2 > \epsilon \right)$ from above by $O \left(\frac{1}{N^2} \int_0^{tN} \mathbb{E}(\eta_s^4(O)) ds \right)$ according to the strong Markov property of our process. For mathematical details, see Section 3.

3 Proofs of Theorems 2.5 and 2.6

In this section we prove Theorems 2.5 and 2.6. We first give the proof of Theorem 2.5.

Proof of Theorem 2.5. Throughout this proof we assume that $d \geq 3$, $\lambda > \frac{1}{2d(2\gamma_d - 1)}$ and η_0 is distributed with $\nu_{\lambda, d}$. According to Proposition 2.3, $\{\eta_u(O)\}_{u \geq 0}$ are uniformly integrable and hence

$$\mathbb{E}_{\nu_{\lambda, d}} \eta(O) = \lim_{t \rightarrow +\infty} \mathbb{E}_{\Gamma} \eta_t(O) = 1.$$

Therefore,

$$\mathbb{E}_{\nu_{\lambda, d}} \left(\frac{1}{N} \int_0^{tN} \eta_u(O) du \right) = t$$

for any $N \geq 1$ and $t > 0$. Hence, to prove Theorem 2.5 we only need to show that

$$\lim_{N \rightarrow +\infty} \frac{1}{N^2} \text{Var}_{\nu_{\lambda, d}} \left(\int_0^{tN} \eta_u(O) du \right) = 0 \quad (3.1)$$

for any $t \geq 0$. Since

$$\begin{aligned} \text{Var}_{\nu_{\lambda,d}} \left(\int_0^{tN} \eta_u(O) du \right) &= 2 \int_0^{tN} \left(\int_0^\theta \text{Cov}_{\nu_{\lambda,d}} (\eta_r(O), \eta_\theta(O)) dr \right) d\theta \\ &= 2 \int_0^{tN} \left(\int_0^\theta \text{Cov}_{\nu_{\lambda,d}} (\eta_0(O), \eta_{\theta-r}(O)) dr \right) d\theta, \end{aligned}$$

to prove Equation (3.1) we only need to show that

$$\lim_{t \rightarrow +\infty} \text{Cov}_{\nu_{\lambda,d}} (\eta_0(O), \eta_t(O)) = 0. \quad (3.2)$$

According to Propositions 2.1 and 2.3,

$$\begin{aligned} \text{Cov}_{\nu_{\lambda,d}} (\eta_0(O), \eta_t(O)) &= \mathbb{E}_{\nu_{\lambda,d}} (\eta_0(O) \eta_t(O)) - 1 \\ &= \mathbb{E}_{\nu_{\lambda,d}} \left(\eta_0(O) \left(\sum_{y \in \mathbb{Z}^d} p_t(O, y) \eta_0(y) \right) \right) - 1 \\ &= \sum_{y \in \mathbb{Z}^d} p_t(O, y) (\mathbb{E}_{\nu_{\lambda,d}} (\eta_0(O) \eta_0(y)) - 1) \\ &= \sum_{y \in \mathbb{Z}^d} p_t(O, y) \text{Cov}_{\nu_{\lambda,d}} (\eta_0(O), \eta_0(y)) = \frac{1}{h_{\lambda,d}} \sum_{y \in \mathbb{Z}^d} p_t(O, y) \Phi(y). \end{aligned} \quad (3.3)$$

Since $d \geq 3$, for any $\epsilon > 0$ there exists $M > 0$ such that $\Phi(y) < \epsilon$ when $\|y\|_1 \geq M$. As a result, by Equation (3.3),

$$\begin{aligned} &\limsup_{t \rightarrow +\infty} \text{Cov}_{\nu_{\lambda,d}} (\eta_0(O), \eta_t(O)) \\ &\leq \limsup_{t \rightarrow +\infty} \frac{\epsilon}{h_{\lambda,d}} \sum_{y: \|y\|_1 \geq M} p_t(O, y) + \frac{1}{h_{\lambda,d}} \sum_{y: \|y\|_1 \leq M} \left(\lim_{t \rightarrow +\infty} p_t(O, y) \right) \\ &= \limsup_{t \rightarrow +\infty} \frac{\epsilon}{h_{\lambda,d}} \sum_{y: \|y\|_1 \geq M} p_t(O, y) \leq \frac{\epsilon}{h_{\lambda,d}}. \end{aligned}$$

Since ϵ is arbitrary, let $\epsilon \rightarrow 0$ and then Equation (3.2) holds. As we have explained, Theorem 2.5 follows from Equation (3.2) and the proof is complete. $\square$

From now on we assume that $d \geq d_0$ and $\lambda \geq \lambda_0$, where d_0 and λ_0 are defined as in Proposition 2.4. As we have introduced at the end of Section 2, our proof follows the strategy introduced in [1], where X_t^N is decomposed as a martingale plus a remainder term. As $N \rightarrow +\infty$, the martingale converges weakly to a Brownian motion and the remainder converges to 0 in probability. In detail, for any $\theta > 0$, we define

$$G_\theta(\eta) = \sum_{x \in \mathbb{Z}^d} g_\theta(x) (\eta(x) - 1),$$

where g_θ is the resolvent function of the simple random walk given by

$$g_\theta(x) = \int_0^{+\infty} e^{-\theta u} p_u(O, x) du.$$

According to the fact that $p_t(O, O) = O(t^{-d/2})$ as $t \rightarrow +\infty$, $\sum_x g_0^k(x) < +\infty$ when $k \geq 2$ and $d \geq 5$. Let

$$M_t^\theta = G_\theta(\eta_t) - G_\theta(\eta_0) - \int_0^t \mathcal{L}G_\theta(\eta_s) ds,$$

then by Dynkin's martingale formula, $\{M_t^\theta\}_{t \geq 0}$ is a martingale with quadratic variation process $\{\langle M^\theta \rangle_t\}_{t \geq 0}$ given by

$$\langle M^\theta \rangle_t = \int_0^t \left(\mathcal{L} \left((G_\theta(\eta_s))^2 \right) - 2G_\theta(\eta_s) \mathcal{L}G_\theta(\eta_s) \right) du.$$

According to the definition of $\mathcal{L}$,

$$\langle M^\theta \rangle_t = \frac{1}{2\lambda d} \int_0^t \sum_{x \in \mathbb{Z}^d} \eta_s^2(x) \left(g_\theta^2(x) + \lambda \sum_{y \sim x} g_\theta^2(y) \right) ds. \quad (3.4)$$

According to the fact that $\theta g_\theta(\cdot) - Lg_\theta(\cdot) = 1_O(\cdot)$,

$$\mathcal{L}G_\theta(\eta) = \theta G_\theta(\eta) - \left(\eta(O) - \theta \sum_x g_\theta(x) \right) = \theta G_\theta(\eta) - (\eta(O) - 1).$$

As a result,

$$X_t^N = \frac{1}{\sqrt{N}} M_{tN}^{1/N} + \frac{1}{\sqrt{N}} R_{tN}^{1/N}, \quad (3.5)$$

where

$$R_t^\theta = -G_\theta(\eta_t) + G_\theta(\eta_0) + \int_0^t \theta G_\theta(\eta_s) ds.$$

To prove Theorem 2.6, we need following lemmas.

Lemma 3.1. *For any $t \geq 0$, $\frac{1}{N} \langle M^{1/N} \rangle_{tN}$ converges to $C_1(\lambda, d)t$ in L^2 as $N \rightarrow +\infty$.*

Lemma 3.2. *For any $t \geq 0$, $\frac{1}{\sqrt{N}} R_{tN}^{1/N}$ converges to 0 in L^2 as $N \rightarrow +\infty$.*

Lemma 3.3. *For any $t \geq 0$,*

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left(\sup_{0 \leq s \leq tN} \frac{1}{N} \left(M_s^{1/N} - M_{s-}^{1/N} \right)^2 \right) = 0,$$

where $M_{s-}^{1/N} = \lim_{u < s, u \rightarrow s} M_u^{1/N}$, i.e., the state of $M^{1/N}$ at the moment just before s .

We first utilize Lemmas 3.1, 3.2 and 3.3 to prove Theorem 2.6.

Proof of Theorem 2.6. According to Lemmas 3.1, 3.3 and Theorem 1.4 of Chapter 7 of [3], $\{\frac{1}{\sqrt{N}} M_{tN}^{1/N}\}_{t \geq 0}$ converges weakly to $\{\sqrt{C_1(\lambda, d)} B_t\}_{t \geq 0}$ as $N \rightarrow +\infty$. Consequently, Theorem 2.6 follows from Equation (3.5) and Lemma 3.2. $\square$

Now we only need to prove Lemmas 3.1, 3.2 and 3.3. We first give the proof of Lemma 3.1.

Proof of Lemma 3.1. Conditioned on $Y_0 = O$, the number of times $\{Y_t\}_{t \geq 0}$ visits O follows a geometric distribution with parameter γ_d and at each time Y_t stays at O for an exponential time with mean 1. Therefore,

$$\int_0^{+\infty} p_s(O, O) ds = \mathbb{E}_O \int_0^{+\infty} 1_{\{X_s=O\}} ds = 1 \times \frac{1}{\gamma_d} = \frac{1}{\gamma_d}.$$

Then by Equation (3.4) and Proposition 2.3,

$$\begin{aligned} \lim_{N \rightarrow +\infty} \mathbb{E}_{\nu_{\lambda,d}} \left(\frac{1}{N} \langle M^{1/N} \rangle_{tN} \right) &= \frac{t(1+2\lambda d)}{2\lambda d} \left(1 + \frac{1}{h_{\lambda,d}} \right) \sum_x g_0^2(x) \\ &= \frac{2\gamma_d t}{h_{\lambda,d}} \int_0^{+\infty} \int_0^{+\infty} p_{\theta+r}(O, O) d\theta dr = C_1(\lambda, d)t. \end{aligned}$$

Hence, to complete the proof we only need to show that

$$\lim_{N \rightarrow +\infty} \frac{1}{N^2} \text{Var}_{\nu_{\lambda,d}} \left(\langle M^{1/N} \rangle_{tN} \right) = 0. \quad (3.6)$$

By Proposition 2.2 and the Markov property of $\{\eta_t\}_{t \geq 0}$, for $s < u$ and $x, w \in \mathbb{Z}^d$,

$$\text{Cov}_{\nu_{\lambda,d}} (\eta_s^2(x), \eta_u^2(w)) = \sum_{y,z} \hat{q}_{u-s}((w, w), (y, z)) \text{Cov}_{\nu_{\lambda,d}} (\eta_0^2(x), \eta_0(y)\eta_0(z)).$$

Therefore, to prove Equation (3.6) we only need to show that

$$\lim_{r \rightarrow +\infty} \sum_{x,w,y,z} V_0(x)V_0(w)\hat{q}_r((w, w), (y, z)) |\text{Cov}_{\nu_{\lambda,d}} (\eta_0^2(x), \eta_0(y)\eta_0(z))| = 0, \quad (3.7)$$

where $V_\theta(x) = g_\theta^2(x) + \sum_{y \sim x} g_\theta^2(y)$, which is decreasing in θ . Since $\sum_x V_0(x) < +\infty$,

$$\sum_{y,z} \hat{q}_r((w, w), (y, z)) = \sum_y q_r(O, y) = \mathbb{E}_1 \left((\eta_r(O))^2 \right) \leq 1 + \frac{1}{h_{\lambda,d}}$$

and

$$\sup_{w,y,z} |\text{Cov}_{\nu_{\lambda,d}} (\eta_0^2(w), \eta_0(y)\eta_0(z))| < 2\mathbb{E}_{\nu_{\lambda,d}} \left((\eta_0(O))^4 \right) < +\infty$$

according to Propositions 2.2 to 2.4 and Cauchy-Schwarz inequality, to prove (3.7) we only need to show that

$$\lim_{r \rightarrow +\infty} \sum_{y,z} \hat{q}_r((w, w), (y, z)) |\text{Cov}_{\nu_{\lambda,d}} (\eta_0^2(x), \eta_0(y)\eta_0(z))| = 0 \quad (3.8)$$

for any $x, w \in \mathbb{Z}^d$ according to the dominated convergence theorem. By Proposition 2.4, for any $\epsilon > 0$, there exists $M > 0$ such that $|\text{Cov}_{\nu_{\lambda,d}} (\eta_0^2(x), \eta_0(y)\eta_0(z))| < \epsilon$ when $\|x - y\|_1 \wedge \|x - z\|_1 \geq M$ and hence

$$\begin{aligned} &\sum_{(y,z): \|x-y\|_1 \wedge \|x-z\|_1 \geq M} \hat{q}_r((w, w), (y, z)) |\text{Cov}_{\nu_{\lambda,d}} (\eta_0^2(x), \eta_0(y)\eta_0(z))| \\ &\leq \epsilon \sum_{(y,z): \|x-y\|_1 \wedge \|x-z\|_1 \geq M} \hat{q}_r((w, w), (y, z)) \leq \epsilon \left(1 + \frac{1}{h_{\lambda,d}} \right). \end{aligned}$$

We claim that

$$\lim_{t \rightarrow +\infty} \sum_v \hat{q}_t((w, w), (y, v)) = \lim_{t \rightarrow +\infty} \sum_v \hat{q}_t((w, w), (v, z)) = 0 \quad (3.9)$$

for any $w, y, z \in \mathbb{Z}^d$. The proof of Equation (3.9) is given later. By Equation (3.9), Proposition 2.4 and Cauchy-Schwarz inequality,

$$\lim_{r \rightarrow +\infty} \sum_{(y, z): \|x-y\|_1 \leq M \text{ or } \|x-z\|_1 \leq M} \hat{q}_r((w, w), (y, z)) |\text{Cov}_{\nu_{\lambda, d}}(\eta_0^2(x), \eta_0(y)\eta_0(z))| = 0$$

and hence

$$\limsup_{r \rightarrow +\infty} \sum_{y, z} \hat{q}_r((w, w), (y, z)) |\text{Cov}_{\nu_{\lambda, d}}(\eta_0^2(x), \eta_0(y)\eta_0(z))| \leq \epsilon(1 + \frac{1}{h_{\lambda, d}}).$$

Since ϵ is arbitrary, let $\epsilon \rightarrow 0$ and then Equation (3.8) holds. □

Now we prove Equation (3.9) to complete the proof of Lemma 3.1.

Proof of Equation (3.9). By Theorem 9.3.1 of [5] and the definition of $\mathcal{L}$,

$$\hat{q}_t((w, w), (y, v)) = e^{-2t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} H^n((w, w), (y, v)),$$

where H is a $(\mathbb{Z}^d)^2 \times (\mathbb{Z}^d)^2$ matrix given by

$$H((x, y), (u, v)) = \begin{cases} \frac{1}{2d} & \text{if } x \neq y, u \sim x \text{ and } v = y, \\ \frac{1}{2d} & \text{if } x \neq y, u = x \text{ and } v \sim y, \\ \frac{1}{2d\lambda} & \text{if } x = y \text{ and } (u, v) = (x, x), \\ \frac{1}{2d} & \text{if } x = y, u \sim x \text{ and } v = u, \\ \frac{1}{2d} & \text{if } x = y, u = x \text{ and } v \sim x, \\ \frac{1}{2d} & \text{if } x = y, u \sim x \text{ and } v = x, \\ 0 & \text{else} \end{cases}$$

and

$$H^n((w, w), (y, v)) = \sum_{\substack{\{(u_i, v_i)\}_{0 \leq i \leq n}: \\ (u_0, v_0) = (w, w), (u_n, v_n) = (y, v)}} \prod_{i=0}^{n-1} H((u_i, v_i), (u_{i+1}, v_{i+1})).$$

As a result,

$$\sum_{v \in \mathbb{Z}^d} \hat{q}_t((w, w), (y, v)) = e^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} \mathbb{E}_{(w, w)} \left(\prod_{i=0}^{n-1} \Theta(\beta_i, \beta_{i+1}) 1_{\{\beta_n(1)=y\}} \right),$$

where $\{\beta_n = (\beta_n(1), \beta_n(2)) : n \geq 0\}$ is a random walk on $(\mathbb{Z}^d)^2$ such that

$$P(\beta_{n+1} = (u, v) | \beta_n = (x, y)) = \begin{cases} \frac{1}{4d} & \text{if } x \neq y, u \sim x \text{ and } v = y, \\ \frac{1}{4d} & \text{if } x \neq y, u = x \text{ and } v \sim y, \\ \frac{1}{6d+1} & \text{if } x = y \text{ and } (u, v) = (x, x), \\ \frac{1}{6d+1} & \text{if } x = y, u \sim x \text{ and } v = u, \\ \frac{1}{6d+1} & \text{if } x = y, u \sim x \text{ and } v = x, \\ \frac{1}{6d+1} & \text{if } x = y, u = x \text{ and } v \sim x, \\ 0 & \text{else} \end{cases}$$

and Θ is a function from $(\mathbb{Z}^d)^2$ to $\mathbb{R}$ such that

$$\Theta((x, y), (u, v)) = \frac{H((x, y), (u, v)) \deg(x, y)}{2},$$

where

$$\deg(x, y) = \begin{cases} 4d & \text{if } x \neq y, \\ 6d + 1 & \text{if } x = y. \end{cases}$$

Then for each $i \geq 0$, $\Theta(\beta_i, \beta_{i+1}) \geq 1$ and $\Theta(\beta_i, \beta_{i+1}) = 1$ when and only when $\beta_i(1) = \beta_i(2)$. Hence, by Hölder's inequality,

$$\begin{aligned} & \sum_{v \in \mathbb{Z}^d} \hat{q}_t((w, w), (y, v)) \\ & \leq \left(\mathbb{E}_{(w, w)} \left(\prod_{i=0}^{+\infty} \Theta^{1+\epsilon}(\beta_i, \beta_{i+1}) \right) \right)^{\frac{1}{1+\epsilon}} e^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} (P(\beta_n(1) = y))^{\epsilon/(1+\epsilon)} \end{aligned} \quad (3.10)$$

for any $\epsilon > 0$. Since $\{\beta_n(1) - \beta_n(2)\}_{n \geq 0}$ is a lazy version of the simple random walk on $\mathbb{Z}^d$ and $H(\beta_i, \beta_{i+1}) > 1$ only when $\beta_i(1) = \beta_i(2)$, by the strong Markov property of $\{\beta_n\}_{n \geq 0}$,

$$\mathbb{E}_{(w, w)} \left(\prod_{i=0}^{+\infty} \Theta^{1+\epsilon}(\beta_i, \beta_{i+1}) \right) = \frac{4d}{6d+1} \left(\frac{6d+1}{4d} \right)^{1+\epsilon} \gamma_d + \sum_{k=1}^{+\infty} (\alpha_\epsilon(d, \lambda))^k \gamma_d,$$

where

$$\alpha_\epsilon(d, \lambda) = \frac{1}{6d+1} \left(\frac{6d+1}{2 \cdot 2d\lambda} \right)^{1+\epsilon} + \frac{2d}{6d+1} \left(\frac{6d+1}{2 \cdot 2d} \right)^{1+\epsilon} + \frac{4d}{6d+1} \left(\frac{6d+1}{2 \cdot 2d} \right)^{1+\epsilon} (1 - \gamma_d).$$

Since $\lambda > \frac{1}{2d(2\gamma_d-1)}$,

$$\lim_{\epsilon \rightarrow 0} \alpha_\epsilon(d, \lambda) = \frac{1}{4d\lambda} + 1/2 + 1 - \gamma_d < 1$$

and hence there exists $\epsilon_0 > 0$ such that $\mathbb{E}_{(w, w)} \left(\prod_{i=0}^{+\infty} \Theta^{1+\epsilon_0}(\beta_i, \beta_{i+1}) \right) < +\infty$. By Equation (3.10), to complete the proof we only need to show that

$$\lim_{t \rightarrow +\infty} e^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} (P(\beta_n(1) = y))^{\epsilon_0/(1+\epsilon_0)} = 0. \quad (3.11)$$

Since $d \geq 3$ and $\{\beta_n(1)\}_{n \geq 1}$ is a lazy version of the simple random walk on $\mathbb{Z}^d$,

$$\lim_{n \rightarrow +\infty} P(\beta_n(1) = y) = 0.$$

Hence, for any $\epsilon > 0$, there exists $M \geq 1$ such that $(P(\beta_n(1) = y))^{\epsilon_0/(1+\epsilon_0)} \leq \epsilon$ when $n \geq M$ and then

$$e^{-2t} \sum_{n=M}^{+\infty} \frac{(2t)^n}{n!} (P(\beta_n(1) = y))^{\epsilon_0/(1+\epsilon_0)} \leq \epsilon e^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} = \epsilon.$$

Since $\lim_{t \rightarrow +\infty} \sum_{n=0}^{M-1} e^{-2t} \frac{(2t)^n}{n!} = 0$, we have

$$\limsup_{t \rightarrow +\infty} e^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} (P(\beta_n(1) = y))^{\epsilon_0/(1+\epsilon_0)} \leq \epsilon.$$

Since ϵ is arbitrary, let $\epsilon \rightarrow 0$ and then Equation (3.11) holds. □

Now we prove Lemma 3.2.

Proof of Lemma 3.2. According to Propositions 2.1 and 2.3, for any moment $c \geq 0$,

$$\begin{aligned}
\mathbb{E}_{\nu_{\lambda,d}} \left(\left(\frac{1}{\sqrt{N}} G_{1/N}(\eta_c) \right)^2 \right) &= \text{Var}_{\nu_{\lambda,d}} \left(\frac{1}{\sqrt{N}} G_{1/N}(\eta_c) \right) = \text{Var}_{\nu_{\lambda,d}} \left(\frac{1}{\sqrt{N}} G_{1/N}(\eta_0) \right) \\
&= \frac{1}{N} \sum_x \sum_y g_{1/N}(x) g_{1/N}(y) \frac{\Phi(y-x)}{h_{\lambda,d}} \\
&= \frac{\gamma_d}{N h_{\lambda,d}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-(r+s)/N} \sum_x \sum_y p_r(0,x) p_s(x,y) p_u(y,0) dr ds du \\
&= \frac{\gamma_d}{N h_{\lambda,d}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-(r+s)/N} p_{r+s+u}(O,O) dr ds du \\
&= \frac{\gamma_d}{N h_{\lambda,d}} \int_0^{+\infty} p_\theta(O,O) \left(\int_0^\theta e^{-v/N} \left(\int_0^v 1 du \right) dv \right) d\theta \\
&= \frac{N\gamma_d}{h_{\lambda,d}} \int_0^{+\infty} p_\theta(O,O) \left(1 - e^{-\theta/N} \left(1 + \frac{\theta}{N} \right) \right) d\theta.
\end{aligned}$$

Since $p_\theta(O,O) = O(\theta^{-d/2})$ as $\theta \rightarrow +\infty$, $\lim_{x \rightarrow +\infty} 1 - e^{-x}(1+x) = 1$ and $\lim_{x \rightarrow 0} \frac{1 - e^{-x}(1+x)}{x^2} = 1/2$, there exist $0 < M_1, C_4 < +\infty$ such that $p_\theta(O,O) \leq C_4 \theta^{-d/2}$ when $\theta \geq M_1$, $1 - e^{-x}(1+x) \leq C_4 x^2$ when $x \leq M_1$ and $1 - e^{-x}(1+x) \leq C_4$ when $x > M_1$. Therefore, for $d \geq 5$,

$$\begin{aligned}
\frac{N\gamma_d}{h_{\lambda,d}} \int_0^{M_1} p_\theta(O,O) \left(1 - e^{-\theta/N} \left(1 + \frac{\theta}{N} \right) \right) d\theta &\leq \frac{C_4 \gamma_d}{N h_{\lambda,d}} \int_0^{M_1} \theta^2 d\theta = O(N^{-1}), \\
\frac{N\gamma_d}{h_{\lambda,d}} \int_{M_1}^{NM_1} p_\theta(O,O) \left(1 - e^{-\theta/N} \left(1 + \frac{\theta}{N} \right) \right) d\theta &\leq \frac{C_4^2 \gamma_d}{N h_{\lambda,d}} \int_{M_1}^{NM_1} \theta^{2-\frac{d}{2}} d\theta = O(N^{-1/2})
\end{aligned}$$

and

$$\frac{N\gamma_d}{h_{\lambda,d}} \int_{NM_1}^{+\infty} p_\theta(O,O) \left(1 - e^{-\theta/N} \left(1 + \frac{\theta}{N} \right) \right) d\theta \leq \frac{C_4^2 N \gamma_d}{h_{\lambda,d}} \int_{NM_1}^{+\infty} \theta^{-d/2} d\theta = O(N^{2-\frac{d}{2}}).$$

In conclusion,

$$\lim_{N \rightarrow +\infty} \mathbb{E}_{\nu_{\lambda,d}} \left(\left(\frac{1}{\sqrt{N}} G_{1/N}(\eta_0) \right)^2 \right) = \lim_{N \rightarrow +\infty} \mathbb{E}_{\nu_{\lambda,d}} \left(\left(\frac{1}{\sqrt{N}} G_{1/N}(\eta_{tN^2}) \right)^2 \right) = 0. \quad (3.12)$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}_{\nu_{\lambda,d}} \left(\left(\frac{1}{\sqrt{N}} \int_0^{tN} \frac{1}{N} G_{1/N}(\eta_s) ds \right)^2 \right) &\leq \frac{tN}{N^3} \int_0^{tN} \left(\mathbb{E}_{\nu_{\lambda,d}} \left((G_{1/N}(\eta_s))^2 \right) \right) ds \\
&= \frac{t^2 N^2}{N^3} \mathbb{E}_{\nu_{\lambda,d}} \left((G_{1/N}(\eta_0))^2 \right).
\end{aligned}$$

Hence, according to Equation (3.12),

$$\lim_{N \rightarrow +\infty} \mathbb{E}_{\nu_{\lambda,d}} \left(\left(\frac{1}{\sqrt{N}} \int_0^{tN} \frac{1}{N} G_{1/N}(\eta_s) ds \right)^2 \right) = 0. \quad (3.13)$$

Lemma 3.2 follows from Equations (3.12) and (3.13). $\square$

At last, we prove Lemma 3.3.

Proof of Lemma 3.3. By Equation (3.5) and the fact that $\{X_t^N\}_{t \geq 0}$ is continuous in t ,

$$\sup_{0 \leq s \leq tN} \frac{1}{N} \left(M_s^{1/N} - M_{s-}^{1/N} \right)^2 \leq \sup_{0 \leq s \leq tN} \frac{1}{N} \sup_x \max_{y \sim x} \{g_{1/N}^2(x) \eta_s^2(x), g_{1/N}^2(x) \eta_s^2(y)\}. \quad (3.14)$$

For any $M > 0$, let

$$\tau_M = \inf \{s : \frac{1}{N} \sup_x \max_{y \sim x} \{g_{1/N}^2(x) \eta_s^2(x), g_{1/N}^2(x) \eta_s^2(y)\} > M\},$$

then

$$\left\{ \sup_{0 \leq s \leq tN} \frac{1}{N} \sup_x \max_{y \sim x} \{g_{1/N}^2(x) \eta_s^2(x), g_{1/N}^2(x) \eta_s^2(y)\} > M \right\} = \{\tau_M \leq tN\}.$$

Conditioned on $\{\tau_M \leq tN\}$, there exists $x_0 \in \mathbb{Z}^d$ such that

$$\max_{y \sim x_0} \{g_{1/N}^2(x_0) \eta_{\tau_M}^2(x_0), g_{1/N}^2(x_0) \eta_{\tau_M}^2(y)\} > NM.$$

If $\{\eta_s(y)\}_{y \sim x_0}$ and $\eta_s(x_0)$ do not jump to 0 during $s \in [\tau_M, \tau_M + 1]$, then

$$\max_{y \sim x_0} \{g_{1/N}^2(x) \eta_s^2(x_0), g_{1/N}^2(x_0) \eta_s^2(y)\} \geq NM e^{\min\{(\frac{1}{2\lambda d} - 1), 0\}}$$

for $s \in [\tau_M, \tau_{M+1}]$ and hence

$$\int_0^{tN+1} \sum_x g_{1/N}^4(x) \left(\eta_s^4(x) + \sum_{y \sim x} \eta_s^4(y) \right) ds \geq N^2 M^2 e^{2 \min\{(\frac{1}{2\lambda d} - 1), 0\}}.$$

Since the state of a vertex jumps to 0 at rate $1/(2\lambda d)$, according to the strong Markov property of $\{\eta_t\}_{t \geq 0}$,

$$\begin{aligned} P \left(\int_0^{tN+1} \sum_x g_{1/N}^4(x) \left(\eta_s^4(x) + \sum_{y \sim x} \eta_s^4(y) \right) ds \geq N^2 M^2 e^{2 \min\{(\frac{1}{2\lambda d} - 1), 0\}} \right) \\ \geq P(\tau_M \leq tN) e^{-\frac{2d+1}{2\lambda d}}. \end{aligned}$$

Then, by Markov's inequality, Proposition 2.4 and the fact that

$$\sum_x g_{1/N}^4(x) \leq \sum_x g_0^4(x) < +\infty,$$

we have

$$\begin{aligned} P \left(\sup_{0 \leq s \leq tN} \frac{1}{N} \sup_x \max_{y \sim x} \{g_{1/N}^2(x) \eta_s^2(x), g_{1/N}^2(x) \eta_s^2(y)\} > M \right) \\ \leq \frac{e^{\frac{2d+1}{2\lambda d}}}{N^2 M^2 e^{2 \min\{(\frac{1}{2\lambda d} - 1), 0\}}} \int_0^{tN+1} \sum_x g_0^4(x) (\mathbb{E}_{\lambda, d}(\eta^4(O)) (2d+1)) ds \leq \frac{C_3}{NM^2}, \quad (3.15) \end{aligned}$$

where $C_3 < +\infty$ is independent of N and M . According to Fubini theorem, for any positive random variable V and any $c > 0$,

$$\mathbb{E}(V 1_{\{V > c\}}) = cP(V > c) + \int_c^{+\infty} P(V \geq u) du.$$

Hence, for any $\epsilon > 0$,

$$\mathbb{E} \left(\sup_{0 \leq s \leq tN} \frac{1}{N} \left(M_s^{1/N} - M_{s-}^{1/N} \right)^2 \right) \leq \epsilon + \epsilon \frac{C_3}{N\epsilon^2} + \int_{\epsilon}^{+\infty} \frac{C_3}{Nu^2} du = \epsilon + \frac{2C_3}{N\epsilon}$$

according to Equations (3.14) and (3.15). As a result,

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left(\sup_{0 \leq s \leq tN} \frac{1}{N} \left(M_s^{1/N} - M_{s-}^{1/N} \right)^2 \right) \leq \epsilon.$$

Since ϵ is arbitrary, let $\epsilon \rightarrow 0$ and the proof is complete. □

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