Supplementary Material to Renewable Composite Quantile Method and Algorithm for Nonparametric Models with Streaming Data

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This supplementary material contains a detailed algorithm for renewable WCQR estimation, the relevant lemmas and technical proofs for the theoretical results.

A Detailed Algorithm for WCQR Estimation

Algorithm A.1 contains details about obtaining the renewable WCQR estimators for $m(x)$ and $\sigma(x)$ from asymmetric models.

B Lemmas and proofs

The proof of Lemma 2 is as follows.

Proof of Lemma 2 Noting that ε is independent with X and $\mathbb{E}[\varepsilon] = 0$, we have

$$E_{WY} := \mathbb{E}[W(X)Y] = \mathbb{E}[W(X)(m(X) + \sigma(X)\varepsilon)] = \mathbb{E}[W(X)m(X)], \tag{B.1}$$

$$E_{WY^2} := \mathbb{E}\left[W(X)Y^2\right] = \mathbb{E}\left[W(X)\left(m^2(X) + \sigma^2(X)\right)\right]. \tag{B.2}$$

Algorithm A.1 Renewable WCQR Estimation for $\{(m(x_i), \sigma(x_i))\}_{i=1}^{\bar{q}}$ from Asymmetric Models

Input Set of grid points $G_* = \{x_i\}_{i=1}^{\bar{q}}$, kernel function $K(\cdot)$, node sets G_{x_i} for $x_i \in G_*$, PLI degree l

Set initial values $N_0 = \widehat{E}_{WY,0} = \widehat{E}_{WY^2,0} = \widehat{f}_{X,0}(x_i) = \widehat{S}_{Y|x,0}(y_{ij}) = 0$ for $y_{ij} \in G_{x_i}$ and $x_i \in G_*$

for $t = 1, 2, \cdots$ **do**

  Obtain the t-th data batch $\mathcal{D}_t = \{X_{tj}, Y_{tj}\}_{j=1}^{n_t}$

  Select the t-th bandwidth h_t

  $N_t = N_{t-1} + n_t$

  $\widehat{E}_{WY,t} = N_{t-1}/N_t \widehat{E}_{WY,t-1} + 1/N_t \sum_{j=1}^{n_t} W(X_{tj}) Y_{tj}$

  $\widehat{E}_{WY^2,t} = N_{t-1}/N_t \widehat{E}_{WY,t-1} + 1/N_t \sum_{j=1}^{n_t} W(X_{tj}) Y_{tj}^2$

  for $x_i \in G_*$ **do**

    $\widehat{f}_{X,t}(x_i) = N_{t-1}/N_t \widehat{f}_{X,t-1}(x_i) + 1/N_t \sum_{j=1}^{n_t} K_{h_t}(X_{tj} - x_i)$,

    For $y_{ij} \in G_{x_i}$, $\widehat{S}_{Y|x_i,t}(y_{ij}) = N_{t-1}/N_t \widehat{S}_{Y|x_i,t-1}(y_{ij}) + 1/N_t \sum_{j=1}^{n_t} I(Y_{tj} < y_{ij}) K_{h_t}(X_{tj} - x_i)$

  end for

  if the estimators for $\{(m(x_i), \sigma(x_i))\}_{x_i \in G_*}$ are needed **then**

    Define $\mathcal{I}_l \widehat{f}_{X,t}(\cdot) = \sum_{x_i \in G_*} \widehat{f}_{X,t}(x_i) L(\cdot, x_i;\ l, G_*)$

    Define $\mathcal{I}_l \widehat{F}_{Y|x_i,t}(\cdot) = \sum_{y_{ij} \in G_{x_i}} \widehat{S}_{Y|x_i,t}(y_{ij}) / \widehat{f}_{X,t}(x_i) L(\cdot, y_{ij};\ l, G_{x_i})$ for $x_i \in G_*$

    for $J \in \{L_\alpha, U_\alpha\}$ **do**

      $\widetilde{r}_t(x_i;\ J) = \int_{\mathbb{R}} y J(\mathcal{I}_l \widehat{F}_{Y|x_i,t}(y)) \mathrm{d}\mathcal{I}_l \widehat{F}_{Y|x_i,t}(y)$ for $x_i \in G_*$

      Define $\mathcal{I}_l \widetilde{r}_t(\cdot\ ;\ J) = \sum_{x_i \in G_*} \widetilde{r}_t(x_i;\ J) L(\cdot, x_i;\ l, G_*)$

    end for

    $\widehat{E}_{WL,t} = \int_{I_*} W(x) \mathcal{I}_l \widetilde{r}_t(x;\ L_\alpha) \mathcal{I}_l \widehat{f}_{X,t}(x) \mathrm{d}x$

    $\widehat{E}_{WU,t} = \int_{I_*} W(x) \mathcal{I}_l \widetilde{r}_t(x;\ U_\alpha) \mathcal{I}_l \widehat{f}_{X,t}(x) \mathrm{d}x$

    $\widehat{w}_t = (\widehat{E}_{WY,t} - \widehat{E}_{WU,t}) / (\widehat{E}_{WL,t} - \widehat{E}_{WU,t})$

    $\widetilde{r}_t(x_i;\ J_{m,\widehat{w}_t}) = \widehat{w}_t \widetilde{r}_t(x_i;\ L_\alpha) + (1 - \widehat{w}_t) \widetilde{r}_t(x_i;\ U_\alpha)$ for $x_i \in G_*$

    $\widetilde{r}_t(x_i;\ J_{\sigma,1}) = -\widetilde{r}_t(x_i;\ L_\alpha) + \widetilde{r}_t(x_i;\ U_\alpha)$ for $x_i \in G_*$

    Define $\mathcal{I}_l \widetilde{r}_t(\cdot;\ J_{m,\widehat{w}_t}) = \sum_{x_i \in G_*} \widetilde{r}_t(x_i;\ J_{m,\widehat{w}_t}) L(\cdot, x_i;\ l, G_*)$

    $\widehat{E}_{Wm^2,t} = \int_{I_*} W(x) \mathcal{I}_l \widetilde{r}_t^2(x;\ J_{m,\widehat{w}_t}) \mathcal{I}_l \widehat{f}_{X,t}(x) \mathrm{d}x$

    $\widehat{E}_{Wr^2,t} = \int_{I_*} W(x) \mathcal{I}_l \widetilde{r}_t^2(x;\ J_{\sigma,1}) \mathcal{I}_l \widehat{f}_{X,t}(x) \mathrm{d}x$

    $\widehat{\theta}_t = \sqrt{(\widehat{E}_{WY^2,t} - \widehat{E}_{Wm^2,t}) / \widehat{E}_{Wr^2,t}}$

    Output $\{\widetilde{r}_t(x_i;\ J_{m,\widehat{w}_t})\}_{x_i \in G_*}$ as the estimators of $\{m(x_i)\}_{x_i \in G_*}$

    Output $\left\{\widehat{\theta}_t \widetilde{r}_t(x_i;\ J_{\sigma,1})\right\}_{x_i \in G_*}$ as the estimators of $\{\sigma(x_i)\}_{x_i \in G_*}$

  end if

end for

By (19), we have

$$\begin{aligned} E_{Wr} &:= \mathbb{E}\left[W(X)r\left(X;\ J\right)\right] \\ &= E_{WY}\int_{[0,1]} J(\tau)\mathrm{d}\tau + E_{W\sigma}\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)\mathrm{d}\tau. \end{aligned} \tag{B.3}$$

with $E_{W\sigma} = \mathbb{E}\left[W\left(X\right)\sigma\left(X\right)\right]$. Based on (B.3), we conclude that if C_{m1} holds, then

$$\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)\mathrm{d}\tau = \left(E_{W\sigma}\right)^{-1}\left(E_{Wr} - E_{WY}\right), \tag{B.4}$$

implying

$$\begin{aligned} &\mathbb{E}\left[W\left(X\right)\sigma\left(X\right)\right]\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)\mathrm{d}\tau \\ &= \mathbb{E}\left[W(X)\left(r\left(X;\ J\right) - Y\right)\right]. \end{aligned} \tag{B.5}$$

Assume that $J_{\sigma}\left(\cdot\right)$ is a weight function satisfying $\mathrm{C}_{\sigma 1}$. Then by (19), we have

$$r\left(x;\ J_{\sigma}\right) = \sigma(x)\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)\mathrm{d}\tau, \tag{B.6}$$

leading to

$$\mathbb{E}\left[W(X)r^2\left(X;\ J_{\sigma}\right)\right] = E_{W\sigma^2}\left(\int_{[0,1]} J_{\sigma}(\tau)F_{\varepsilon}^{-1}\left(\tau\right)\mathrm{d}\tau\right)^2 \tag{B.7}$$

with $E_{W\sigma^2} := \mathbb{E}\left[W(X)\sigma^2(X)\right]$. The equality (B.2) implies

$$E_{W\sigma^2} = \mathbb{E}\left[W\left(X\right)\left(Y^2 - m^2\left(X\right)\right)\right]. \tag{B.8}$$

Combining (B.8) and (B.7), we obtain

$$\left(\int_{[0,1]} J(\tau)F_{\varepsilon}^{-1}\left(\tau\right)\mathrm{d}\tau\right)^2 = \frac{\mathbb{E}\left[W\left(X\right)\left(Y^2 - m^2\left(X\right)\right)\right]}{\mathbb{E}\left[W(X)r^2\left(X;\ J_{\sigma}\right)\right]}. \tag{B.9}$$

□

By Lemma 1, we can obtain the following result.

Lemma B.1 *Let* $1 \leq l \leq q-1$ *and* $\mathcal{I}_l F_{Y|x}\left(\cdot\right)$ *be given in* (10). *With* $F_{Y|x}^{(l+1)}\left(\cdot\right)$ *existing and continuous on* $I\left(G_x\right) := \left[\min G_x, \max G_x\right]$, *it holds that*

$$\|\mathcal{I}_l F_{Y|x} - F_{Y|x}\|_{\infty, I(G_x)} \leq \|F_{Y|x}^{(l+1)}\|_{\infty, I(G_x)}\left|\Delta\left(G_x\right)\right|^{l+1}.$$

For positive integers l and r, denote

$$k_{l,r} = \int_{\mathbb{R}} u^l K^r\left(u\right)\mathrm{d}u. \tag{B.10}$$

The following lemma is useful to deduce the asymptotic properties of $\widehat{F}_{Y|x}\left(y\right)$.

Lemma B.2 *Assuming that $g:\mathbb{R}\to\mathbb{R}$ and $f_X(\cdot)$ has derivatives up to order $r+1$, it holds that*

$$\mathbb{E}\left[g(X)K_h(X-x)\right]=\sum_{l=0}^{r}\frac{k_{l,1}h^l}{l!}\left(gf_X\right)^{(l)}(x)+O\left(k_{r+1,1}M_{r+1}h^{r+1}\right), \tag{B.11}$$

$$\mathbb{E}\left[g(X)K_h^2(X-x)\right]=\frac{1}{h}\sum_{l=0}^{r}\frac{k_{l,2}h^l}{l!}\left(gf_X\right)^{(l)}(x)+O\left(k_{r+1,2}M_{r+1}h^{r}\right) \tag{B.12}$$

with $M_{r+1}=\left\|\left(gf_X\right)^{(r+1)}\right\|_{I_*,\infty}$.

Proof By Taylor's expansion, we have

$$\begin{aligned}\mathbb{E}\left[g(X)K_h(X-x)\right]&=\int_{\mathbb{R}}g(u)f_X(u)K_h(u-x)\,\mathrm{d}u\\&=\int_{\mathbb{R}}\left(gf_X\right)(x+vh)K(v)\,\mathrm{d}v\\&=\sum_{l=0}^{r}(l!)^{-1}\left(gf_X\right)^{(l)}(x)h^l\int_{\mathbb{R}}v^lK(v)\,\mathrm{d}v\\&\quad+O\left(M_{r+1}h^{r+1}\int_{\mathbb{R}}v^{r+1}K(v)\,\mathrm{d}v\right)\end{aligned} \tag{B.13}$$

leading to (B.11). Similarly, we have

$$\begin{aligned}\mathbb{E}\left[g(X)K_h^2(X-x)\right]&=\int_{\mathbb{R}}g(u)f_X(u)K_h^2(u-x)\,\mathrm{d}u\\&=\frac{1}{h}\int_{\mathbb{R}}\left(gf_X\right)(x+vh)K^2(v)\,\mathrm{d}v\\&=\frac{1}{h}\sum_{l=0}^{r}\frac{h^l}{l!}\left(gf_X\right)^{(l)}(x)\int_{\mathbb{R}}v^lK^2(v)\,\mathrm{d}v\\&\quad+O\left(M_{r+1}h^{r}\int_{\mathbb{R}}v^{r+1}K^2(v)\,\mathrm{d}v\right)\end{aligned} \tag{B.14}$$

leading to (B.12).

□

Denote

$$\boldsymbol{y}=\{y_i\}_{i=1}^q,\quad \boldsymbol{F}_{Y|x,t}(\boldsymbol{y})=\left\{F_{Y|x,t}(y_i)\right\}_{i=1}^q,\quad \widehat{\boldsymbol{F}}_{Y|x,t}(\boldsymbol{y})=\left\{\widehat{F}_{Y|x,t}(y_i)\right\}_{i=1}^q.$$

The following lemma give the asymptotic properties of $\widehat{\boldsymbol{F}}_{Y|x,t}(\boldsymbol{y})$.

Lemma B.3 *Under Assumptions 1 - 3, it holds that*

$$\begin{aligned}\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_th_s}\right\}^{-1/2}&\left\{\widehat{\boldsymbol{F}}_{Y|x,t}(\boldsymbol{y})-\boldsymbol{F}_{Y|x}(\boldsymbol{y})-\boldsymbol{B}_{F,x}(\boldsymbol{y})\sum_{s=1}^{t}\frac{n_sh_s^2}{N_t}\right\}\\&\xrightarrow{\mathrm{d}}N\left(\boldsymbol{0},\boldsymbol{\Sigma}_{F,x}(\boldsymbol{y})\right),\end{aligned} \tag{B.15}$$

where

$$\boldsymbol{B}_{F,x}(\boldsymbol{y}) = \left\{B_{F,x}(y_i)\right\}_{i=1}^{q}, \quad \boldsymbol{\Sigma}_{F,x}(\boldsymbol{y}) = \left[C_{F,x}\left(y_i, y_j\right)\right]_{i,j=1,\cdots,q} \tag{B.16}$$

with $B_{F,x}(y_i)$ *and* $C_{F,x}\left(y_i, y_j\right)$ *given by* (35) *and* (36), *respectively.*

Proof For $y \in \mathbb{R}$, denote

$$M_{x,t}(y) = \widehat{S}_{Y|x,t}(y) - F_{Y|x}(y)\widehat{f}_{X,t}(x). \tag{B.17}$$

Then, we have

$$\widehat{F}_{Y|x,t} - F_{Y|x}(y) = \frac{M_{x,t}(y)}{\widehat{f}_{X,t}(x)}. \tag{B.18}$$

Under Assumptions 1 and 2, it follows from Lemma B.2 with $r = 3$ that

$$\begin{aligned}
\mathbb{E}\left[\widehat{S}_{Y|x,t}(y)\right] &= N_t^{-1}\sum_{s=1}^{t}\sum_{j=1}^{n_s}\mathbb{E}\left[\mathbb{E}\left[I\left(Y_{sj} < y\right)|X_{sj}\right]K_{h_s}\left(X_{sj} - x\right)\right] \\
&= \sum_{s=1}^{t}\frac{n_s}{N_t}\mathbb{E}\left[F_{Y|X}(y)K_{h_s}(X - x)\right] \\
&= F_{Y|x}(y)f_X(x) + \frac{k_{2,1}}{2}\partial_x^2\left\{F_{Y|x}(y)f_X(x)\right\}\sum_{s=1}^{t}\frac{n_s h_s^2}{N_t} \\
&\quad + O\left(\sum_{s=1}^{t}\frac{n_s}{N_t}h_s^4\right),
\end{aligned} \tag{B.19}$$

$$\begin{aligned}
\mathbb{E}\left[\widehat{f}_{X,t}(x)\right] &= N_t^{-1}\sum_{s=1}^{t}\sum_{j=1}^{n_s}\mathbb{E}\left[K_{h_s}\left(X_{sj} - x\right)\right] \\
&= f_X(x) + \frac{k_{2,1}}{2}f_X^{(2)}(x)\sum_{s=1}^{t}\frac{n_s h_s^2}{N_t} + O\left(\sum_{s=1}^{t}\frac{n_s}{N_t}h_s^4\right).
\end{aligned} \tag{B.20}$$

Then we have

$$\mathbb{E}\left[M_{x,t}(y)\right] - f_X(x)B_{F,x}(y)\sum_{s=1}^{t}\frac{n_s h_s^2}{N_t} = O\left(\sum_{s=1}^{t}\frac{n_s}{N_t}h_s^4\right) \tag{B.21}$$

with

$$B_{F,x}(y) = \frac{k_{2,1}}{2}\left\{\partial_x^2 F_{Y|x}(y) + 2\partial_x F_{Y|x}(y)f_X'(x)/f_X(x)\right\}. \tag{B.22}$$

Under Assumptions 1, 2 and 3, It is easy to obtain that

$$\mathrm{Var}\left[\widehat{f}_{X,t}(x)\right] = \frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t}\mathrm{Var}\left[K_{h_s}(X_s - x)\right] = O\left(\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t h_s}\right) = o(1). \tag{B.23}$$

Combining (B.20) and (B.23), we have

$$\begin{aligned}
\widehat{f}_{X,t}(x) &= f_X(x) + \frac{k_{2,1}}{2}f_X^{(2)}(x)\sum_{s=1}^{t}\frac{n_s h_s^2}{N_t} \\
&\quad + O\left(\sum_{s=1}^{t}\frac{n_s}{N_t}h_s^4\right) + O_p\left(\sqrt{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t h_s}}\right).
\end{aligned} \tag{B.24}$$

For x and $y \in \mathbb{R}$, denote

$$D_{F,x}(y) = I(Y < y) - F_{Y|x}(y). \tag{B.25}$$

For u, y_1 and y_2 in $\mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}\left[D_{F,x}(y_1) D_{F,x}(y_2) | X = u\right] \\
&= \mathbb{E}\left[I(Y < y_1) I(Y < y_2) | X = u\right] - \mathbb{E}\left[I(Y < y_1) | X = u\right] F_{Y|x}(y_2) \\
&\quad - F_{Y|x}(y_1) \mathbb{E}\left[I(Y < y_2) | X = u\right] + F_{Y|x}(y_1) F_{Y|x}(y_2) \\
&= F_{Y|u}(y_1 \wedge y_2) - F_{Y|u}(y_1) F_{Y|x}(y_2) - F_{Y|x}(y_1) F_{Y|u}(y_2) \\
&\quad + F_{Y|x}(y_1) F_{Y|x}(y_2).
\end{aligned} \tag{B.26}$$

Noting that the samples are i.i.d., we have

$$\operatorname{Cov}\left(M_{x,t}(y_1), M_{x,t}(y_2)\right) = N_t^{-2} \sum_{s=1}^{t} n_s C_s(y_1, y_2) \tag{B.27}$$

where

$$\begin{aligned}
C_s(y_1, y_2) &= \operatorname{Cov}\left(D_{F,x}(y_1) K_{h_s}(X - x), D_{F,x}(y_2) K_{h_s}(X - x)\right) \\
&= \mathbb{E}\left[D_{F,x}(y_1) D_{F,x}(y_2) K_{h_s}^2(X - x)\right] - \prod_{i=1,2} \mathbb{E}\left[D_{F,x}(y_i) K_{h_s}(X - x)\right]
\end{aligned} \tag{B.28}$$

with

$$\begin{aligned}
&\mathbb{E}\left[D_{F,x}(y_1) D_{F,x}(y_2) K_{h_s}^2(X - x)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[D_{F,x}(y_1) D_{F,x}(y_2) | X\right] K_{h_s}^2(X - x)\right] \\
&= \frac{1}{h_s} k_{0,2} \left\{F_{Y|x}(y_1 \wedge y_2) - F_{Y|x}(y_1) F_{Y|x}(y_2)\right\} f_X(x) + O(h_s),
\end{aligned} \tag{B.29}$$

$$\begin{aligned}
\mathbb{E}\left[D_{F,x}(y_i) K_{h_s}(X - x)\right] &= \mathbb{E}\left[\mathbb{E}\left[F_{Y|X}(y_i) - F_{Y|x}(y) | X\right] K_{h_s}(X - x)\right] \\
&= O\left(h^2\right),
\end{aligned} \tag{B.30}$$

where the last equalities in (B.29) and (B.30) are obtained by using Lemma B.2 and (B.26) under the conditions of Assumptions 1 and 2. Inserting (B.29) and (B.30) into (B.28) and further inserting the deduced equation into (B.27), we obtain

$$\begin{aligned}
&\operatorname{Cov}\left(M_{x,t}(y_1), M_{x,t}(y_2)\right) \\
&= \frac{k_{0,2}}{N_t} \left\{F_{Y|x}(y_1 \wedge y_2) - F_{Y|x}(y_1) F_{Y|x}(y_2)\right\} f_X(x) \sum_{s=1}^{t} \frac{n_s}{N_t h_s} \\
&\quad + o\left(\frac{1}{N_t} \sum_{s=1}^{t} \frac{n_s}{N_t h_s}\right).
\end{aligned} \tag{B.31}$$

By Assumption 3, we have

$$\left\{\frac{1}{N_t} \sum_{s=1}^{t} \frac{n_s}{N_t h_s}\right\}^{-1/2} \sum_{s=1}^{t} \frac{n_s}{N_t} h_s^4 \to 0. \tag{B.32}$$

Combining (B.21), (B.31) and (B.32), and by Lyapunov's central limit theorem, we have

$$\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_th_s}\right\}^{-1/2}\left\{\left\{M_{x,t}\left(y_i\right)\right\}_{i=1}^{q}-f_X\left(x\right)\left\{B_{F,x}\left(y_i\right)\right\}_{i=1}^{q}\sum_{s=1}^{t}\frac{n_sh_s^2}{N_t}\right\} \xrightarrow{\mathrm{d}} N\left(0,f_X^2\left(x\right)\boldsymbol{\Sigma}_{F,x}\left(\boldsymbol{y}\right)\right). \tag{B.33}$$

By (B.24) and Slutsky's lemma, (B.33) leads to

$$\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_th_s}\right\}^{-1/2}\left\{\left\{\frac{M_{x,t}\left(y_i\right)}{\widehat{f}_{X,t}\left(x\right)}\right\}_{i=1}^{q}-\frac{f_X\left(x\right)}{\widehat{f}_{X,t}\left(x\right)}\left\{B_{F,x}\left(y_i\right)\right\}_{i=1}^{q}\sum_{s=1}^{t}\frac{n_sh_s^2}{N_t}\right\} \xrightarrow{\mathrm{d}} N\left(0,\boldsymbol{\Sigma}_{F,x}\left(\boldsymbol{y}\right)\right). \tag{B.34}$$

Let $x \in I_*$ and from Assumption 1 we know that $f_X\left(x\right) > 0$. Then we have

$$\frac{f_X\left(x\right)}{\widehat{f}_{X,t}\left(x\right)} = 1 + O_p\left(\sum_{s=1}^{t}\frac{n_sh_s^2}{N_t}+\sqrt{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_th_s}}\right). \tag{B.35}$$

Combining (B.18), (B.32), (B.34) and (B.35), we obtain (B.15). □

Based on Lemma B.3, we can immediately obtain the asymptotic distribution of $\mathcal{I}_l\widehat{F}_{Y|x,t}\left(y\right)$ stated in the following two corollaries.

Corollary B.4 *Under Assumptions 1 - 3, it holds that*

$$\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_th_s}\right\}^{-1/2}\left\{\mathcal{I}_l\widehat{F}_{Y|x,t}\left(y\right)-\mathcal{I}_lF_{Y|x}\left(y\right)-\mathcal{I}_lB_{F,x}\left(y\right)\sum_{s=1}^{t}\frac{n_sh_s^2}{N_t}\right\} \xrightarrow{\mathrm{d}} N\left(\mathbf{0},\mathcal{I}_l^2C_{F,x}\left(y\right)\right), \tag{B.36}$$

where

$$\mathcal{I}_lB_{F,x}\left(y\right) = \sum_{y_i\in G_x}B_{F,x}\left(y_i\right)L\left(y,y_i;\ l,G_x\right), \tag{B.37}$$

$$\mathcal{I}_l^2C_{F,x}\left(y\right) = \sum_{y_i,y_j\in G_x}L\left(y,y_i;\ l,G_x\right)L\left(y,y_j;\ l,G_x\right)C_{F,x}\left(y_i,y_j\right). \tag{B.38}$$

Corollary B.5 *Under Assumptions 1 - 3, for any measurable set $I \subset \mathbb{R}$, it holds that*

$$\int_I\left|\mathcal{I}_l\widehat{F}_{Y|x,t}-\mathcal{I}_lF_{Y|x}\right|^2\mathrm{d}\mu = \mu\left(I\right)O_p\left(\sum_{s=1}^{t}\frac{n_sh_s^4}{N_t}+\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_th_s}\right), \tag{B.39}$$

where μ is the Lebesgue measure on $\mathbb{R}$.

Combining Lemma B.1 and Corollary B.4, we obtain Theorem 3.

For any CDF $F : \mathbb{R} \to \mathbb{R}$ and any measurable function $J_0 : [0,1] \to \mathbb{R}$, define the functional Γ by

$$\Gamma\left(J_0;\ F\right) = \int_{\mathbb{R}}\left(J_0\circ F\right)\mathrm{I_d}\mathrm{d}F. \tag{B.40}$$

Lemma B.6 *Let F and H be CDFs from $\mathbb{R}$ to $\mathbb{R}$. Assume that F is continuous and strictly increasing on an open set containing $[\underline{\tau}, \overline{\tau}] \subset (0,1)$, and $J : [0,1] \to \mathbb{R}$ is piecewise continuously differentiable with a support in $[\underline{\tau}, \overline{\tau}]$. Then for any $\delta > 0$, it holds that*

$$\Gamma(J;\ H) = \Gamma(J;\ F) - \int_{F^{-1}([\underline{\tau},\overline{\tau}])} (J \circ F)(H - F)\,\mathrm{d}\mu + O\left(\int_{I_F^\delta} (H - F)^2\,\mathrm{d}\mu\right), \tag{B.41}$$

where $I_F^\delta = \left(F^{-1}(\underline{\tau}) - \delta, F^{-1}(\overline{\tau}) + \delta\right)$.

Proof By the formula of integration by parts, we have (see Section 8.1.1 Lemma B in Serfling 1980)

$$\Gamma(J;\ H) - \Gamma(J;\ F) = -\int_{\mathbb{R}} (\Phi \circ H - \Phi \circ F)\,\mathrm{d}\mu \tag{B.42}$$

with $\Phi(y) := \int_0^y J(u)\mathrm{d}u$ for $y \in \mathbb{R}$. Further noting that $(J \circ F)(y) = 0$ for $y \neq F^{-1}([\underline{\tau}, \overline{\tau}])$, we have

$$\Gamma(J;\ H) - \Gamma(J;\ F) + \int_{F^{-1}([\underline{\tau},\overline{\tau}])} (J \circ F)(H - F)\,\mathrm{d}\mu = -\int_{I_{HF}} W_{H,F}(H - F)\,\mathrm{d}\mu, \tag{B.43}$$

where $I_{HF} = F^{-1}([\underline{\tau}, \overline{\tau}]) \cup H^{-1}([\underline{\tau}, \overline{\tau}])$ and

$$W_{H,F}(y) = \begin{cases} \dfrac{\Phi(H(y)) - \Phi(F(y))}{H(y) - F(y)} - J(F(y)), & \text{if } H(y) \neq F(y), \\ 0, \quad \text{if } H(y) = F(y). \end{cases} \tag{B.44}$$

Since $J(\cdot)$ is piecewise continuously differentiable with a compact support, then we have

$$M_{J'} := \sup_{\tau \in [0,1] \setminus I_{\mathrm{nc}}} |J'(\tau)| < +\infty, \tag{B.45}$$

where $I_{\mathrm{nc}} := \{\tau_1', \cdots, \tau_m'\}_{i=1}^m$ the set consists of all the discontinuity points of $J'(\cdot)$. Since F is strictly increasing, we have that the set $I_{F,0} := \left\{y \in \mathbb{R} : F_{Y|x}(y) \notin \{\tau_i'\}_{i=1}^m, F_{Y|x}(y) \in (\overline{\tau}, \underline{\tau})\right\}$ is open. Assuming that

$$\int_{I_{HF}} (H - F)^2\,\mathrm{d}\mu \to 0, \tag{B.46}$$

then $H \to F$ on I_{HF} almost everywhere (a.e.) with respect to (w.r.t.) μ, which further implies that

$$\limsup_{H(y) \to F(y)} \frac{|W_{H,F}(y)(H(y) - F(y))|}{(H(y) - F(y))^2} \leq M_{J'} \tag{B.47}$$

holding a.e. for $y \in I_{F,0} \cap I_{HF}$ w.r.t. μ. Therefore we can deduce

$$\int_{I_{HF}} W_{H,F}(H - F)\,\mathrm{d}\mu = \int_{I_{F,0} \cap I_{HF}} W_{H,F}(H - F)\,\mathrm{d}\mu = O\left(\int_{I_{HF}} (H - F)^2\,\mathrm{d}\mu\right), \tag{B.48}$$

where the last equality is obtained by combining (B.47) and the Lebesgue's dominated convergence theorem.

Under the assumed conditions on F, there exists $\epsilon > 0$ such that

$$E^{\epsilon} := \left(F^{-1}\left(\underline{\tau} - \epsilon\right) - \epsilon, F^{-1}\left(\overline{\tau} + \epsilon\right) + \epsilon\right) \subset \left(F^{-1}\left(\underline{\tau}\right) - \delta, F^{-1}\left(\overline{\tau}\right) + \delta\right) = I_F^{\delta}. \tag{B.49}$$

By Markov's inequality, we have

$$\mu\left(\left\{y \in I_F^{\delta} : |H(y) - F(y)| \geq \epsilon\right\}\right) \leq \epsilon^{-2} \int_{I_F^{\delta}} (H - F)^2 \,\mathrm{d}\mu. \tag{B.50}$$

Thus given $\int_{I_F^{\delta}} (H - F)^2 \,\mathrm{d}\mu \leq \epsilon^3$, we have

$$\mu\left(\left\{y \in I_F^{\delta} : |H(y) - F(y)| \geq \epsilon\right\}\right) \leq \epsilon, \tag{B.51}$$

implying $I_{HF} \subset E^{\epsilon}$ and thus $I_{HF} \subset I_F^{\delta}$, which further leads to

$$\int_{I_{HF}} (H - F)^2 \,\mathrm{d}\mu = O\left(\int_{I_F^{\delta}} (H - F)^2 \,\mathrm{d}\mu\right). \tag{B.52}$$

Combining (B.43), (B.48) and (B.52), we obtain (B.41). □

The proof of Theorem 4 is as follows.

Proof of Theorem 4 By the definition in (B.40), we can write

$$\widetilde{r}_t(x;\ J) = \Gamma\left(J;\ \mathcal{I}_l \widehat{F}_{Y|x,t}\right), \quad r(x;\ J) = \Gamma\left(J;\ F_{Y|x}\right). \tag{B.53}$$

Then we have the decomposition

$$\widetilde{r}_t(x;\ J) - r(x;\ J) = \Delta\Gamma + R_{m,x} \tag{B.54}$$

with

$$\Delta\Gamma = \Gamma\left(J;\ \mathcal{I}_l \widehat{F}_{Y|x,t}\right) - \Gamma\left(J;\ \mathcal{I}_l F_{Y|x}\right), \tag{B.55}$$

$$R_{m,x} = \Gamma\left(J;\ \mathcal{I}_l F_{Y|x}\right) - \Gamma\left(J;\ F_{Y|x}\right). \tag{B.56}$$

Under Assumption 4, it follows from Lemma B.6 that

$$\Delta\Gamma = -A_t + O\left(E^2_{\mathcal{I}_l,t}\right) \tag{B.57}$$

where

$$A_t = \int_{I(G_x)} J\left(F_{Y|x}(y)\right)\left(\mathcal{I}_l \widehat{F}_{Y|x,t}(y) - \mathcal{I}_l F_{Y|x}(y)\right) \mathrm{d}y, \tag{B.58}$$

$$E^2_{\mathcal{I}_l,t} = \int_{I(G_x)} \left|\mathcal{I}_l \widehat{F}_{Y|x,t}(y) - \mathcal{I}_l F_{Y|x}(y)\right|^2 \mathrm{d}y. \tag{B.59}$$

Denoting

$$\omega_i = \int_{I(G_x)} J\left(F_{Y|x}(y)\right) L\left(y, y_i;\ l, G_x\right) \mathrm{d}y \quad \text{for} \quad y_i \in G_x,$$

we rewrite (B.58) into

$$A_t = \sum_{y_i \in G_x} \omega_i \left(\widehat{F}_{Y|x}(y_i) - F_{Y|x}(y_i)\right). \tag{B.60}$$

By Lemma B.3, we can obtain

$$\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t h_s}\right\}^{-1/2}\left\{-A_t+\sum_{y_i\in G_x}\omega_i B_{F,x}\left(y_i\right)\sum_{s=1}^{t}\frac{n_s h_s^2}{N_t}\right\} \xrightarrow{\mathrm{d}} N\left(0,\sum_{y_i,y_j\in G_x}\omega_i\omega_j C_{F,x}\left(y_i,y_j\right)\right). \tag{B.61}$$

By Corollary B.5, we have

$$E^2_{\mathcal{I}_l,t}=O_p\left(\sum_{s=1}^{t}\frac{n_s h_s^4}{N_t}+\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t h_s}\right). \tag{B.62}$$

Further using Assumption 3, we obtain

$$\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t h_s}\right\}^{-1/2} E^2_{\mathcal{I}_l,t}=o_p\left(1\right). \tag{B.63}$$

Combining (B.57), (B.61) and (B.63), we obtain

$$\left\{\frac{1}{N_t}\sum_{s=1}^{t}\frac{n_s}{N_t h_s}\right\}^{-1/2}\left\{\Delta\Gamma-B_{\mathcal{I}_l,m,x}\sum_{s=1}^{t}\frac{n_s h_s^2}{N_t}\right\} \xrightarrow{\mathrm{d}} N\left(0,\Sigma_{\mathcal{I}_l,m,x}\right). \tag{B.64}$$

with

$$\begin{aligned} B_{\mathcal{I}_l,m,x} &= -\sum_{y_i\in G_x}\omega_i B_{F,x}\left(y_i\right)\\ &= -\int_{I(G_x)} J\left(F_{Y|x}\left(y\right)\right)\sum_{y_i\in G_x} B_{F,x}\left(y_i\right)L\left(y,y_i;\ l,G_x\right)\mathrm{d}y\\ &= -\int_{I(G_x)} J\left(F_{Y|x}\left(y\right)\right)\mathcal{I}_l B_{F,x}\left(y\right)\mathrm{d}y\\ &= -\int_{I(G_x)} J\left(F_{Y|x}\left(y\right)\right)B_{F,x}\left(y\right)\mathrm{d}y+O\left(\left|\Delta\left(G_x\right)\right|^{l+1}\right) \end{aligned} \tag{B.65}$$

and

$$\begin{aligned} \Sigma_{\mathcal{I}_l,m,x} &= \sum_{y_i,y_j\in G_x}\omega_i\omega_j C_{F,x}\left(y_i,y_j\right)\\ &= \int_{I(G_x)}\int_{I(G_x)} J\left(F_{Y|x}\left(z_1\right)\right)J\left(F_{Y|x}\left(z_2\right)\right)\mathcal{I}_l^2 C_{F,x}\left(z_1,z_2\right)\mathrm{d}z_1\mathrm{d}z_2. \end{aligned} \tag{B.66}$$

By Lemma B.6, we have

$$R_{\mathcal{I}_l,m,x}=E_1+E_2, \tag{B.67}$$

where

$$E_1=\int_{I(G_x)} J\left(F_{Y|x}\left(y\right)\right)\left(\mathcal{I}_l F_{Y|x}\left(y\right)-F_{Y|x}\left(y\right)\right)\mathrm{d}y \tag{B.68}$$

and

$$E_2=O\left(\int_{[y_1,y_1]}\left|\mathcal{I}_l F_{Y|x}\left(y\right)-F_{Y|x}\left(y\right)\right|^2\mathrm{d}y\right) \tag{B.69}$$

Further using Lemma 1 and Assumption 4, we obtain

$$E_1 = -\int_{I(G_x)} J\left(F_{Y|x}(y)\right) R_{\mathcal{I}_l,F,x}(y)\,\mathrm{d}, E_2 = \quad O\left(|\Delta(G_x)|^{2l+2}\right). \tag{B.70}$$

Then we have

$$R_{\mathcal{I}_l,m,x} = -\int_{I(G_x)} J\left(F_{Y|x}(y)\right) R_{\mathcal{I}_l,F,x}(y)\,\mathrm{d}y + O\left(|\Delta(G_x)|^{2l+2}\right). \tag{B.71}$$

Combining (B.54), (B.64) and (B.71), we obtain (38).

□

The proof of Corollary 5 is as follows.

Proof of Corollary 5 The expansion of $B_{\mathcal{I}_l,m,x}$ can be obtained by using Lemma 1 and Assumption 1. The estimation of $R_{\mathcal{I}_l,m,x}$ is obtained by using the expression of $R_{\mathcal{I}_l,F,x}(y)$ in Theorem 3 and Assumption 1.

Now we deduce the expansion of $\Sigma_{\mathcal{I}_l,F,x}$. By Lemma 1 and Assumption 1, we obtain

$$C_{JF,x}(z_1,z_2) = J\left(F_{Y|x}(z_1)\right) J\left(F_{Y|x}(z_2)\right) C_{F,x}(z_1,z_2) + O\left(|\Delta(G_x)|^{l+1}\right)$$

with $C_{F,x}(z_1,z_2)$ given in (36). Then we express $\Sigma_{\mathcal{I}_l,m,x}$ as

$$\Sigma_{\mathcal{I}_l,m,x} = \Sigma_{m,x} + O\left(|\Delta(G_x)|^{l+1}\right) \tag{B.72}$$

with

$$\Sigma_{m,x} = \int_{I(G_x)}\int_{I(G_x)} J\left(F_{Y|x}(z_1)\right) J\left(F_{Y|x}(z_2)\right) C_{F,x}(z_1,z_2)\,\mathrm{d}z_1\mathrm{d}z_2.$$

Recalling the expression of $C_{F,x}(z_1,z_2)$ in (36) and using the substitutions $\tau_i = F_{Y|x}(z_i)$ for $i=1,2$, we obtain

$$\begin{aligned}\Sigma_{m,x} &= \frac{k_{0,2}}{f_X(x)}\int_{[\underline{\tau},\overline{\tau}]}\int_{[\underline{\tau},\overline{\tau}]} \frac{J(\tau_1)J(\tau_2)(\tau_1\wedge\tau_2 - \tau_1\tau_2)}{F'_{Y|x}\left(F^{-1}_{Y|x}(\tau_1)\right)F'_{Y|x}\left(F^{-1}_{Y|x}(\tau_2)\right)}\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &= \frac{k_{0,2}\sigma^2(x)}{f_X(x)}\int_{[\underline{\tau},\overline{\tau}]}\int_{[\underline{\tau},\overline{\tau}]} \frac{J(\tau_1)J(\tau_2)(\tau_1\wedge\tau_2 - \tau_1\tau_2)}{f_\varepsilon\left(F^{-1}_{\varepsilon}(\tau_1)\right)f_\varepsilon\left(F^{-1}_{\varepsilon}(\tau_2)\right)}\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &= \frac{k_{0,2}\sigma^2(x)}{f_X(x)}\int_{[\underline{\tau},\overline{\tau}]}\int_{[\underline{\tau},\overline{\tau}]} S_J(\tau_1,\tau_2)\,\mathrm{d}\tau_1\mathrm{d}\tau_2,\end{aligned} \tag{B.73}$$

where the integrating range $[\underline{\tau},\overline{\tau}]$ is obtained by using Assumption 4. Inserting (B.73) into (B.72), we complete the proof. □

The proof of Lemma 7 is as follows.

Proof of Lemma 7 By straightforward calculations, we have

$$\begin{aligned}
\mathcal{E}_t\left(x;\ H_t\right) &= \left(\sum_{s=1}^{t} \frac{n_s h_s^2}{N_t}\right)^2 \left(B_{m,x}\right)^2 + \frac{1}{N_t} \sum_{s=1}^{t} \frac{n_s}{N_t h_s} \Sigma_{m,x} \\
&= N_t^{-2} \left\{ \left(\sum_{s=1}^{t} n_s h_s^2\right)^2 \left(B_{m,x}\right)^2 + \sum_{s=1}^{t} \frac{n_s}{h_s} \Sigma_{m,x} \right\} \\
&= N_t^{-2} \left\{ \left(\sum_{s=1}^{t-1} n_s h_s^2 + n_t h_t^2\right)^2 \left(B_{m,x}\right)^2 + \left(\sum_{s=1}^{t-1} \frac{n_s}{h_s} + \frac{n_t}{h_t}\right) \Sigma_{m,x} \right\}, \\
&= \left(\frac{N_{t-1}}{N_t}\right)^2 N_{t-1}^{-2} \left\{ \left(\sum_{s=1}^{t-1} n_s h_s^2\right)^2 \left(B_{m,x}\right)^2 + \sum_{s=1}^{t-1} \frac{n_s}{h_s} \Sigma_{m,x} \right\} \\
&\quad + N_t^{-2} n_t^2 \left\{ h_t^4 \left(B_{m,x}\right)^2 + \frac{1}{n_t h_t} \Sigma_{m,x} + 2 h_t^2 \sum_{s=1}^{t-1} \frac{n_s h_s^2}{n_t} \left(B_{m,x}\right)^2 \right\}
\end{aligned} \tag{B.74}$$

leading to

$$\mathcal{E}_t\left(x;\ H_t\right) = \left(\frac{N_{t-1}}{N_t}\right)^2 \mathcal{E}_{t-1}\left(x;\ H_{t-1}\right) + \left(\frac{n_t}{N_t}\right)^2 E_t\left(x, h_t;\ H_{t-1}\right), \tag{B.75}$$

where

$$E_t\left(x, h_t;\ H_{t-1}\right) = h_t^4 \left(B_{m,x}\right)^2 + \frac{1}{n_t h_t} \Sigma_{m,x} + 2 h_t^2 \sum_{s=1}^{t-1} \frac{n_s h_s^2}{n_t} \left(B_{m,x}\right)^2. \tag{B.76}$$

□

The proof of Theorem 8 is as follows.

Proof of Theorem 8 Denote by $\mathbb{L}^2$ the space consists of all the square integrable functions which are defined on $[0,1]$ and have a support in $[\underline{\tau}, \overline{\tau}]$. Then we have $\mathbb{J}_z\left(\underline{\tau}, \overline{\tau}\right) \subset \mathbb{L}^2$. We can express the functional $V : \mathbb{L}^2 \to \mathbb{R}$ as

$$V(J) = A(J, J) \tag{B.77}$$

with $A : \mathbb{L}^2 \times \mathbb{L}^2 \to \mathbb{R}$ a bilinear form defined by

$$A(J_1, J_2) = \int_{[\underline{\tau}, \overline{\tau}]^2} J_1\left(\tau_1\right) J_2\left(\tau_2\right) \frac{\tau_1 \wedge \tau_2 - \tau_1 \tau_2}{f_\varepsilon\left(F_\varepsilon^{-1}\left(\tau_1\right)\right) f_\varepsilon\left(F_\varepsilon^{-1}\left(\tau_2\right)\right)} \mathrm{d}\tau_1 \mathrm{d}\tau_2. \tag{B.78}$$

In the following, assume that $J(\cdot)$ minimizes $V\left(\cdot\right)$ over $\mathbb{J}_z\left(\underline{\tau}, \overline{\tau}\right)$. Recalling the constrains in (20), the equality

$$\left.\frac{\mathrm{d}V(J + \delta g)}{\mathrm{d}\delta}\right|_{\delta=0} = 0 \tag{B.79}$$

holds for any $g\left(\cdot\right) \in \mathbb{L}^2$ satisfying

$$\int_{[\underline{\tau}, \overline{\tau}]} g(\tau) \mathrm{d}\tau = 0, \quad \int_{[\underline{\tau}, \overline{\tau}]} g(\tau) F_\varepsilon^{-1}(\tau) \mathrm{d}\tau = 0. \tag{B.80}$$

By straightforward calculations, we have

$$\begin{aligned}\left.\frac{\mathrm{d}V(J+\delta g)}{\mathrm{d}\delta}\right|_{\delta=0} &= A(J,g) = \int_{[\underline{\tau},\overline{\tau}]^2} J(\tau_1)g(\tau_2)\frac{\tau_1\wedge\tau_2-\tau_1\tau_2}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_1)\right)f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)}\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &= \int_{[\underline{\tau},\overline{\tau}]}\frac{g(\tau_2)}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)}\left\{\int_{[\underline{\tau},\overline{\tau}]}J(\tau_1)\frac{\tau_1\wedge\tau_2-\tau_1\tau_2}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_1)\right)}\mathrm{d}\tau_1\right\}\mathrm{d}\tau_2\\ &= 0.\end{aligned} \tag{B.81}$$

Note that $\mathbb{L}^2$ is a Hilbert space, and thus it follows from (B.80) and (B.81) that there exist two constants C_1 and C_2 such that

$$\frac{1}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)}\left\{\int_{[\underline{\tau},\overline{\tau}]}J(\tau_1)\frac{\tau_1\wedge\tau_2-\tau_1\tau_2}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_1)\right)}\mathrm{d}\tau_1\right\} = C_1 + C_2F_\varepsilon^{-1}(\tau_2), \tag{B.82}$$

i.e.,

$$\int_{[\underline{\tau},\overline{\tau}]}J(\tau_1)\frac{\tau_1\wedge\tau_2-\tau_1\tau_2}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_1)\right)}\mathrm{d}\tau_1 = \left\{C_1 + C_2F_\varepsilon^{-1}(\tau_2)\right\}f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right). \tag{B.83}$$

Taking the operator $\frac{\partial}{\partial\tau_2}$ twice on both sides of (B.83), we deduce

$$\begin{aligned}-\frac{J(\tau_2)}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)} &= \left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\right)^2\left\{\left(C_1+C_2F_\varepsilon^{-1}(\tau_2)\right)f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)\right\}\\ &= C_1\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\right)^2 f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right) + C_2f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\right)^2F_\varepsilon^{-1}(\tau_2)\\ &\quad + 2C_2\frac{\mathrm{d}}{\mathrm{d}\tau_2}F_\varepsilon^{-1}(\tau_2)\frac{\mathrm{d}}{\mathrm{d}\tau_2}f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right)\\ &\quad + C_2F_\varepsilon^{-1}(\tau_2)\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\right)^2 f_\varepsilon\left(F_\varepsilon^{-1}(\tau_2)\right).\end{aligned} \tag{B.84}$$

By the rules of finding derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}F_\varepsilon^{-1}(\tau) = \left(\frac{1}{f_\varepsilon}\right)\left(F_\varepsilon^{-1}(\tau)\right), \quad \frac{\mathrm{d}}{\mathrm{d}\tau}f_\varepsilon\left(F_\varepsilon^{-1}(\tau)\right) = \left(\frac{f_\varepsilon'}{f_\varepsilon}\right)\left(F_\varepsilon^{-1}(\tau)\right), \tag{B.85}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}F_\varepsilon^{-1}(\tau) = -\left(\frac{1}{f_\varepsilon^3}\right)\left(F_\varepsilon^{-1}(\tau)\right), \tag{B.86}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^2 f_\varepsilon\left(F_\varepsilon^{-1}(\tau)\right) = \left(\frac{f_\varepsilon''f_\varepsilon-(f_\varepsilon')^2}{f_\varepsilon^3}\right)\left(F_\varepsilon^{-1}(\tau)\right) \tag{B.87}$$

Using the equalities in (B.85), (B.86) and (B.87), we can write (B.84) into

$$\begin{aligned}-\frac{J(\tau)}{f_\varepsilon\left(F_\varepsilon^{-1}(\tau)\right)} &= C_1\left(\frac{f_\varepsilon''f_\varepsilon-(f_\varepsilon')^2}{f_\varepsilon^3}\right)\left(F_\varepsilon^{-1}(\tau)\right) - C_2\left(\frac{f_\varepsilon'}{f_\varepsilon^2}\right)\left(F_\varepsilon^{-1}(\tau)\right)\\ &\quad + 2C_2\left(\frac{f_\varepsilon'}{f_\varepsilon^2}\right)\left(F_\varepsilon^{-1}(\tau)\right)\\ &\quad + C_2F_\varepsilon^{-1}(\tau)\left(\frac{f_\varepsilon''f_\varepsilon-(f_\varepsilon')^2}{f_\varepsilon^3}\right)\left(F_\varepsilon^{-1}(\tau)\right),\end{aligned} \tag{B.88}$$

which further leads to

$$\begin{aligned}
J(\tau) = & -\left(C_1 + C_2 F_\varepsilon^{-1}(\tau)\right)\left(\frac{f_\varepsilon'' f_\varepsilon - \left(f_\varepsilon'\right)^2}{f_\varepsilon^2}\right)\left(F_\varepsilon^{-1}(\tau)\right) - C_2\left(\frac{f_\varepsilon'}{f_\varepsilon}\right)\left(F_\varepsilon^{-1}(\tau)\right) \\
= & -\left(C_1 + C_2 F_\varepsilon^{-1}(\tau)\right)(\log f_\varepsilon)''\left(F_\varepsilon^{-1}(\tau)\right) - C_2 (\log f_\varepsilon)'\left(F_\varepsilon^{-1}(\tau)\right) \\
= & -C_1 (\log f_\varepsilon)''\left(F_\varepsilon^{-1}(\tau)\right) \\
& -C_2\left\{F_\varepsilon^{-1}(\tau)(\log f_\varepsilon)''\left(F_\varepsilon^{-1}(\tau)\right) + (\log f_\varepsilon)'\left(F_\varepsilon^{-1}(\tau)\right)\right\} \\
= & \, C_1 \Psi_1(\tau) + C_2 \Psi_2(\tau)
\end{aligned} \tag{B.89}$$

with Ψ_1 and Ψ_2 given in (49).

Recalling (20) and (21), the constants (C_1, C_2) should satisfy

$$\begin{aligned}
C_1 A_{01} + C_2 A_{02} = 1, \quad C_1 A_{11} + C_2 A_{12} = 0, \quad \text{for} \;\; z = m, \\
C_1 A_{01} + C_2 A_{02} = 0, \quad C_1 A_{11} + C_2 A_{12} = 1, \quad \text{for} \;\; z = \sigma.
\end{aligned} \tag{B.90}$$

By this we complete the proof.

□

The proof of Corollary 9 is as follows.

Proof of Corollary 9 By (49), we have

$$\begin{aligned}
\Psi_i(\tau) = & -\left\{(\delta_{1i} + \delta_{2i}\mathrm{I_d})\log f_\varepsilon\right\}''\left(F_\varepsilon^{-1}(\tau)\right) \\
= & -\left\{(\delta_{1i} + \delta_{2i}\mathrm{I_d})(\log f_\varepsilon)'' + 2\delta_{2i}(\log f_\varepsilon)'\right\}\left(F_\varepsilon^{-1}(\tau)\right)
\end{aligned} \tag{B.91}$$

By the model (1), we have

$$F_{Y|x}^{-1}(\tau) = m(x) + \sigma(x) F_\varepsilon^{-1}(\tau), \tag{B.92}$$

$$\sigma^l(x) F_{Y|x}^{(l)}(y) = f_\varepsilon^{(l-1)}\left(\frac{y - m(x)}{\sigma(x)}\right) \quad \text{for} \;\; l = 1, 2, 3. \tag{B.93}$$

Using the relations in (B.92) and (B.93), we can obtain

$$\begin{aligned}
(\log f_\varepsilon)'\left(\frac{y - m(x)}{\sigma(x)}\right) & = \frac{f_\varepsilon'}{f_\varepsilon}\left(\frac{y - m(x)}{\sigma(x)}\right) = \sigma(x)\frac{F_{Y|x}''(y)}{F_{Y|x}'(y)} \\
& = \sigma(x)\left(\log F_{Y|x}'\right)'(y),
\end{aligned} \tag{B.94}$$

$$\begin{aligned}
(\log f_\varepsilon)''\left(\frac{y - m(x)}{\sigma(x)}\right) & = \frac{f_\varepsilon'' f_\varepsilon - \left(f_\varepsilon'\right)^2}{f_\varepsilon^2}\left(\frac{y - m(x)}{\sigma(x)}\right) \\
& = \sigma^2(x)\frac{F_{Y|x}^{(3)} F_{Y|x}^{(1)} - \left(F_{Y|x}^{(2)}\right)^2}{\left(F_{Y|x}^{(1)}\right)^2}(y) \\
& = \sigma^2(x)\left(\log F_{Y|x}'\right)''(y).
\end{aligned} \tag{B.95}$$

Then we can rewrite (B.91) into

$$\begin{aligned}\Psi_i\left(F_{Y|x}(y)\right) &= -\left\{(\delta_{1i}+\delta_{2i}\mathrm{I_d})(\log f_\varepsilon)'' + 2\delta_{2i}(\log f_\varepsilon)'\right\}\left(\frac{y-m(x)}{\sigma(x)}\right)\\ &= -\left\{\left(\delta_{1i}+\delta_{2i}\frac{\mathrm{I_d}-m(x)}{\sigma(x)}\right)\sigma^2(x)\left(\log F'_{Y|x}\right)'' + 2\delta_{2i}\sigma(x)\left(\log F'_{Y|x}\right)'\right\}(y)\\ &= -\sigma(x)\left\{(\delta_{1i}\sigma(x)-\delta_{2i}m(x)+\delta_{2i}\mathrm{I_d})\log F'_{Y|x}\right\}''(y).\end{aligned} \tag{B.96}$$

By (B.96), we have

$$\Psi_1\left(F_{Y|x}(y)\right) = -\sigma^2(x)\left(\log F'_{Y|x}\right)''(y), \tag{B.97}$$

leading to

$$a_1\Psi_1(\tau) = -\int_{I_*}\left(\log F'_{Y|x}\right)''\left(F^{-1}_{Y|x}(\tau)\right)\mathrm{d}x = \tilde{\Psi}_1(\tau) \tag{B.98}$$

with $a_1 = \int_{I_*}\sigma^{-2}(x)\,\mathrm{d}x$. Similarly, it follows from (B.96) that

$$\begin{aligned}\Psi_2\left(F_{Y|x}(y)\right) &= \sigma(x)\left\{m(x)\left(\log F'_{Y|x}\right)'' - \left(\mathrm{I_d}\log F'_{Y|x}\right)''\right\}(y)\\ &= -\frac{m(x)}{\sigma(x)}\Psi_1\left(F_{Y|x}(y)\right) - \sigma(x)\left(\mathrm{I_d}\log F'_{Y|x}\right)''(y),\end{aligned} \tag{B.99}$$

leading to

$$\frac{1}{\sigma(x)}\Psi_2(\tau) + \frac{m(x)}{\sigma^2(x)}\Psi_1(\tau) = -\left(\mathrm{I_d}\log F'_{Y|x}\right)''\left(F^{-1}_{Y|x}(\tau)\right), \tag{B.100}$$

which further implies

$$\begin{aligned}a_2\Psi_2(\tau)+a_3\Psi_1(\tau) &= -\int_{I_*}\left(\mathrm{I_d}\log F'_{Y|x}\right)''\left(F^{-1}_{Y|x}(\tau)\right)\mathrm{d}x\\ &= \tilde{\Psi}_2(\tau)\end{aligned} \tag{B.101}$$

with $a_2 = \int_{I_*} 1/\sigma(x)\mathrm{d}x$ and $a_3 = \int_{I_*} m(x)/\sigma^2(x)\mathrm{d}x$. Combining (B.98) and (B.101), we conclude that

$$\Psi_1(\tau) = a_1^{-1}\tilde{\Psi}_1(\tau), \quad \Psi_2(\tau) = -a_2^{-1}a_3\tilde{\Psi}_2(\tau) + a_2^{-1}\tilde{\Psi}_2(\tau), \tag{B.102}$$

i.e., $\{\Psi_1, \Psi_2\}$ can be linearly expressed by $\left\{\tilde{\Psi}_1, \tilde{\Psi}_2\right\}$. Then by combining Lemma 2 and Theorem 8, we complete the proof. □

References

Serfling, R.J. 1980. *Approximation theorems of mathematical statistics.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York.