

An iterative generalized quasi-boundary value regularization method for the backward problem of time fractional diffusion-wave equation in a cylinder

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Abstract

In this paper, we consider the backward problem for a time fractional diffusion-wave equation in a cylinder. The ill-posedness and a conditional stability of the inverse problem are proved. Based on the generalized quasi-boundary value regularization method, we propose an iterative generalized quasi-boundary value regularization method to deal with the inverse problem, and this iterative method has a higher convergence rate. The convergence rates of the regularized solutions under an *a priori* regularization parameter choice rule and an *a posteriori* regularization parameter choice rule are obtained. Numerical examples illustrate the effectiveness and stability of our proposed method.

Keywords: backward problem; time fractional diffusion-wave equation; iterative generalized quasi-boundary value regularization method; convergence rates

1. Introduction

In recent years, the fractional diffusion-wave equation has attracted more and more attention because it has been successfully applied to some important practical fields such as physics, chemistry, biology, finance, etc [1, 2]. Due to the memory and genetic properties of the fractional derivative, the fractional diffusion-wave equation has its unique advantages in describing superdiffusion phenomena [3, 4, 5].

The direct problem for the time fractional diffusion-wave equation is well-posed in the sense of Hadamard, and has been studied with relatively well-developed results [6, 7, 8]. However, the inverse problem for the time fractional diffusion-wave equation is usually ill-posed in the sense of Hadamard. This means that the existence, uniqueness and stability of the solution of the inverse problem cannot be satisfied simultaneously. The solution to most inverse problems exists, but the solution is not continuously dependent

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on the measurement data, i.e., a small noise in the measurement data can cause a large perturbation in the solution. In other words, the ill-posedness can lead to unstable result of the direct computation. Therefore, an appropriate regularization should be proposed to obtain a stable solution of the inverse problem.

Currently, some regularization methods are used for the inverse problem of the time fractional diffusion or diffusion-wave equation. Wei and Wang [9] proposed a modified quasi-boundary value method to identify a space-dependent source term of the time fractional diffusion equation. Tuan et al. [10] solved the backward problem for the inhomogeneous time fractional diffusion equation using the Tikhonov regularization method. Tuan et al. [11] proposed a modified regularization method to study the inverse initial value problem of the time fractional diffusion-wave equation. Yang et al. [12] used the truncated regularization method to determine the initial value of the inhomogeneous time fractional diffusion-wave equation. Han et al. [13] studied the backward problem of the time fractional diffusion equation using the fractional Landweber iterative regularization method. Yang et al. [14] proposed an iterated fractional Tikhonov regularization method for solving the backward problem of the spherically symmetric time fractional diffusion equation. Wei and Luo [15] proposed a generalized quasi-boundary value regularization method to deal with the inverse source problem in the time fractional diffusion-wave equation. For other regularization methods to solve the inverse problem of the time-fractional diffusion or diffusion-wave equation, refer to [16, 17, 18, 19]. The available regularization methods can be divided into two categories. One is non-iterative regularization methods, such as Tikhonov regularization method, truncation regularization method, and quasi-boundary value regularization method. The other category is iterative regularization methods, such as iterated Tikhonov regularization method and Landweber iterative regularization method. In this paper, we combine the generalized quasi-boundary value regularization method and iteration, and thus propose an iterative generalized quasi-boundary value regularization method for solving the backward problem of a time fractional diffusion-wave equation in a cylinder. We learned from [15] that the generalized quasi-boundary value regularization method has a higher convergence rate compared to the Larventiew regularization method and the modified quasi-boundary value regularization method. For the iterative generalized quasi-boundary value regularization method, we can obtain some higher convergence rates than the generalized quasi-boundary value regularization method in [15].

We focus on the following inhomogeneous time fractional diffusion-wave equation in a cylinder

$$D_t^\alpha u(r, z, t) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + f(r, z, t), \quad (1.1)$$

with the initial conditions

$$u(r, z, 0) = \phi(r, z), \quad 0 \leq r \leq R, 0 \leq z \leq L, \quad (1.2)$$

$$u_t(r, z, 0) = \psi(r, z), \quad 0 \leq r \leq R, 0 \leq z \leq L, \quad (1.3)$$

and the boundary conditions

$$\lim_{r \rightarrow 0} u(r, z, t) \text{ bounded}, \quad 0 \leq z \leq L, 0 \leq t \leq T, \quad (1.4)$$

$$u(R, z, t) = 0, \quad 0 \leq z \leq L, 0 \leq t \leq T, \quad (1.5)$$

$$u(r, 0, t) = u(r, L, t) = 0, \quad 0 \leq r \leq R, 0 \leq t \leq T, \quad (1.6)$$

where $f(r, z, t)$ is source term, the boundary condition (1.4) is equivalent to the symmetry boundary condition $u_r(0, z, t) = 0$ [20], $D_t^\alpha u(x, t)$ represents the Caputo fractional derivative of order α defined by

$$D_t^\alpha u(r, z, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u_{\tau\tau}(r, z, \tau)}{(t-\tau)^{\alpha-1}} d\tau, \quad 1 < \alpha < 2,$$

and $\phi(r, z)$ need to satisfy the compatibility conditions

$$\lim_{r \rightarrow 0} \phi(r, z) = \lim_{r \rightarrow 0} u(r, z, 0),$$

$$\phi(R, z) = u(R, z, 0),$$

$$\phi(r, 0) = u(r, 0, 0),$$

$$\phi(r, L) = u(r, L, 0).$$

If all the data $\phi(r, z)$, $\psi(r, z)$ and $f(r, z, t)$ are known, problem (1.1)-(1.6) is a direct problem for the time fractional diffusion-wave equation. The inverse problem here is to recover the initial value $\phi(r, z)$ based on the final time data

$$u(r, z, T) = g(r, z), \quad 0 \leq r \leq R, 0 \leq z \leq L. \quad (1.7)$$

Since the measurement is inevitably contaminated by noise, we assume that $g^\delta(r, z)$ is $g(r, z)$ with measurement noise and satisfies

$$\|g^\delta(r, z) - g(r, z)\| \leq \delta, \quad (1.8)$$

where $\|\cdot\|$ is the $L_r^2([0, R] \times [0, L])$ norm and $\delta > 0$ denotes a noise level.

The remainder of this paper is composed of five sections. In Section 2, we give some preliminary material. The ill-posedness and a conditional stability of the inverse problem are analyzed in Section 3. In Section 4, we propose an iterative generalized quasi-boundary value regularization method and provide the convergence rates under two regularization parameter choice rules. In Section 5, numerical examples are shown to verify the effectiveness and stability of our method. Finally, a brief conclusion is given in Section 6.

2. Preliminaries

Throughout this paper, we use $L_r^2(\Omega)$ to denote the Hilbert space of Lebesgue measurable function $v(r, z)$ with weight r on $\Omega = [0, R] \times [0, L]$. Denote the inner product and norm on $L_r^2(\Omega)$ by $(\cdot, \cdot)$ and $\|\cdot\|_{L_r^2(\Omega)}$, respectively, as follows

$$(u, v)_r = \int_{\Omega} u(r, z)v(r, z)rdrdz,$$

$$\|v\|_{L_r^2(\Omega)} = \left(\int_{\Omega} v^2(r, z)rdrdz \right)^{\frac{1}{2}}.$$

Then, we give some useful definitions and lemmas.

Definition 2.1 [21, 22]: The two parameter Mittag-Leffler function is defined as

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.1 [22]: If $1 < \alpha < 2$, $\beta \in \mathbb{R}$, $\eta > 0$, there holds

$$E_{\alpha, \beta}(-\eta) = \frac{1}{\Gamma(\beta - \alpha)\eta} + \frac{1}{O(\eta^2)}, \quad \eta \rightarrow \infty.$$

Lemma 2.2 [21]: If $1 < \alpha < 2$, $\beta \in \mathbb{R}$, $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, there exists a positive constant C which depends on α , β , and μ , such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

Lemma 2.3 [23]: For $0 < \zeta < 1$ and $k \geq 1$, define $p_k(\zeta) = \sum_{i=0}^{k-1} (1 - \zeta)^i$ and $r_k(\zeta) = 1 - \zeta p_k(\zeta) = (1 - \zeta)^k$. Then

$$\begin{aligned} p_k(\zeta)\zeta^\mu &\leq k^{1-\mu}, \quad 0 \leq \mu \leq 1, \\ r_k(\zeta)\zeta^\nu &\leq \theta_\nu(k+1)^{-\nu}, \end{aligned}$$

where

$$\theta_\nu = \begin{cases} 1, & 0 \leq \nu \leq 1, \\ \nu^\nu, & \nu > 1. \end{cases}$$

3. Ill-posedness and conditional stability

In this section, we formulate the inverse problem into an integral equation by the series expression of the weak solution for the direct problem. Further, we consider the ill-posedness and a conditional stability of the inverse problem.

If the source term $f(r, z, t)$, initial conditions, and boundary conditions as (1.1)–(1.6) are known, applying the method of separation of variables and Laplace transform of Mittag-Leffler function, we can obtain the unique solution of the direct problem as follows

$$u(r, z, t) = \sum_{m,n=1}^{\infty} [\phi_{mn} E_{\alpha,1}(-\lambda_{mn} t^{\alpha}) + \psi_{mn} t E_{\alpha,2}(-\lambda_{mn} t^{\alpha}) + \int_0^t f_{mn}(\tau) (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (t-\tau)^{\alpha}) d\tau] \omega_{mn}(r, z),$$

where $\phi_{mn} = (\phi(r, z), \omega_{mn}(r, z))_r$, $\psi_{mn} = (\psi(r, z), \omega_{mn}(r, z))_r$, $f_{mn}(\tau) = (f(r, z, \tau), \omega_{mn}(r, z))_r$, $\lambda_{mn} = (\frac{\mu_n}{R})^2 + (\frac{m\pi}{L})^2$. The eigenfunctions $\omega_{mn}(r, z) = \frac{2}{\sqrt{LR} J_1(\mu_n)} J_0(\frac{\mu_n r}{R}) \sin(\frac{m\pi}{L} z)$ form a standard orthogonal basis in $L_r^2(\Omega)$. The $J_0(x)$ and $J_1(x)$ denote the order 0 and order 1 Bessel functions, and $\{\mu_n\}_{n=1}^{\infty}$ are the sequence of the zeros of $J_0(x)$.

Using the final time data (1.7), we have

$$\begin{aligned} u(r, z, T) &= \sum_{m,n=1}^{\infty} [\phi_{mn} E_{\alpha,1}(-\lambda_{mn} T^{\alpha}) + \psi_{mn} T E_{\alpha,2}(-\lambda_{mn} T^{\alpha}) \\ &\quad + \int_0^T f_{mn}(\tau) (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (T-\tau)^{\alpha}) d\tau] \omega_{mn}(r, z) \\ &= g(r, z). \end{aligned}$$

Denote

$$\begin{aligned} h(r, z) &= g(r, z) - \sum_{m,n=1}^{\infty} [\psi_{mn} T E_{\alpha,2}(-\lambda_{mn} T^{\alpha}) \\ &\quad - \int_0^T f_{mn}(\tau) (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (T-\tau)^{\alpha}) d\tau] \omega_{mn}(r, z), \end{aligned}$$

then, the inverse problem is to find coefficients $\{\phi_{mn}\}_{m,n=1}^{\infty}$ from the following equation

$$\sum_{m,n=1}^{\infty} \phi_{mn} E_{\alpha,1}(-\lambda_{mn} T^{\alpha}) \omega_{mn}(r, z) = h(r, z). \quad (3.1)$$

Let $g_{mn} = (g(r, z), \omega_{mn}(r, z))_r$, $h_{mn} = (h(r, z), \omega_{mn}(r, z))_r$, we have

$$\phi_{mn} E_{\alpha,1}(-\lambda_{mn} T^{\alpha}) = h_{mn},$$

where $h_{mn} = g_{mn} - \psi_{mn} T E_{\alpha,2}(-\lambda_{mn} T^{\alpha}) - \int_0^T f_{mn}(\tau) (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (T-\tau)^{\alpha}) d\tau$.

According to (3.1), the inverse problem can be transformed into solving the following integral equation

$$\int_0^R \int_0^L \sum_{m,n=1}^{\infty} E_{\alpha,1}(-\lambda_{mn} T^{\alpha}) \xi \omega_{mn}(\xi, \eta) \omega_{mn}(r, z) \phi(\xi, \eta) d\xi d\eta = h(r, z). \quad (3.2)$$

Regarding the properties of the function $E_{\alpha,1}(-\lambda_{mn}T^\alpha)$, we give the following Lemmas.
Lemma 3.1: For $1 < \alpha < 2$ and any fixed $T > 0$, there exists at most finite points such that $E_{\alpha,1}(-\lambda_{mn}T^\alpha) = 0$. Denote the point set which makes $E_{\alpha,1}(-\lambda_{mn}T^\alpha) = 0$ is $I_1 = \{(mn)_1, (mn)_2, \dots, (mn)_J\}$.

Proof: By Lemma 2.1, we know that there exists $C_0 > 0$ such that

$$E_{\alpha,1}(-\lambda_{mn}T^\alpha) \leq \frac{1}{2\Gamma(1-\alpha)\lambda_{mn}T^\alpha} < 0, \quad \lambda_{mn}T^\alpha > C_0,$$

for $1 < \alpha < 2$, so we know $E_{\alpha,1}(-\lambda_{mn}T^\alpha) = 0$ only if $\lambda_{mn}T^\alpha \leq C_0$. Because $\lambda_{mn} \rightarrow +\infty$ when $m \rightarrow +\infty$ or $n \rightarrow +\infty$, there are only finite λ_{mn} satisfying $\lambda_{mn}T^\alpha \leq C_0$.

Lemma 3.2: For any λ_{mn} satisfying $\lambda_{mn} \geq \lambda_{11} > 0$ and $1 < \alpha < 2$, there exist positive constants $\underline{C}$ and $\overline{C}$ which depend on α, T and λ_{11} , such that

$$\frac{\underline{C}}{\lambda_{mn}} \leq |E_{\alpha,1}(-\lambda_{mn}T^\alpha)| \leq \frac{\overline{C}}{\lambda_{mn}}, \quad mn \notin I_1.$$

Proof: From Lemma 2.2, we have

$$|E_{\alpha,\beta}(-\lambda_{mn}T^\alpha)| \leq \frac{C}{1 + \lambda_{mn}T^\alpha} \leq \frac{\overline{C}}{\lambda_{mn}}.$$

By lemma 3.1 and its proof process, we know that there exists $C_0 > 0$ such that

$$|E_{\alpha,1}(-\lambda_{mn}T^\alpha)| \geq \left| \frac{1}{2\Gamma(1-\alpha)\lambda_{mn}T^\alpha} \right|, \quad \lambda_{mn}T^\alpha > C_0,$$

then there exist $\underline{C} > 0$ such that

$$|E_{\alpha,1}(-\lambda_{mn}T^\alpha)| \geq \frac{\underline{C}}{\lambda_{mn}}, \quad mn \notin I_1.$$

Thus, we have

$$\frac{\underline{C}}{\lambda_{mn}} \leq |E_{\alpha,1}(-\lambda_{mn}T^\alpha)| \leq \frac{\overline{C}}{\lambda_{mn}}, \quad mn \notin I_1.$$

From Lemma 3.1, we can easily get that there exists at most a finite point set $I_1 = \{(mn)_1, (mn)_2, \dots, (mn)_J\}$ such that $E_{\alpha,1}(-\lambda_{mn}T^\alpha) = 0$ for $mn \in I_1$. In particular, note that the point set I_1 may be empty, which can be considered as a special case.

We consider the general case, i.e., $I_1 \neq \emptyset$, then equation (3.2) can be rewritten as

$$\int_0^R \int_0^L \sum_{m,n=1, mn \notin I_1}^{\infty} E_{\alpha,1}(-\lambda_{mn}T^\alpha) \xi \omega_{mn}(\xi, \eta) \omega_{mn}(r, z) \phi(\xi, \eta) d\xi d\eta = h(r, z). \quad (3.3)$$

Similar to reference [24], we known that the equation (3.3) has infinitely solutions only when $h_{mn} = 0$ for $mn \in I_1$ and the condition

$$\sum_{m,n=1, mn \notin I_1}^{\infty} \left(\frac{h_{mn}}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)} \right)^2 < +\infty$$

holds, but in $L_r^2(\Omega)$, it exists a unique solution as

$$\tilde{\phi}(r, z) = \sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)} \omega_{mn}(r, z).$$

Denote $\omega_{mn}^*(r, z) = \begin{cases} \omega_{mn}(r, z), & E_{\alpha,1}(-\lambda_{mn}T^\alpha) > 0 \\ -\omega_{mn}(r, z), & E_{\alpha,1}(-\lambda_{mn}T^\alpha) < 0 \end{cases}$, we have

$$\tilde{\phi}(r, z) = \sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} \omega_{mn}^*(r, z). \quad (3.4)$$

From Lemma 3.2, we note that in (3.4)

$$\frac{1}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} \geq \frac{\lambda_{mn}}{C} = [(\frac{\mu_n}{R})^2 + (\frac{m\pi}{L})^2] \frac{1}{C} \rightarrow \infty, \quad \text{as } m, n \rightarrow \infty.$$

which indicates that the inverse problem is ill-posed, i.e., a small measurement noise in the data $g(r, z)$ can cause a large perturbation in the solution $\tilde{\phi}(r, z)$. Hence, it is necessary to use a regularization method to recover the stability of the solution of the inverse problem. Before that, we give a conditional stability theorem for the inverse problem.

Theorem 3.1: For any $\phi(r, z) \in \text{span}\{\omega_{mn}(r, z), mn \in I_1\}^\perp$ satisfying an *a priori* bound condition

$$\|\phi(r, z)\|_p \leq E, \quad p > 0, \quad (3.5)$$

where

$$\begin{aligned} \|\phi(r, z)\|_p &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (\lambda_{mn})^{\frac{p}{2}} \phi_{mn} \omega_{mn}(r, z) \right\|, \\ &= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (\lambda_{mn})^p \phi_{mn}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

then we have

$$\|\phi(r, z)\| \leq C_1 E^{\frac{2}{p+2}} \|h(r, z)\|^{\frac{p}{p+2}},$$

where $C_1 = (\frac{1}{C})^{\frac{p}{p+2}}$ is a constant.

Proof: For any $\phi(r, z) \in \text{span}\{\omega_{mn}(r, z), mn \in I_1\}^\perp$, we have

$$\phi(r, z) = \sum_{m,n=1, mn \notin I_1}^{\infty} \phi_{mn} \omega_{mn}(r, z).$$

Using the Hölder inequality, we obtain

$$\|\phi(r, z)\|^2 = \sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}^2}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|^2}$$

$$\begin{aligned}
&= \sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}^{\frac{4}{p+2}}}{|E_{\alpha,1}(-\lambda_{mn}T^{\alpha})|^2} h_{mn}^{\frac{2p}{p+2}} \\
&\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}^2}{|E_{\alpha,1}(-\lambda_{mn}T^{\alpha})|^{p+2}} \right)^{\frac{2}{p+2}} \left(\sum_{m,n=1, mn \notin I_1}^{\infty} h_{mn}^2 \right)^{\frac{p}{p+2}}. \quad (3.6)
\end{aligned}$$

From Lemma 3.2, there holds

$$\begin{aligned}
&\sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}^2}{|E_{\alpha,1}(-\lambda_{mn}T^{\alpha})|^{p+2}} \leq \sum_{m,n=1, mn \notin I_1}^{\infty} \frac{h_{mn}^2}{|E_{\alpha,1}(-\lambda_{mn}T^{\alpha})|^2} \left(\frac{\lambda_{mn}}{\underline{C}} \right)^p \\
&= \sum_{m,n=1, mn \notin I_1}^{\infty} \lambda_{mn}^p \phi_{mn}^2 \left(\frac{1}{\underline{C}} \right)^p = \left(\frac{1}{\underline{C}} \right)^p \|\phi(r, z)\|_p^2 \leq \left(\frac{1}{\underline{C}} \right)^p E^2. \quad (3.7)
\end{aligned}$$

Combining (3.6) and (3.7), we can obtain

$$\|\phi(r, z)\|^2 \leq \left(\frac{1}{\underline{C}} \right)^{\frac{2p}{p+2}} E^{\frac{4}{p+2}} \|h(r, z)\|^{\frac{2p}{p+2}}.$$

4. Iterative regularization method and convergence rate

In this section, we propose an iterative generalized quasi-boundary value regularization method to solve the backward problem of time-fractional diffusion-wave equation in a cylinder. The convergence rates of the regularized solution under an *a priori* regularization parameter choice rule and an *a posteriori* regularization parameter choice rule are given.

We define a linear operator $\mathcal{A} : L_r^2(\Omega) \rightarrow L_r^2(\Omega)$ as follows

$$\mathcal{A}^{\gamma} v(r, z) = \sum_{m,n=1}^{\infty} \lambda_{mn}^{\gamma} v_{mn} \omega_{mn}(r, z),$$

where $\gamma \geq 0$ is a real number and $v_{mn} = (v(r, z), \omega_{mn}(r, z))_r$.

Let the function $u_{\mu}^{\delta,k}(r, z, t)$ be the solution of the following iterative regularization problem

$$\left\{ \begin{array}{ll} D_t^{\alpha} u_{\mu}^{\delta,k}(r, z, t) = \frac{\partial^2 u_{\mu}^{\delta,k}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{\mu}^{\delta,k}}{\partial r} + \frac{\partial^2 u_{\mu}^{\delta,k}}{\partial z^2} + f(r, z, t), & 0 < r < R, 0 < z < L, 0 < t < T, \\ u_{\mu}^{\delta,k}(r, z, 0) = \phi_{\mu}^{\delta,k}(r, z), & 0 \leq r \leq R, 0 \leq z \leq L, \\ (u_{\mu}^{\delta,k})_t(r, z, 0) = \psi(r, z), & 0 \leq r \leq R, 0 \leq z \leq L, \\ \lim_{r \rightarrow 0} u_{\mu}^{\delta,k}(r, z, t) \text{ bounded}, & 0 \leq z \leq L, 0 \leq t \leq T, \\ u_{\mu}^{\delta,k}(R, z, t) = 0, & 0 \leq z \leq L, 0 \leq t \leq T, \\ u_{\mu}^{\delta,k}(r, 0, t) = u_{\mu}^{\delta,k}(r, L, t) = 0, & 0 \leq r \leq R, 0 \leq t \leq T, \\ u_{\mu}^{\delta,k}(r, z, T) = g^{\delta}(r, z) - \mu \mathcal{A}^{\gamma} \phi_{\mu}^{\delta,k}(r, z) + \mu \mathcal{A}^{\gamma} \phi_{\mu}^{\delta,k-1}(r, z), & 0 \leq r \leq R, 0 \leq z \leq L, \end{array} \right.$$

for $k = 1, 2, 3, \dots$, and the initial value $\phi_\mu^{\delta,0}(r, z) = 0$, where $\mu > 0$ is a regularization parameter and k is the number of iterations. If we fix $k = 1$, this iterative regularization method is transformed into the generalized quasi-boundary value regularization method in [15].

Denote $(\phi_\mu^{k,\delta})_{mn} = (\phi_\mu^{\delta,k}(r, z), \omega_{mn}(r, z))_r$, we have

$$\begin{aligned} u_\mu^{\delta,k}(r, z, T) &= \sum_{m,n=1}^{\infty} [(\phi_\mu^{k,\delta})_{mn} E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \psi_{mn} T E_{\alpha,2}(-\lambda_{mn} T^\alpha) \\ &\quad + \int_0^T f_{mn}(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (T - \tau)^\alpha) d\tau] \omega_{mn}(r, z). \end{aligned}$$

Then, based on the iterative regularization form, we have

$$\begin{aligned} u_\mu^{\delta,k}(r, z, T) &= \sum_{m,n=1}^{\infty} [(\phi_\mu^{k,\delta})_{mn} E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \psi_{mn} T E_{\alpha,2}(-\lambda_{mn} T^\alpha) \\ &\quad + \int_0^T f_{mn}(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (T - \tau)^\alpha) d\tau] \omega_{mn}(r, z) \\ &= \sum_{m,n=1}^{\infty} [g_{mn}^\delta - \mu \lambda_{mn}^\gamma (\phi_\mu^{k,\delta})_{mn} + \mu \lambda_{mn}^\gamma (\phi_\mu^{k-1,\delta})_{mn}] \omega_{mn}(r, z), \end{aligned}$$

where $g_{mn} = (g(r, z), \omega_{mn}(r, z))_r$. Further, we can deduce that

$$\begin{aligned} \phi_\mu^{\delta,k}(r, z) &= \sum_{m,n=1}^{\infty} \left[\frac{h_{mn}^\delta}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} + \frac{\mu \lambda_{mn}^\gamma (\phi_\mu^{k-1,\delta})_{mn}}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} \right] \omega_{mn}(r, z) \\ &= \sum_{m,n=1}^{\infty} \sum_{i=0}^{k-1} \left(\frac{\mu \lambda_{mn}^\gamma}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} \right)^i \frac{h_{mn}^\delta}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} \omega_{mn}(r, z) \\ &= \sum_{m,n=1}^{\infty} \left(1 - \left(\frac{\mu \lambda_{mn}^\gamma}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} \right)^k \right) \frac{h_{mn}^\delta}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} \omega_{mn}(r, z), \end{aligned} \quad (4.1)$$

where $h_{mn}^\delta = g_{mn}^\delta - \psi_{mn} T E_{\alpha,2}(-\lambda_{mn} T^\alpha) - \int_0^T f_{mn}(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn} (T - \tau)^\alpha) d\tau$.

From (4.1), we can find that $\left(\frac{\mu \lambda_{mn}^\gamma}{E_{\alpha,1}(-\lambda_{mn} T^\alpha) + \mu \lambda_{mn}^\gamma} \right)^k$ may tend to infinity when $E_{\alpha,1}(-\lambda_{mn} T^\alpha) < 0$, which causes trouble in obtaining the error estimate between $\phi_\mu^{\delta,k}(r, z)$ and $\tilde{\phi}(r, z)$. Thus, we modify the regularized solution $\phi_\mu^{\delta,k}(r, z)$ as

$$\tilde{\phi}_\mu^{\delta,k}(r, z) = \sum_{m,n=1, mn \notin I_1}^{\infty} \left(1 - \left(\frac{\mu \lambda_{mn}^\gamma}{|E_{\alpha,1}(-\lambda_{mn} T^\alpha)| + \mu \lambda_{mn}^\gamma} \right)^k \right) \frac{h_{mn}^\delta}{|E_{\alpha,1}(-\lambda_{mn} T^\alpha)|} \omega_{mn}^*(r, z).$$

For simplicity, let

$$F_{\mu,\gamma}^k(\lambda_{mn}) = 1 - \left(\frac{\mu \lambda_{mn}^\gamma}{|E_{\alpha,1}(-\lambda_{mn} T^\alpha)| + \mu \lambda_{mn}^\gamma} \right)^k,$$

then we obtain the regularized solution as

$$\tilde{\phi}_\mu^{\delta,k}(r, z) = \sum_{m,n=1, mn \notin I_1}^{\infty} F_{\mu,\gamma}^k(\lambda_{mn}) \frac{h_{mn}^\delta}{|E_{\alpha,1}(-\lambda_{mn} T^\alpha)|} \omega_{mn}^*(r, z). \quad (4.2)$$

4.1. a priori regularization parameter choice rule

We first prove some useful lemmas, and to ensure that the following lemmas and theorems are strictly valid, assume that $\lambda_{mn} \geq 1$. This holds easily because $\lambda_{mn} = (\frac{\mu_n}{R})^2 + (\frac{m\pi}{L})^2$, so it is sufficient to take $L \leq \pi$ to satisfy $\lambda_{mn} \geq 1$.

Lemma 4.1: For constants $\mu > 0$, $\gamma \geq 0$, $T > 0$, $k > 0$, we have

$$\sup_{m,n \in \mathbb{N}} \frac{F_{\mu,\gamma}^k(\lambda_{mn})}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} \leq \frac{kC_2}{\mu^{\frac{1}{1+\gamma}}}.$$

Proof: By Lemma 2.3, we have

$$F_{\mu,\gamma}^k(\lambda_{mn}) \leq \frac{k|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)| + \mu\lambda_{mn}^\gamma},$$

then

$$\begin{aligned} \frac{F_{\mu,\gamma}^k(\lambda_{mn})}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} &= \frac{k}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)| + \mu\lambda_{mn}^\gamma} \\ &\leq \frac{k}{\frac{\underline{C}}{\lambda_{mn}} + \mu\lambda_{mn}^\gamma} \\ &= \frac{k\lambda_{mn}}{\underline{C} + \mu\lambda_{mn}^{1+\gamma}}. \end{aligned} \tag{4.3}$$

Introduce a new variable $x = \lambda_{mn}$, let

$$A(x) = \frac{kx}{\underline{C} + \mu x^{1+\gamma}}. \tag{4.4}$$

For $\gamma \geq 0$, the function $A(x)$ is continuous. Since $A(x) \geq 0$, $\lim_{x \rightarrow 0} A(x) = 0$ and $\lim_{x \rightarrow +\infty} A(x) = 0$, the maximum point satisfies $A'(x) = 0$. Thus, we obtain

$$x^* = \left(\frac{\underline{C}}{\mu\gamma}\right)^{\frac{1}{1+\gamma}},$$

and

$$A(x) \leq A(x^*) = \frac{k(1+\gamma)^{-1}\gamma^{\frac{\gamma}{1+\gamma}}\underline{C}^{-\frac{\gamma}{1+\gamma}}}{\mu^{\frac{1}{1+\gamma}}}. \tag{4.5}$$

Denote $C_2 = (1+\gamma)^{-1}\gamma^{\frac{\gamma}{1+\gamma}}\underline{C}^{-\frac{\gamma}{1+\gamma}}$, from (4.3) and (4.5), we have

$$\sup_{m,n \in \mathbb{N}} \frac{F_{\mu,\gamma}^k(\lambda_{mn})}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} \leq \frac{kC_2}{\mu^{\frac{1}{1+\gamma}}}.$$

Lemma 4.2: For constants $\mu > 0$, $\gamma \geq 0$, $T > 0$, $k > 0$, we have

$$\sup_{m,n \in \mathbb{N}} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))\lambda_{mn}^{-\frac{p}{2}} \leq \begin{cases} \underline{C}^{-k}\mu^k, & p \geq 2k(1+\gamma), \\ C_3^k k^{-k} \mu^{\frac{p}{2(1+\gamma)}}, & 0 < p < 2k(1+\gamma). \end{cases}$$

Proof: By $\lambda_{mn} \geq 1$ and Lemma 3.2, we have

$$\begin{aligned}
(1 - F_{\mu,\gamma}^k(\lambda_{mn}))\lambda_{mn}^{-\frac{p}{2}} &= \left(\frac{\mu\lambda_{mn}^\gamma}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)| + \mu\lambda_{mn}^\gamma} \right)^k \lambda_{mn}^{-\frac{p}{2}} \\
&\leq \left(\frac{\mu\lambda_{mn}^\gamma}{\frac{C}{\lambda_{mn}} + \mu\lambda_{mn}^\gamma} \right)^k \lambda_{mn}^{-\frac{p}{2}} \\
&= \left(\frac{\mu\lambda_{mn}^{1+\gamma-\frac{p}{2k}}}{C + \mu\lambda_{mn}^{1+\gamma}} \right)^k.
\end{aligned} \tag{4.6}$$

Let

$$B(x) = \frac{\mu x^{1+\gamma-\frac{p}{2k}}}{C + \mu x^{1+\gamma}}. \tag{4.7}$$

If $1 + \gamma - \frac{p}{2k} \leq 0$, that is $p \geq 2k(1 + \gamma)$, we have

$$B(x) \leq C^{-1}\mu. \tag{4.8}$$

If $1 + \gamma - \frac{p}{2k} > 0$, that is $0 < p < 2k(1 + \gamma)$, the function $B(x)$ is continuous. Since $B(x) \geq 0$, $\lim_{x \rightarrow 0} B(x) = 0$ and $\lim_{x \rightarrow +\infty} B(x) = 0$, the maximum point satisfies $B'(x^*) = 0$. Thus, we obtain

$$x^* = \left(\frac{C(2k(1 + \gamma) - p)}{p\mu} \right)^{\frac{1}{1+\gamma}},$$

and

$$B(x) \leq B(x^*) = \frac{1}{2k(1 + \gamma)} p^{\frac{p}{2k(1+\gamma)}} \mu^{\frac{p}{2k(1+\gamma)}} C^{-\frac{p}{2k(1+\gamma)}} (2k(1 + \gamma) - p)^{1 - \frac{p}{2k(1+\gamma)}}. \tag{4.9}$$

Denote $C_3 = \frac{1}{2(1+\gamma)} p^{\frac{p}{2k(1+\gamma)}} C^{-\frac{p}{2k(1+\gamma)}} (2k(1 + \gamma) - p)^{1 - \frac{p}{2k(1+\gamma)}}$, from (4.6) to (4.9), we have

$$\sup_{m,n \in \mathbb{N}} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))\lambda_{mn}^{-\frac{p}{2}} \leq \begin{cases} C^{-k}\mu^k, & p \geq 2k(1 + \gamma), \\ C_3^k k^{-k} \mu^{\frac{p}{2(1+\gamma)}}, & 0 < p < 2k(1 + \gamma). \end{cases}$$

Theorem 4.1: Let the exact solution $\tilde{\phi}(r, z)$ be given by (3.4) and the regularized solution $\tilde{\phi}_\mu^{\delta,k}(r, z)$ be given by (4.2). If the noise data $g^\delta(r)$ satisfies (1.8) and the exact solution $\tilde{\phi}(r, z)$ satisfies the *a priori* bound condition (3.5), the following convergence rates are satisfied.

(1) If $p \geq 2k(1 + \gamma)$ and choose $\mu = \left(\frac{\delta}{E}\right)^{\frac{1+\gamma}{1+k+k\gamma}}$, we have a convergence rate

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}(r, z)\| \leq (kC_2 + C^{-k}) E^{\frac{1}{1+k+k\gamma}} \delta^{\frac{k+k\gamma}{1+k+k\gamma}}.$$

(2) If $0 < p < 2k(1 + \gamma)$ and choose $\mu = \left(\frac{\delta}{E}\right)^{\frac{2(1+\gamma)}{p+2}}$, we have a convergence rate

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}(r, z)\| \leq (kC_2 + C_3^k k^{-k}) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof: Using the triangle inequality, we obtain

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}(r, z)\| \leq \|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}_\mu^k(r, z)\| + \|\tilde{\phi}_\mu^k(r, z) - \tilde{\phi}(r, z)\|.$$

From (1.8), (4.2) and Lemma 4.1, we have

$$\begin{aligned} \|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}_\mu^k(r, z)\| &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} F_{\mu,\gamma}^k(\lambda_{mn}) \frac{1}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} (h_{mn}^\delta - h_{mn}) \omega_{mn}^*(r, z) \right\| \\ &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} F_{\mu,\gamma}^k(\lambda_{mn}) \frac{1}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} (g_{mn}^\delta - g_{mn}) \omega_{mn}^*(r, z) \right\| \\ &\leq \sup_{m,n \in \mathbb{N}} \frac{F_{\mu,\gamma}^k(\lambda_{mn})}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (g_{mn}^\delta - g_{mn}) \omega_{mn}^*(r, z) \right\| \\ &\leq \frac{kC_2}{\mu^{\frac{1}{1+\gamma}}} \delta. \end{aligned} \tag{4.10}$$

From (3.4), (3.5), (4.2) and Lemma 4.2, we have

$$\begin{aligned} \|\tilde{\phi}_\mu^k(r, z) - \tilde{\phi}(r, z)\| &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (F_{\mu,\gamma}^k(\lambda_{mn}) - 1) \frac{h_{mn}}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)|} \omega_{mn}^*(r, z) \right\| \\ &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} \lambda_{mn}^{\frac{p}{2}} \tilde{\phi}_{mn} \omega_{mn}^*(r, z) \right\| \\ &\leq \sup_{m,n \in \mathbb{N}} (1 - F_{\mu,\gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} \lambda_{mn}^{\frac{p}{2}} \tilde{\phi}_{mn} \omega_{mn}^*(r, z) \right\| \\ &\leq E \sup_{m,n \in \mathbb{N}} (1 - F_{\mu,\gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} \\ &\leq \begin{cases} \underline{C}^{-k} \mu^k E, & p \geq 2k(1 + \gamma), \\ C_3^k k^{-k} \mu^{\frac{p}{2(1+\gamma)}} E, & 0 < p < 2k(1 + \gamma). \end{cases} \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we obtain

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}(r, z)\| \leq \frac{kC_2}{\mu^{\frac{1}{1+\gamma}}} \delta + \begin{cases} \underline{C}^{-k} \mu^k E, & p \geq 2k(1 + \gamma), \\ C_3^k k^{-k} \mu^{\frac{p}{2(1+\gamma)}} E, & 0 < p < 2k(1 + \gamma). \end{cases}$$

Choose μ by

$$\mu = \begin{cases} \left(\frac{\delta}{E}\right)^{\frac{1+\gamma}{1+k+k\gamma}}, & p \geq 2k(1 + \gamma), \\ \left(\frac{\delta}{E}\right)^{\frac{2(1+\gamma)}{p+2}}, & 0 < p < 2k(1 + \gamma), \end{cases}$$

then we have

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}(r, z)\| \leq \begin{cases} (kC_2 + \underline{C}^{-k}) E^{\frac{1}{1+k+k\gamma}} \delta^{\frac{k+k\gamma}{1+k+k\gamma}}, & p \geq 2k(1 + \gamma), \\ (kC_2 + C_3^k k^{-k}) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2k(1 + \gamma). \end{cases}$$

4.2. a posteriori regularization parameter choice rule

We consider an *a posteriori* regularization parameter choice rule which is in accordance with the Morozov discrepancy principle.

First, we define a linear operator $\mathcal{B} : L_r^2(\Omega) \rightarrow L_r^2(\Omega)$ and an orthogonal operator $P : L_r^2(\Omega) \rightarrow \text{span}\{\omega_{mn}(r, z), mn \in I_1\}^\perp$ as follows

$$\begin{aligned}\mathcal{B}v(r, z) &= \sum_{m,n=1}^{\infty} E_{\alpha,1}(-\lambda_{mn}T^\alpha)v_{mn}\omega_{mn}(r, z), \\ Pv(r, z) &= \sum_{m,n=1, mn \notin I_1}^{\infty} v_{mn}\omega_{mn}(r, z),\end{aligned}$$

where $v_{mn} = (v(r, z), \omega_{mn}(r, z))_r$. Then, the Morozov difference principle here is to find the regularization parameter μ so that

$$\|\mathcal{B}\tilde{\phi}_\mu^{\delta,k}(r, z) - Ph^\delta(r, z)\| = \tau\delta, \quad (4.12)$$

where $\tau > 1$ is a constant and

$$\begin{aligned}Ph(r, z)^\delta &= \sum_{m,n=1, mn \notin I_1}^{\infty} [g_{mn}^\delta - \psi_{mn}TE_{\alpha,2}(-\lambda_{mn}T^\alpha) \\ &\quad - \int_0^T f_{mn}(\tau)(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{mn}(T-\tau)^\alpha)d\tau]\omega_{mn}(r, z).\end{aligned} \quad (4.13)$$

According to the following lemma, we know that there exists a unique solution for (4.10) if $Ph(r, z)^\delta > \tau\delta > 0$.

Lemma 4.3: Set

$$G(\mu) = \|\mathcal{B}\tilde{\phi}_\mu^{\delta,k}(r, z) - Ph^\delta(r, z)\|.$$

If $\|h^\delta(r, z)\| > 0$, then the following results are obtained

- (a) $G(\mu)$ is a continuous function;
- (b) $\lim_{\mu \rightarrow 0} G(\mu) = 0$;
- (c) $\lim_{\mu \rightarrow +\infty} G(\mu) = \|h^\delta(r, z)\|$;
- (d) $G(\mu)$ is a strictly increasing function over $(0, +\infty)$.

Proof: From (4.2) and (4.13), we have

$$\begin{aligned}G(\mu) &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (F_{\mu,\gamma}^k(\lambda_{mn}) - 1)h_{mn}^\delta\omega_{mn}^*(r, z) \right\| \\ &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} \left(\frac{\mu\lambda_{mn}^\gamma}{|E_{\alpha,1}(-\lambda_{mn}T^\alpha)| + \mu\lambda_{mn}^\gamma} \right)^k h_{mn}^\delta\omega_{mn}^*(r, z) \right\|.\end{aligned} \quad (4.14)$$

This implies that

$$\begin{aligned} G(\mu) &\leq \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} h_{mn}^{\delta} \omega_{mn}^*(r, z) \right\|, \\ &= \|h^{\delta}(r, z)\|. \end{aligned}$$

From (4.14), the results are obtained.

Lemma 4.4: For constants $\mu > 0$, $\gamma \geq 0$, $T > 0$, $k > 0$, we have

$$\sup_{m,n \in \mathbb{N}} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})| \leq \begin{cases} \overline{C} \underline{C}^{-k} \mu^k, & p \geq 2k(1 + \gamma) - 2, \\ \overline{C} C_4^k k^{-k} \mu^{\frac{p+2}{2(1+\gamma)}}, & 0 < p < 2k(1 + \gamma) - 2. \end{cases}$$

Proof: By $\lambda_{mn} \geq 1$ and Lemma 3.2, we have

$$\begin{aligned} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})| &= \left(\frac{\mu \lambda_{mn}^{\gamma}}{|E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})| + \mu \lambda_{mn}^{\gamma}} \right)^k \lambda_{mn}^{-\frac{p}{2}} |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})| \\ &\leq \left(\frac{\mu \lambda_{mn}^{\gamma}}{\frac{\underline{C}}{\lambda_{mn}} + \mu \lambda_{mn}^{\gamma}} \right)^k \lambda_{mn}^{-\frac{p}{2}} \frac{\overline{C}}{\lambda_{mn}} \\ &= \left(\frac{\mu \lambda_{mn}^{1+\gamma-\frac{p+2}{2k}}}{\underline{C} + \mu \lambda_{mn}^{1+\gamma}} \right)^k \overline{C}. \end{aligned} \quad (4.15)$$

Let

$$H(x) = \frac{\mu x^{1+\gamma-\frac{p+2}{2k}}}{\underline{C} + \mu x^{1+\gamma}}. \quad (4.16)$$

If $1 + \gamma - \frac{p+2}{2k} \leq 0$, that is $p \geq 2k(1 + \gamma) - 2$, we have

$$H(x) \leq \underline{C}^{-1} \mu. \quad (4.17)$$

If $1 + \gamma - \frac{p+2}{2k} > 0$, that is $0 < p < 2k(1 + \gamma) - 2$, the function $H(x)$ is continuous. Since $H(x) \geq 0$, $\lim_{x \rightarrow 0} H(x) = 0$ and $\lim_{x \rightarrow +\infty} H(x) = 0$, the maximum point satisfies $H'(x^*) = 0$. Thus, we obtain

$$x^* = \left(\frac{\underline{C}(2k(1 + \gamma) - p - 2)}{(p + 2)\mu} \right)^{\frac{1}{1+\gamma}},$$

and

$$H(x) \leq H(x^*) = \frac{1}{2k(1 + \gamma)} (p + 2)^{\frac{p+2}{2k(1+\gamma)}} \mu^{\frac{p+2}{2k(1+\gamma)}} \underline{C}^{-\frac{p+2}{2k(1+\gamma)}} (2k(1 + \gamma) - p - 2)^{1 - \frac{p+2}{2k(1+\gamma)}}. \quad (4.18)$$

Denote $C_4 = \frac{1}{2(1+\gamma)} (p + 2)^{\frac{p+2}{2k(1+\gamma)}} \underline{C}^{-\frac{p+2}{2k(1+\gamma)}} (2k(1 + \gamma) - p - 2)^{1 - \frac{p+2}{2k(1+\gamma)}}$, from (4.15) to (4.18), we have

$$\sup_{m,n \in \mathbb{N}} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})| \leq \begin{cases} \overline{C} \underline{C}^{-k} \mu^k, & p \geq 2k(1 + \gamma) - 2, \\ \overline{C} C_4^k k^{-k} \mu^{\frac{p+2}{2(1+\gamma)}}, & 0 < p < 2k(1 + \gamma) - 2. \end{cases}$$

Theorem 4.2: Let the exact solution $\tilde{\phi}(r, z)$ be given by (3.4) and the regularized solution $\tilde{\phi}_\mu^{\delta, k}(r, z)$ be given by (4.2). Assume that the noise data $g^\delta(r)$ satisfies (1.8) and the exact solution $\tilde{\phi}(r, z)$ satisfies the *a priori* bound condition (3.5). The regularization parameter μ is chosen by the Morozov discrepancy principle (4.12), then the following convergence rates are satisfied.

(1) If $p \geq 2k(1 + \gamma) - 2$, we have a convergence rate

$$\|\tilde{\phi}_\mu^{\delta, k}(r, z) - \tilde{\phi}(r, z)\| \leq C_5 E^{\frac{1}{k+k\gamma}} \delta^{\frac{k+k\gamma-1}{k+k\gamma}},$$

where $C_5 = KC_2(\frac{\overline{C}C^{-k}}{\tau-1})^{\frac{1}{k+k\gamma}} + (1 + \tau)^{\frac{k+k\gamma-1}{k+k\gamma}} \underline{C}^{\frac{1-k-k\gamma}{k+k\gamma}}$.

(2) If $0 < p < 2k(1 + \gamma) - 2$, we have a convergence rate

$$\|\tilde{\phi}_\mu^{\delta, k}(r, z) - \tilde{\phi}(r, z)\| \leq C_6 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}},$$

where $C_6 = KC_2(\frac{\overline{C}C_4^k k^{-k}}{\tau-1})^{\frac{2}{p+2}} + (1 + \tau)^{\frac{p}{p+2}} \underline{C}^{\frac{-2p}{p+2}}$.

Proof: From (4.12) and Lemma 4.4, we have

$$\begin{aligned} \tau\delta &= \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (F_{\mu, \gamma}^k(\lambda_{mn}) - 1) h_{mn}^\delta \omega_{mn}^*(r, z) \right\| \\ &\leq \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) (h_{mn}^\delta - h_{mn}) \omega_{mn}^*(r, z) \right\| \\ &\quad + \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) h_{mn} \omega_{mn}^*(r, z) \right\| \\ &\leq \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (h_{mn}^\delta - h_{mn}) \omega_{mn}^*(r, z) \right\| + \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (F_{\mu, \gamma}^k(\lambda_{mn}) - 1) h_{mn} \omega_{mn}^*(r, z) \right\| \\ &\leq \delta + \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) h_{mn} \omega_{mn}^*(r, z) \right\| \\ &= \delta + \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} \lambda_{mn}^{\frac{p}{2}} |E_{\alpha, 1}(-\lambda_{mn} T^\alpha)| \tilde{\phi}_{mn} \omega_{mn}^*(r, z) \right\| \\ &\leq \delta + \sup_{m, n \in \mathbb{N}} (1 - F_{\mu, \gamma}^k(\lambda_{mn})) \lambda_{mn}^{-\frac{p}{2}} |E_{\alpha, 1}(-\lambda_{mn} T^\alpha)| \left\| \sum_{m,n=1, mn \notin I_1}^{\infty} \lambda_{mn}^{\frac{p}{2}} \tilde{\phi}_{mn} \omega_{mn}^*(r, z) \right\| \\ &\leq \delta + \begin{cases} \overline{C} \underline{C}^{-k} \mu^k E, & p \geq 2k(1 + \gamma) - 2, \\ \overline{C} C_4^k k^{-k} \mu^{\frac{p+2}{2(1+\gamma)}} E, & 0 < p < 2k(1 + \gamma) - 2. \end{cases} \end{aligned}$$

Therefore, we have

$$\frac{1}{\mu} \leq \begin{cases} (\frac{\overline{C} \underline{C}^{-k}}{\tau-1})^{\frac{1}{k}} (\frac{E}{\delta})^{\frac{1}{k}}, & p \geq 2k(1 + \gamma) - 2, \\ (\frac{\overline{C} C_4^k k^{-k}}{\tau-1})^{\frac{2(1+\gamma)}{p+2}} (\frac{E}{\delta})^{\frac{2(1+\gamma)}{p+2}}, & 0 < p < 2k(1 + \gamma) - 2. \end{cases} \quad (4.19)$$

Combining (4.10) and (4.19), we obtain

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}_\mu^k(r, z)\| \leq \begin{cases} KC_2 \left(\frac{\overline{C}C^{-k}}{\tau-1} \right)^{\frac{1}{k+k\gamma}} E^{\frac{1}{k+k\gamma}} \delta^{\frac{k+k\gamma-1}{k+k\gamma}}, & p \geq 2k(1+\gamma) - 2, \\ KC_2 \left(\frac{\overline{C}C_4^k k^{-k}}{\tau-1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2k(1+\gamma) - 2. \end{cases} \quad (4.20)$$

From (3.4) and (4.2), we have

$$\|\tilde{\phi}_\mu^k(r, z) - \tilde{\phi}(r, z)\|^2 = \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 \tilde{\phi}_{mn}^2.$$

Case 1: $p \geq 2k(1+\gamma) - 2$, we have

$$\begin{aligned} \|\tilde{\phi}_\mu^k(r, z) - \tilde{\phi}(r, z)\|^2 &= \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 \tilde{\phi}_{mn}^2 \\ &= \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^{2\frac{k+k\gamma-1}{k+k\gamma}} h_{mn}^{2\frac{k+k\gamma-1}{k+k\gamma}} \dots \\ &\quad (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^{2\frac{1}{k+k\gamma}} |E_{\alpha,1}(-\lambda_{mn}T^\alpha)|^{-2\frac{k+k\gamma-1}{k+k\gamma}} \tilde{\phi}_{mn}^{2\frac{1}{k+k\gamma}} \\ &\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 h_{mn}^2 \right)^{\frac{k+k\gamma-1}{k+k\gamma}} \dots \\ &\quad \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 |E_{\alpha,1}(-\lambda_{mn}T^\alpha)|^{-2(k+k\gamma-1)} \tilde{\phi}_{mn}^2 \right)^{\frac{1}{k+k\gamma}} \\ &:= (Q_1)^{\frac{k+k\gamma-1}{k+k\gamma}} (Q_2)^{\frac{1}{k+k\gamma}}. \end{aligned} \quad (4.21)$$

Then

$$\begin{aligned} Q_1^{\frac{1}{2}} &= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 h_{mn}^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 (h_{mn} - h_{mn}^\delta)^2 \right)^{\frac{1}{2}} + \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 (h_{mn}^\delta)^2 \right)^{\frac{1}{2}} \\ &\leq \delta + \tau\delta. \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} Q_2^{\frac{1}{2}} &= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu,\gamma}^k(\lambda_{mn}))^2 |E_{\alpha,1}(-\lambda_{mn}T^\alpha)|^{-2(k+k\gamma-1)} \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} \left(\frac{C}{\lambda_{mn}} \right)^{-2(k+k\gamma-1)} \lambda_{mn}^{-p} \lambda_{mn}^p \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} \underline{C}^{-2(k+k\gamma-1)} \lambda_{mn}^{2k+2k\gamma-2-p} \lambda_{mn}^p \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} \underline{C}^{-2(k+k\gamma-1)} \lambda_{mn}^p \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}} \\
&\leq \underline{C}^{1-k-k\gamma} E.
\end{aligned} \tag{4.23}$$

From (4.21) to (4.23), we get

$$\|\tilde{\phi}_{\mu}^k(r, z) - \tilde{\phi}(r, z)\| \leq (1 + \tau) \underline{C}^{\frac{k+k\gamma-1}{k+k\gamma}} E^{\frac{1-k-k\gamma}{k+k\gamma}} \delta^{\frac{k+k\gamma-1}{k+k\gamma}}. \tag{4.24}$$

Case 2: $0 < p < 2k(1 + \gamma) - 2$, we have

$$\begin{aligned}
\|\tilde{\phi}_{\mu}^k(r, z) - \tilde{\phi}(r, z)\|^2 &= \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 \tilde{\phi}_{mn}^2 \\
&= \sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^{2\frac{p}{p+2}} h_{mn}^{2\frac{p}{p+2}} \dots \\
&\quad (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^{2\frac{2}{p+2}} |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})|^{-2\frac{p}{p+2}} \tilde{\phi}_{mn}^{2\frac{2}{p+2}} \\
&\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 h_{mn}^2 \right)^{\frac{p}{p+2}} \dots \\
&\quad \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})|^{-p} \tilde{\phi}_{mn}^2 \right)^{\frac{2}{p+2}} \\
&:= (Q_3)^{\frac{p}{p+2}} (Q_4)^{\frac{2}{p+2}}.
\end{aligned} \tag{4.25}$$

Then

$$\begin{aligned}
Q_3^{\frac{1}{2}} &= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 h_{mn}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 (h_{mn} - h_{mn}^{\delta})^2 \right)^{\frac{1}{2}} + \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 (h_{mn}^{\delta})^2 \right)^{\frac{1}{2}} \\
&\leq \delta + \tau \delta.
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
Q_4^{\frac{1}{2}} &= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} (1 - F_{\mu, \gamma}^k(\lambda_{mn}))^2 |E_{\alpha, 1}(-\lambda_{mn} T^{\alpha})|^{-p} \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{m,n=1, mn \notin I_1}^{\infty} \left(\frac{\underline{C}}{\lambda_{mn}} \right)^{-p} \lambda_{mn}^{-p} \lambda_{mn}^p \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{m,n=1, mn \notin I_1}^{\infty} \underline{C}^{-p} \lambda_{mn}^p \tilde{\phi}_{mn}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \underline{C}^{-p} E. \quad (4.27)$$

From (4.25) to (4.27), we get

$$\|\tilde{\phi}_\mu^k(r, z) - \tilde{\phi}(r, z)\| \leq (1 + \tau)^{\frac{p}{p+2}} \underline{C}^{\frac{-2p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (4.28)$$

Combining (4.24) and (4.28), we have

$$\|\tilde{\phi}_\mu^k(r, z) - \tilde{\phi}(r, z)\| \begin{cases} (1 + \tau)^{\frac{k+k\gamma-1}{k+k\gamma}} \underline{C}^{\frac{1-k-k\gamma}{k+k\gamma}} E^{\frac{1}{k+k\gamma}} \delta^{\frac{k+k\gamma-1}{k+k\gamma}}, & p \geq 2k(1 + \gamma) - 2, \\ (1 + \tau)^{\frac{p}{p+2}} \underline{C}^{\frac{-2p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2k(1 + \gamma) - 2. \end{cases} \quad (4.29)$$

Combining (4.20) and (4.29), we obtain

$$\|\tilde{\phi}_\mu^{\delta,k}(r, z) - \tilde{\phi}(r, z)\| \leq \begin{cases} [KC_2(\frac{\overline{C}\underline{C}^{-k}}{\tau-1})^{\frac{1}{k+k\gamma}} + (1 + \tau)^{\frac{k+k\gamma-1}{k+k\gamma}} \underline{C}^{\frac{1-k-k\gamma}{k+k\gamma}}] E^{\frac{1}{k+k\gamma}} \delta^{\frac{k+k\gamma-1}{k+k\gamma}}, & p \geq 2k(1 + \gamma) - 2, \\ [KC_2(\frac{\overline{C}C_4^k k^{-k}}{\tau-1})^{\frac{2}{p+2}} + (1 + \tau)^{\frac{p}{p+2}} \underline{C}^{\frac{-2p}{p+2}}] E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2k(1 + \gamma) - 2. \end{cases}$$

Remark 4.1: For the iterative generalized quasi-boundary value regularization method, there is no saturation in the convergence rates. If the smoothness index p of the unknown initial value $\phi(r, z)$ is large enough, we can choose a slightly larger k such that the convergence rate $O(\delta^{\frac{k+k\gamma}{1+k+k\gamma}})$ is better than $O(\delta^{\frac{1+\gamma}{2+\gamma}})$ under the *a priori* regularization parameter choice rule and the convergence rate $O(\delta^{\frac{k+k\gamma-1}{k+k\gamma}})$ is better than $O(\delta^{\frac{\gamma}{1+\gamma}})$ under the *a posteriori* regularization parameter choice rule. $O(\delta^{\frac{1+\gamma}{2+\gamma}})$ and $O(\delta^{\frac{\gamma}{1+\gamma}})$ are the convergence rates of the generalized quasi-boundary value regularization method under the *a priori* regularization parameter choice rule and the *a posteriori* regularization parameter choice rule, respectively. This is obvious because both functions $\frac{k+k\gamma}{1+k+k\gamma}$ and $\frac{k+k\gamma-1}{k+k\gamma}$ are strictly increasing functions with respect to $k \in \mathbb{N}$.

5. Numerical implementation

In this section, we present some examples to illustrate the effectiveness and stability of the proposed method. To avoid the 'inverse crime', the finite difference methods [25] are used to calculate the direct problem. The finite difference schemes are sketched as follows.

First, we denote the discrete points in the time interval $[0, T]$ as $t_n = n\tau$ ($n = 0, 1, \dots, N$) with the time step size $\tau = \frac{T}{N}$, the grid points in the space interval $[0, R]$ as $r_i = ih_r$ ($i = 0, 1, \dots, M$) and $[0, L]$ as $z_j = jh_z$ ($j = 0, 1, \dots, M$), where $h_r = \frac{R}{M}$ and $h_z = \frac{L}{M}$ represents the space step size. The value of each grid point is $u_{i,j}^n = u(r_i, z_j, t_n)$.

Second, based on the finite difference scheme, we discretize the equation $D_t^\alpha u(r, z, t) = \frac{\partial u^2}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial u^2}{\partial z^2} + f(r, z, t)$ as

$$\begin{aligned} & \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} [b_0^{(\alpha)} \delta_t u_{i,j}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)}) \delta_t u_{i,j}^{k-\frac{1}{2}} - b_{n-1}^{(\alpha)} \psi_{i,j}] \\ &= \frac{1}{h_r^2} (u_{i+1,j}^{n-\frac{1}{2}} - 2u_{i,j}^{n-\frac{1}{2}} + u_{i-1,j}^{n-\frac{1}{2}}) + \frac{1}{2r_i h_r} (u_{i+1,j}^{n-\frac{1}{2}} - u_{i-1,j}^{n-\frac{1}{2}}) + \frac{1}{h_z^2} (u_{i,j+1}^{n-\frac{1}{2}} - 2u_{i,j}^{n-\frac{1}{2}} + u_{i,j-1}^{n-\frac{1}{2}}) \\ &+ f(r_i, z_j, t_{n-\frac{1}{2}}), \quad i, j = 1, 2, \dots, M-1, \end{aligned} \quad (5.1)$$

where $b_k^{(\alpha)} = (k+1)^{2-\alpha} - k^{2-\alpha} (k \geq 0)$, $u_{i,j}^{n-\frac{1}{2}} = \frac{1}{2}(u_{i,j}^n + u_{i,j}^{n-1})$, $\delta_t u_{i,j}^{n-\frac{1}{2}} = \frac{1}{\tau}(u_{i,j}^n - u_{i,j}^{n-1})$. Denote $p = \frac{\tau^\alpha \Gamma(3-\alpha)}{h_r^2}$, $q = \frac{\tau^\alpha \Gamma(3-\alpha)}{h_z^2}$, $s = \frac{\tau^\alpha \Gamma(3-\alpha)}{h_r}$ and (5.1) can be simplified as

$$\begin{aligned} & (\frac{s}{4r_i} - \frac{p}{2}) u_{i-1,j}^n + (1+p+q) u_{i,j}^n - (\frac{s}{4r_i} + \frac{p}{2}) u_{i+1,j}^n - \frac{q}{2} u_{i,j-1}^n - \frac{q}{2} u_{i,j+1}^n \\ &= (\frac{p}{2} - \frac{s}{4r_i}) u_{i-1,j}^{n-1} + (1-p-q) u_{i,j}^{n-1} + (\frac{p}{2} + \frac{s}{4r_i}) u_{i+1,j}^{n-1} + \frac{q}{2} u_{i,j-1}^{n-1} + \frac{q}{2} u_{i,j+1}^{n-1} \\ &+ \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)}) (u_{i,j}^k - u_{i,j}^{k-1}) + \tau b_{n-1}^{(\alpha)} \psi_{i,j} + \frac{\tau^\alpha \Gamma(3-\alpha)}{2} (f_{i,j}^n + f_{i,j}^{n-1}). \end{aligned}$$

Then we have the following matrix equation

$$\begin{aligned} AU^1 &= BU^0 + \tau b_0^{(\alpha)} C + \frac{\tau^\alpha \Gamma(3-\alpha)}{2} (F^1 + F^0), \\ AU^n &= BU^{n-1} + \beta_1 U^{n-1} + \beta_2 U^{n-2} + \dots + \beta_n U^0 + \tau b_{n-1}^{(\alpha)} C + \frac{\tau^\alpha \Gamma(3-\alpha)}{2} (F^n + F^{n-1}), \\ n &= 2, 3, \dots, N, \end{aligned}$$

where $\beta_1 = b_0^{(\alpha)} - b_1^{(\alpha)}$, $\beta_l = 2b_{l-1}^{(\alpha)} - b_l^{(\alpha)} - b_{l-2}^{(\alpha)} (l = 2, 3, \dots, n-1)$, $\beta_n = b_{n-1}^{(\alpha)} - b_{n-2}^{(\alpha)}$, vectors

$$\begin{aligned} U^n &= (u_{1,1}^n, u_{1,2}^n, \dots, u_{1,M-1}^n, u_{2,1}^n, u_{2,2}^n, \dots, u_{2,M-1}^n, \dots, u_{M-1,1}^n, u_{M-1,2}^n, \dots, u_{M-1,M-1}^n), \\ C &= (\psi_{1,1}, \psi_{1,2}, \dots, \psi_{1,M-1}, \psi_{2,1}, \psi_{2,2}, \dots, \psi_{2,M-1}, \dots, \psi_{M-1,1}, \psi_{M-1,2}, \dots, \psi_{M-1,M-1}), \\ F^n &= (f_{1,1}^n, f_{1,2}^n, \dots, f_{1,M-1}^n, f_{2,1}^n, f_{2,2}^n, \dots, f_{2,M-1}^n, \dots, f_{M-1,1}^n, f_{M-1,2}^n, \dots, f_{M-1,M-1}^n), \\ n &= 0, 1, \dots, N, \end{aligned}$$

A and B are matrices

$$A = \begin{bmatrix} A_{1,1} & -(\frac{s}{4r_1} + \frac{p}{2})I & & \\ (\frac{s}{4r_2} - \frac{p}{2})I & A_{2,2} & -(\frac{s}{4r_2} + \frac{p}{2})I & \\ & \ddots & \ddots & \ddots \\ & & (\frac{s}{4r_{M-2}} - \frac{p}{2})I & A_{M-2,M-2} & -(\frac{s}{4r_{M-2}} + \frac{p}{2})I \\ & & & (\frac{s}{4r_{M-1}} - \frac{p}{2})I & A_{M-1,M-1} \end{bmatrix} \in R^{(M-1)^2 \times (M-1)^2},$$

$$B = \begin{bmatrix} B_{1,1} & (\frac{p}{2} + \frac{s}{4r_1})I & & \\ (\frac{p}{2} - \frac{s}{4r_2})I & B_{2,2} & (\frac{p}{2} + \frac{s}{4r_2})I & \\ & \ddots & \ddots & \ddots \\ & & (\frac{p}{2} - \frac{s}{4r_{M-2}})I & B_{M-2,M-2} & (\frac{p}{2} + \frac{s}{4r_{M-2}})I \\ & & & (\frac{p}{2} - \frac{s}{4r_{M-1}})I & B_{M-1,M-1} \end{bmatrix} \in R^{(M-1)^2 \times (M-1)^2},$$

$$A_{i,i} = \begin{bmatrix} (1+p+q) & -\frac{q}{2} & & \\ -\frac{q}{2} & (1+p+q) & -\frac{q}{2} & \\ & \ddots & \ddots & \ddots \\ & & -\frac{q}{2} & (1+p+q) & -\frac{q}{2} \\ & & & -\frac{q}{2} & (1+p+q) \end{bmatrix} \in R^{(M-1) \times (M-1)},$$

$$B_{i,i} = \begin{bmatrix} (1-p-q) & \frac{q}{2} & & \\ \frac{q}{2} & (1-p-q) & \frac{q}{2} & \\ & \ddots & \ddots & \ddots \\ & & \frac{q}{2} & (1-p-q) & \frac{q}{2} \\ & & & \frac{q}{2} & (1-p-q) \end{bmatrix} \in R^{(M-1) \times (M-1)},$$

$i = 1, 2, \dots, M-1$, and $I \in R^{(M-1) \times (M-1)}$ is identity matrix.

Final, the final time data $u(r, z, T) = g(r, z)$ is obtained.

After obtaining the final time data $g(r, z)$, we generate noisy data $g^\delta(r, z)$ by adding a random disturbance, i.e.

$$g^\delta = g + \epsilon g \cdot (2\text{rand}(\text{size}(g)) - 1),$$

and the corresponding noise level is calculated by $\delta = \|g^\delta - g\|$.

To show the accuracy of the regularization method, we calculate the relative error between the exact solution and the regularized solution by

$$e_r = \frac{\|\tilde{\phi}_\mu^{k,\delta}(r, z) - \tilde{\phi}(r, z)\|}{\|\tilde{\phi}(r, z)\|}.$$

In numerical experiments, we always take $\alpha = 1.5$, $R = \pi$, $L = \pi$, $T = 1$, $N = 10$, $M = 40$. The exact solution $\tilde{\phi}(r, z)$ is given by (3.4) and the regularized solution $\tilde{\phi}_\mu^{k,\delta}(r, z)$ is given (4.2). The regularization parameter μ under the a priori choice rule is given by Theorem 4.1, and the regularization parameter μ under the a posteriori choice rule is chosen by (4.12) with $\tau = 1.1$. Moreover, to demonstrate the advantages of our proposed iterative generalized quasi-boundary value regularization method (IGM), we

compare it with the generalized quasi-boundary value regularization method (GM). We have mentioned that when $k = 1$, IGM agree with GM. Here we choose $k = 20$ in IGM, and $\gamma = 0.5$.

Example 5.1: Take the initial values

$$\begin{aligned}\phi(r, z) &= (r^2 - \pi^2)(z^2 - \pi z), \\ \psi(r, z) &= 0,\end{aligned}$$

and the source term

$$f(r, z, t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}(r^2 - \pi^2)(z^2 - \pi z) - 4(t^2 + 1)(z^2 - \pi z) - 2(t^2 + 1)(r^2 - \pi^2).$$

Then the exact solution is given by

$$u(r, z, t) = (t^2 + 1)(r^2 - \pi^2)(z^2 - \pi z).$$

Example 5.2: Take the initial values

$$\begin{aligned}\phi(r, z) &= \begin{cases} (\frac{1}{2} + \frac{r}{\pi}) \sin(z), & 0 \leq r < \frac{\pi}{2}, \\ (2 - \frac{2r}{\pi}) \sin(z), & \frac{\pi}{2} \leq r \leq \pi, \end{cases} \\ \psi(r, z) &= \begin{cases} (1 + \frac{2r}{\pi}) \sin(z), & 0 \leq r < \frac{\pi}{2}, \\ (4 - \frac{4r}{\pi}) \sin(z), & \frac{\pi}{2} \leq r \leq \pi, \end{cases}\end{aligned}$$

and the source term

$$f(r, z, t) = 2t^{2-\alpha}r \sin(z).$$

Figs 1-6 give the comparisons and the absolute error between the exact solution $\phi(r, z)$ and the regularized solutions of GM and IGM under the *a priori* and *a posteriori* regularization parameter choice rules with $\epsilon = 0.001$ for Examples 5.1-5.2. Tables 1-2 show the relative errors of GM and IGM under the *a priori* and *a posteriori* regularization parameter choice rules with different noise levels of Examples 5.1-5.2.

From Figs 1-6 and Tables 1-2, we can find that IGM is even comparable to GM. It can also be concluded that the smaller ϵ , the more effective the regularization method is. In addition, it seems that the results obtained from the *a priori* regularization parameter choice rule and the *a posteriori* regularization parameter choice rule are very close to each other. This is because the *a priori* regularization parameter choice rule is related to the *a priori* bound condition. If a better *a priori* bound condition is chosen, the result is closer to the *a posteriori* regularization parameter choice rule. However, if a smaller or larger *a priori* bound condition is chosen, the result is worse. In summary, numerical examples verify the effectiveness and stability of our iterative regularization method.

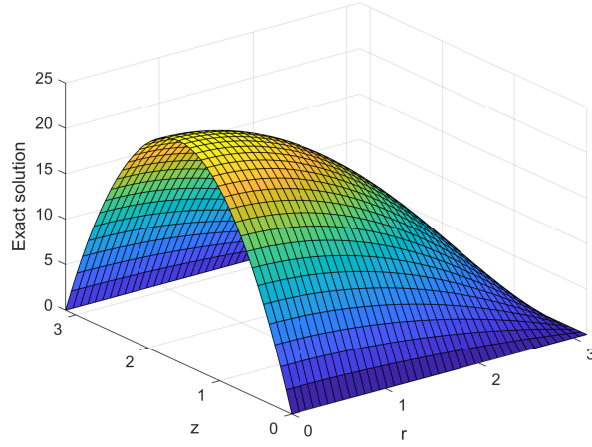
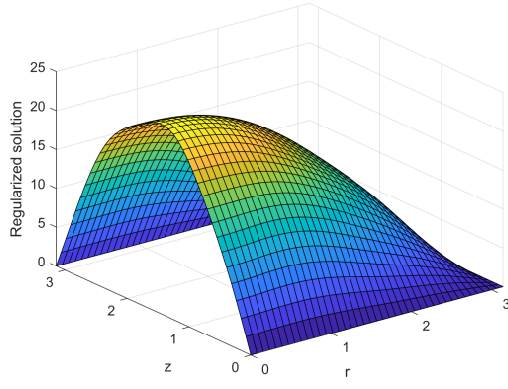
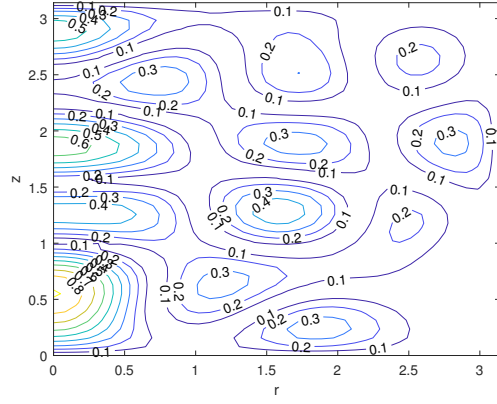


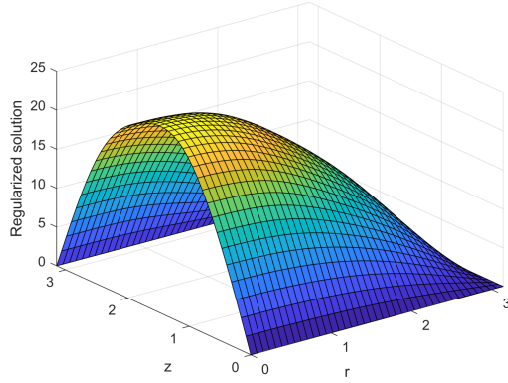
Fig 1: Exact solution $\phi(r, z)$ for Example 5.1.



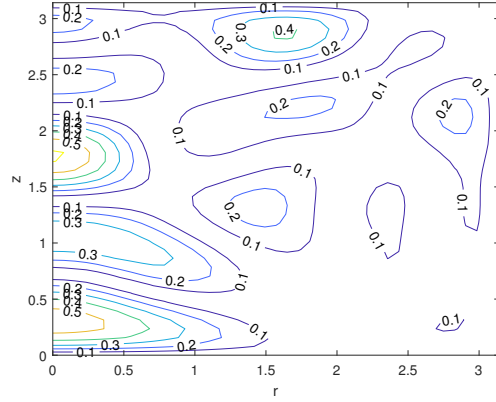
(a) GM



(b) Absolute error

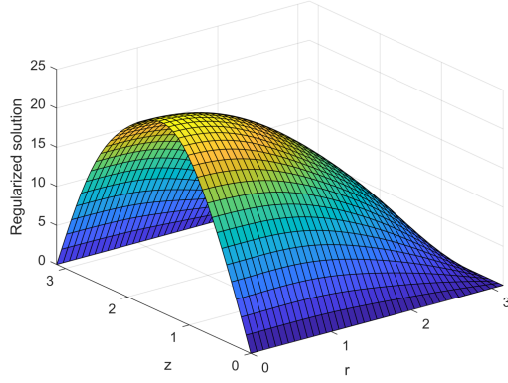


(c) IGM

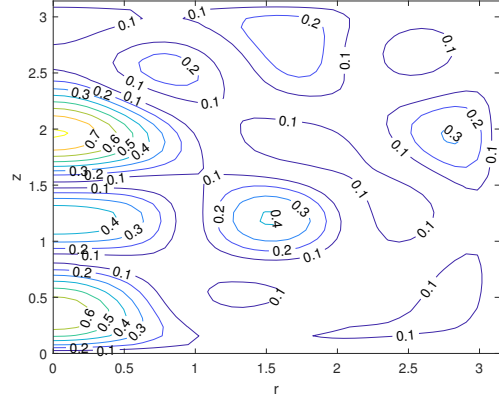


(d) Absolute error

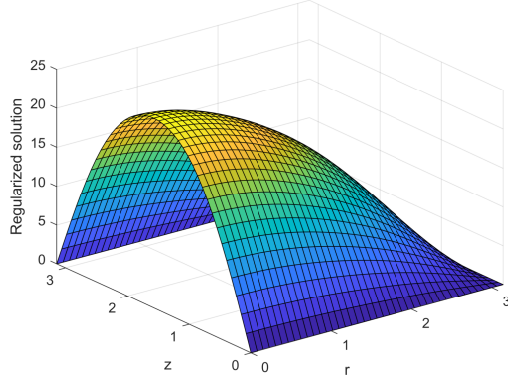
Fig 2: The comparison of regularized solutions of GM and IGM under *a priori* case for Example 5.1.



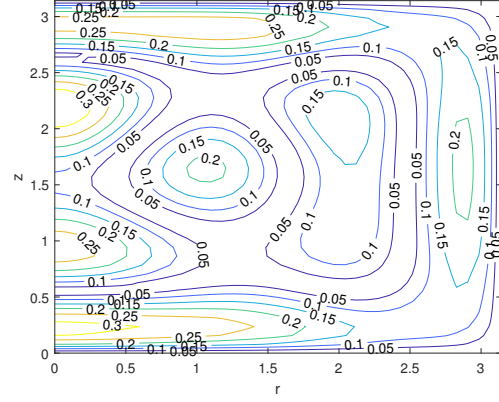
(a) GM



(b) Absolute error



(c) IGM



(d) Absolute error

Fig 3: The comparison of regularized solutions of GM and IGM under *a posteriori* case for Example 5.1.

Table 1: The relative errors for different ϵ under *a priori* case and *a posteriori* case of Example 5.1

	ϵ	0.001	0.01	0.05
GM	e_r^{pri}	0.0145	0.0644	0.1583
IGM	e_r^{pri}	0.0095	0.0361	0.1072
GM	e_r^{post}	0.0136	0.0520	0.1419
IGM	e_r^{post}	0.0087	0.0337	0.0981

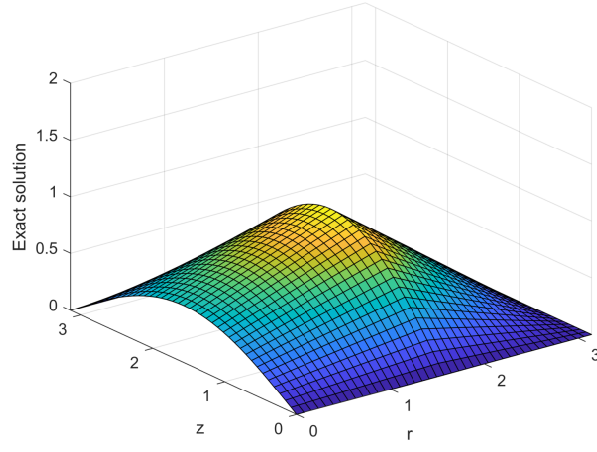
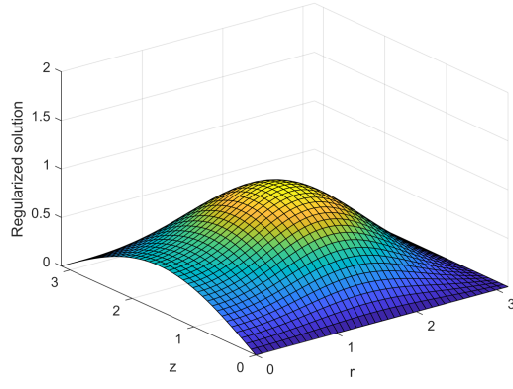
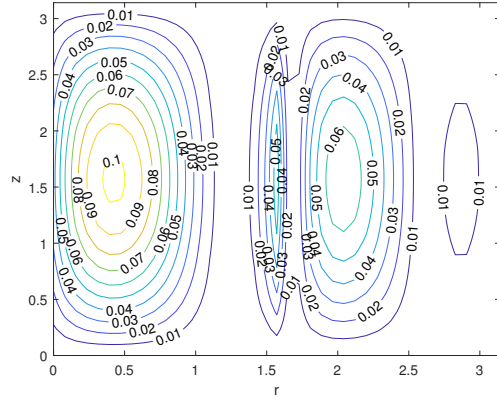


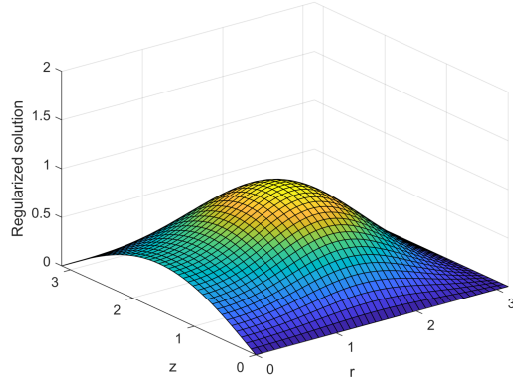
Fig 4: Exact solution $\phi(r, z)$ for Example 5.2.



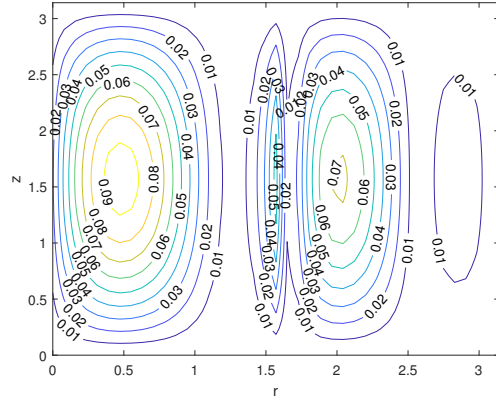
(a) GM



(b) Absolute error

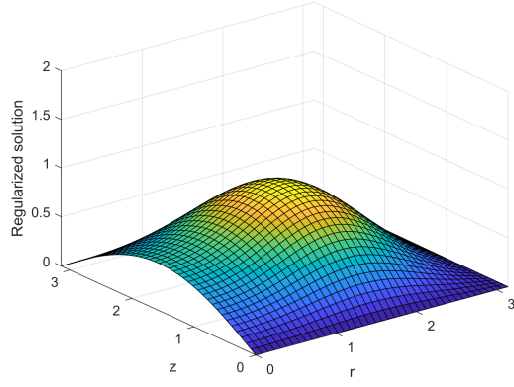


(c) IGM

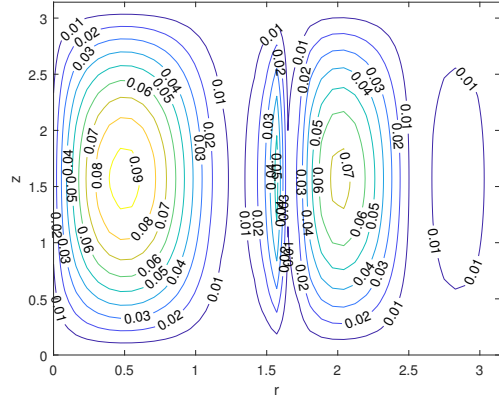


(d) Absolute error

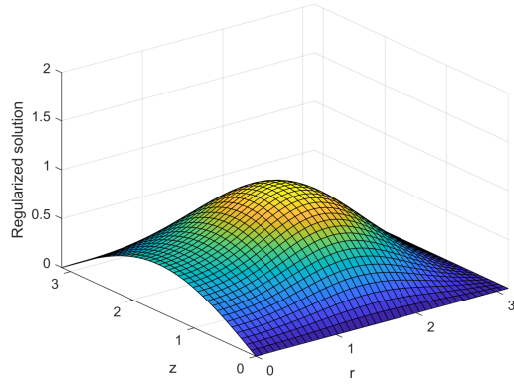
Fig 5: The comparison of regularized solutions of GM and IGM under *a priori* case for Example 5.2.



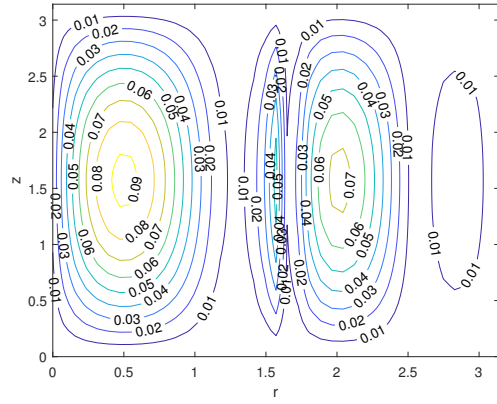
(a) GM



(b) Absolute error



(c) IGM



(d) Absolute error

Fig 6: The comparison of regularized solutions of GM and IGM under *a posteriori* case for Example 5.2.

Table 2: The relative errors for different ϵ under *a priori* case and *a posteriori* case of Example 5.2

	ϵ	0.001	0.01	0.05
GM	e_r^{pri}	0.0782	0.0864	0.1874
IGM	e_r^{pri}	0.0754	0.0934	0.1728
GM	e_r^{post}	0.0751	0.1051	0.2313
IGM	e_r^{post}	0.0746	0.0998	0.1527

6. Conclusion

In this paper, we propose an iterative generalized quasi-boundary value regularization method for solving the backward problem of time fractional diffusion-wave equation in a cylinder from final time data. The ill-posedness and a conditional stability of the inverse problem are proved. Based on an *a priori* assumption for the exact solution, we obtain the convergence rates of the regularized solutions under an *a priori* regularization parameter choice rule and an *a posteriori* regularization parameter choice rule. Numerical examples illustrate the effectiveness and stability of our proposed method.

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- Consent to participate: Not applicable
- Consent for publication: Not applicable
- Availability of data and materials: Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
- Code availability: Not applicable
- Authors' contributions: Chengxin Shi wrote the main manuscript text and Hao Cheng revised the manuscript. All authors reviewed the manuscript.

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