1 Magnetophoretic force

The force on a magnetic particle is

$$F = \frac{V\chi}{\mu_0}(\vec{B} \cdot \nabla)\vec{B} \tag{1}$$

where V is the volume of the particle, χ is the magnetic susceptibility of the particle (assuming the magnetic susceptibility of the surrounding buffer is zero), $\mu_0 = 4\pi \times 10^{-7}$ (T m A^{-1}), and B is the magnetic field (T). Here we are also assuming that the particles exhibit magnetic saturation. For SPIONs like ferumoxytol, these saturate on the order of millitesla, and our magnetic fields at the relevant distances are on the order of 0.5 T.

The measured magnetic susceptibilities of the superparamagnetic particles are mass susceptibilities χ_m, given in units of m^3 kg^{-1}, so we multiply them by their density ρ.

$$\begin{aligned}
F &= \frac{V\rho\chi_m}{\mu_0}(\vec{B} \cdot \nabla)\vec{B} \\
\vec{B} \cdot \nabla &= \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \\
&= B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \\
(\vec{B} \cdot \nabla)\vec{B} &= \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \\
&= \begin{bmatrix} B_x \frac{\partial B_x}{\partial x} + B_y \frac{\partial B_x}{\partial y} + B_z \frac{\partial B_x}{\partial z} \\ B_x \frac{\partial B_y}{\partial x} + B_y \frac{\partial B_y}{\partial y} + B_z \frac{\partial B_y}{\partial z} \\ B_x \frac{\partial B_z}{\partial x} + B_y \frac{\partial B_z}{\partial y} + B_z \frac{\partial B_z}{\partial z} \end{bmatrix}
\end{aligned}$$

Therefore the magnetophoretic force in the x direction is $F_{m,x} \propto \left(B_x \frac{\partial B_x}{\partial x} + B_y \frac{\partial B_x}{\partial y} + B_z \frac{\partial B_x}{\partial z} \right)$.

2 Magnetic field gradient

The magnetic field gradient ∇B is

$$\nabla B = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial x} & \frac{\partial B_z}{\partial x} \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{bmatrix}$$

The magnitude of the magnetic field gradient is given by the Frobenius norm

$$\|\nabla B\| = \sqrt{\left(\frac{\partial B_x}{\partial x}\right)^2 + \left(\frac{\partial B_y}{\partial x}\right)^2 + \left(\frac{\partial B_z}{\partial x}\right)^2 + \left(\frac{\partial B_x}{\partial y}\right)^2 + \left(\frac{\partial B_y}{\partial y}\right)^2 + \left(\frac{\partial B_z}{\partial y}\right)^2 + \left(\frac{\partial B_x}{\partial z}\right)^2 + \left(\frac{\partial B_y}{\partial z}\right)^2 + \left(\frac{\partial B_z}{\partial z}\right)^2}$$