

Supplementary Information for Time inhomogeneous quantum dynamical maps

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1 Proof of Proposition 2

The simplest way to solve (2.24) is to pass to the Laplace Transform (LT) domain:

$$\tilde{F}_s := \int_0^\infty e^{-ts} F_t dt. \quad (1.1)$$

Taking LT of (2.24) one obtains

$$s\tilde{\Lambda}_s^{(\ell)} - \text{id} = -\tilde{Z}_s \circ \tilde{\Lambda}_s^{(\ell)} + \tilde{\Phi}_s \circ \tilde{\Lambda}_s^{(\ell)}, \quad (1.2)$$

and using

$$s\tilde{\Lambda}_s^{(0)} - \text{id} = -\tilde{Z}_s \circ \tilde{\Lambda}_s^{(0)}, \quad (1.3)$$

one finds

$$s\tilde{\Lambda}_s^{(\ell)} = (s + \tilde{Z}_s)^{-1} \circ \tilde{\Phi}_s \circ \tilde{\Lambda}_s^{(\ell-1)} = \tilde{\Lambda}_s^{(0)} \circ \tilde{\Phi}_s \circ \tilde{\Lambda}_s^{(\ell-1)}, \quad (1.4)$$

and hence going back to the time domain one finally obtains $\Lambda_t^{(\ell)} = \Lambda_t^{(0)} * \Phi_t * \Lambda_t^{(\ell-1)}$ which implies (3.23).

2 Proof of Proposition 3

One has

$$(A \circledast [B \circledast C])_{t,t_0} = \int_{t_0}^t d\tau A_{t,\tau} \circ [B \circledast C]_{\tau,t_0} = \int_{t_0}^t d\tau A_{t,\tau} \circ \int_{t_0}^\tau du B_{\tau,u} \circ C_{u,t_0}, \quad (2.1)$$

and hence using

$$\int_{t_0}^t d\tau \int_{t_0}^\tau du \dots = \int_{t_0}^t du \int_u^t d\tau \dots, \quad (2.2)$$

one obtains

$$(A \circledast [B \circledast C])_{t,t_0} = \int_{t_0}^t du \left\{ \int_u^t d\tau A_{t,\tau} \circ B_{\tau,u} \right\} \circ C_{u,t_0} = \int_{t_0}^t du [A \circledast B]_{t,u} \circ C_{u,t_0} = ([A \circledast B] \circledast C)_{t,t_0}, \quad (2.3)$$

which ends the proof. □

3 Proof of Proposition 4

One finds

$$\Lambda_{t,t_0}^{(\ell)} = \Lambda_{t,t_0}^{(0)} \circ C_{t,t_0}^{(\ell)}, \quad (3.1)$$

where C_{t,t_0} satisfies

$$\partial_t C_{t,t_0}^{(\ell)} = (\Lambda_{t,t_0}^{(0)})^{-1} \circ \Phi_t \circ \Lambda_{t,t_0}^{(\ell-1)}, \quad C_{t_0,t_0}^{(\ell)} = 0, \quad (3.2)$$

and hence

$$C_{t,t_0}^{(\ell)} = \int_{t_0}^t (\Lambda_{\tau,t_0}^{(0)})^{-1} \circ \Phi_\tau \circ \Lambda_{\tau,t_0}^{(\ell-1)}, \quad (3.3)$$

which eventually gives rise to

$$\Lambda_{t,t_0}^{(\ell)} = \int_{t_0}^t \Lambda_{t,\tau}^{(0)} \circ \Phi_\tau \circ \Lambda_{\tau,t_0}^{(\ell-1)} d\tau = \Lambda_{t,t_0}^{(0)} \circ (\Phi_t \circ \Lambda_{t,t_0}^{(\ell-1)}), \quad (3.4)$$

where we have used the semigroup property

$$\Lambda_{t,t_0}^{(0)} \circ (\Lambda_{\tau,t_0}^{(0)})^{-1} = \Lambda_{t,t_0}^{(0)} \circ \Lambda_{t_0,\tau}^{(0)} = \Lambda_{t,\tau}^{(0)}. \quad (3.5)$$

Simple iteration leads to

$$\Lambda_t^{(\ell+1)} = \Lambda_t^{(0)} \circ (\Phi \circ \Lambda_t^{(\ell)}) = \Lambda_t^{(0)} \circ \underbrace{(\Phi_t \circ \Lambda_t^{(0)}) \circ \dots \circ (\Phi_t \circ \Lambda_{t,t_0}^{(0)})}_{\ell \text{ terms}}. \quad (3.6)$$

which proves (3.10). $\square$

4 Derivation of (5.3)

Defining $\mathcal{P}_{t,t_0} = \Phi_{t,t_0} \circ \Lambda_{t,t_0}^{(0)}$ one has

$$\Lambda_{t,t_0} = \Lambda_{t,t_0}^{(0)} + \left(\mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \dots \right) \circ \Lambda_{t,t_0}^{(0)}, \quad (4.1)$$

and hence

$$\partial_t \Lambda_{t,t_0} = \partial_t \Lambda_{t,t_0}^{(0)} + \partial_t \left(\mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \dots \right) \circ \Lambda_{t,t_0}^{(0)}, \quad (4.2)$$

due to $\mathcal{P}_{t,t} = 0$. Now,

$$\begin{aligned} & \partial_t \left(\mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \dots \right) \circ \Lambda_{t,t_0}^{(0)} \\ &= \left(\partial_t \mathcal{P}_{t,t_0} + [\partial_t \mathcal{P}_{t,t_0}] \circ \mathcal{P}_{t,t_0} + [\partial_t \mathcal{P}_{t,t_0}] \circ \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \dots \right) \circ \Lambda_{t,t_0}^{(0)} \\ &= [\partial_t \mathcal{P}_{t,t_0}] \circ \Lambda_{t,t_0}^{(0)} + [\partial_t \mathcal{P}_{t,t_0}] \circ \left(\mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} \circ \mathcal{P}_{t,t_0} + \dots \right) \circ \Lambda_{t,t_0}^{(0)} \\ &= [\partial_t \mathcal{P}_{t,t_0}] \circ \Lambda_{t,t_0}^{(0)}, \end{aligned} \quad (4.3)$$

which proves (5.3).