Simulating single-photon detector array sensors for depth imaging : supplemental document

1 Derivation of photons detected per-pulse-per-pixel

The number of photons returned from a target is described by a photon channel which models the loss of signal photons as a series of sequential processes. Consider a laser pulse of initial energy E_0 having a divergence θ projected over a range R, through an atmosphere of attenuation length C_{atm}. The energy density ρ_E at the target is,

$$\rho_E = \frac{E_0 e^{\frac{-R}{C_{atm}}}}{\pi R^2 \tan^2(\theta)}. \tag{1}$$

For an imaging sensor with pixels of effective size $W_p \times H_p$ at the focal plane of a collecting lens of focal length f and f-number f_{no}, the energy E_1 available to each pixel is,

$$E_1 = \rho_E \left(\frac{R^2 W_p H_p}{f^2} \right). \tag{2}$$

Assuming Lambertian reflection, for a target of reflectively Γ the scattered energy E_2 which arrives at the aperture of the lens for each pixel is,

$$E_2 = \frac{\Gamma E_1 e^{\frac{-R}{C_{atm}}}}{2\pi R^2}. \tag{3}$$

Each pixel then captures a fraction of this scattered energy according to the aperture of the lens and the quantum efficiency q of the detector,

$$E_3 = q E_2 \pi \left(\frac{f}{2 f_{no}} \right)^2. \tag{4}$$

Combining Eqs. 1-4 and dividing by $\frac{hc}{\lambda}$ where λ is the wavelength of the illuminating light, the number of photons-per-pulse P_{pp} captured by the detector is,

$$P_{pp} = \frac{\lambda E_0}{hc} \frac{q \Gamma e^{\frac{-2R}{C_{atm}}}}{8} \frac{W_p H_p}{f_{no}^2 \pi R^2 \tan^2(\theta)}. \tag{5}$$

2 Experimental parameters used in simulating the resolution test target

Symbol	**Parameter**	**Value**	**Unit**
E_0	Energy per pulse	1	nJ
ν	Repetition rate	2.25	MHz
λ	Wavelength	671	nm
σ'	Pulse FWHM	600	ps
R	Range	14.73	m
C_{atm}	Attenuation	6.2	km
θ	Divergence	0.02	radians
Γ	Reflectivity	0.09	–
C_{bckg}	Solar background	0	W
f_{no}	f-number	2.0	-
C_{dc}	Dark counts	126	Hz
η	Exposure time	1000	μs
q	Quantum efficiency	0.26	–
W_p/H_p	Pixel size (width/height)	9.2	μm
ω	Bin width	50	ps
j	Jitter	200	ps

Table 1: The parameters used to model the resolution test target. Note that the target was effectively normal to all pixels and so a constant reflectivity value was used.

3 Match filtering as an unbiased estimator

Consider a single Gaussian signal on top of a uniform background $g(t)$ match filtered with a Gaussian kernel $f(t)$. The depth estimate $\hat{\mu}$ is then given by,

$$\hat{\mu} = \text{argmax}[f(t) * g(t)], \tag{6}$$

where

$$\begin{aligned} f(t) &= \frac{1}{\sigma'\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma'}\right)^2\right], \\ g(t) &= \frac{1}{\sigma'\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu'}{\sigma'}\right)^2\right] + A\text{rect}\left[\frac{t-\mu'}{T}\right], \end{aligned} \tag{7}$$

Formally, the background is treated as a uniform value of amplitude A over the time domain T centered upon the true depth value μ'. Under the conditions that the signal $g(t)$ contains only a single peak that is situated sufficiently far (i.e. ¿¿ $3\sigma'$) from the boundaries of the domain of t, then Eq. 6 is equivalent to,

$$\max[f(t) * g(t)]. \tag{8}$$

Thus,

$$\begin{aligned}
\max[f(t) * g(t)] &= \max\left[\int_{-\infty}^{\infty} f(t)g(t)dt\right], \\
\max[f(t) * g(t)] &= \max\left\{B\exp\left[-\frac{1}{2}\left(\frac{\mu-\mu'}{\sigma'}\right)^2\right] + \int_{-\infty}^{\infty} f(t)\mathrm{rect}\left[\frac{t-\mu'}{T}\right]dt\right\}.
\end{aligned} \tag{9}$$

Here B is a normalization constant. Equation 9 implies that under the prior stated conditions,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\mathrm{rect}\left[\frac{t-\mu'}{T}\right]dt = C \implies \max[f(t) * g(t)] \iff \mu &= \mu' \\
\hat{\mu} = \mu &= \mu'
\end{aligned} \tag{10}$$

where C is an integration constant. Hence, Eq. 10 implies that match filtering by a Gaussian kernel is an unbaised estimator for the case of a single Gaussian signal on a uniform background.

Further, in the case of a Gaussian signal, in the absence of background ($A = 0$), the minimum variance unbiased estimator (i.e. the estimator which saturates the Cramér-Rao bound) is given by the maximum likelyhood [1]. This operation is equivalent to Eq. 6 with a kernel $h(t) = \log[f(t)]$. However, for $A \neq 0$, $h(t)$ is biased towards the center of the domain T and as such is no longer consistent with the conditions for the Cramér-Rao bound. Consider the kernel $f(t)$ then,

$$\log[A + f(t)] = \log(A) + \frac{f(t)}{A} + o\left(\frac{f(t)}{A}\right) \tag{11}$$

Here $o()$ represents the higher order terms in the Taylor series. Hence, as A (i.e. the amplitude of background) increase, the contribution of these terms diminishes. Hence

$$\log[A + f(t)] \propto D + \frac{f(t)}{A} \tag{12}$$

Specifically, D is a constant only tied to the amplitude of the background while the form of the kernel $f(t)$ remains unchanged. Consequently, $f(t)$ represents an operation analogous to the maximum likelyhood for cases with $A \neq 0$.

4 Experimental parameters used in simulating the landrover

Symbol	Parameter	Value	Unit
E_0	Energy per pulse	14	μJ
ν	Repetition rate	33	kHz
λ	Wavelength	532	nm
σ'	Pulse FWHM	3.5	ns
R	Range	1.4	km
C_{atm}	Attenuation	6.2	km
θ	Divergence	1.07	milliradian
Γ	Body reflectivity	0.065	–
	Tyres and trim reflectivity	0.029	–
	Wall reflectivity	0.081	–
	Ground reflectivity	0.066	–
	Seats reflectivity	0.04	–
	Heatlights reflectivity	0.25	–
	Numberplate reflectivity	0.8	–
C_{bckg}	Solar background	0.5	W
f_{no}	f-number	10.0	-
C_{dc}	Dark counts	126	Hz
η	Exposure time	83	μs
q	Quantum efficiency	0.26	–
W_p/H_p	Pixel size	9.2	μm
ω	Bin width	50	ps
j	Jitter	1.5	ns

Table 2: The parameters used to model the Landrover. Note that the reflectivities represent the base values prior to their modification based on the orientation of the surface relative to the camera.

References

[1] H. Cramer, "Mathematical methods of statistics, princeton univ," *Press, Princeton, NJ*, 1946.